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On the identification of the tip vortex core

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An overview is given for identification criteria of tip vortices, trailing from lifting surfaces of aircraft. Q-distribution is used as the main vortex identification method in this work. According to the definition of Q-criterion, the vortex core is bounded by a surface on which the norm of the vorticity tensor is equal to the norm of the strain-rate tensor. Moreover, following conditions are satisfied inside of the vortex core: (i) net (non-zero) vorticity tensor; (ii) the geometry of the identified vortex core should be Galilean invariant. Based on the existing analytical vortex models, a vortex center of a two-dimensional vortex is defined as a point, where the Q-distribution reaches a maximum value and it is much greater than the norm of the strain-rate tensor (for an axisymmetric 2D vortex, the norm of the vorticity tensor tends to zero at the vortex center). Since the existence of the vortex axis is discussed by various authors and it seems to be a fairly natural requirement in the analysis of vortices, the above-mentioned conditions (i), (ii) can be supplemented with a third condition (iii): the vortex core in a three-dimensional flow must contain a vortex axis. Flows, having axisymmetric or non-axisymmetric (in particular, elliptic) vortex cores in 2D cross-sections, are analyzed. It is shown that in such cases Q-distribution can be used to obtain not only the boundary of the vortex core, but also to determine the axis of the vortex. These concepts are illustrated using the numerical simulation results for a finite span wing flow-field, obtained using the Reynolds-Averaged Navier – Stokes (RANS) equations with k-ω turbulence model.

Keywords: tip vortex, vortex core, Q-criterion, vortex axis, numerical modeling

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МАТЕМАТИЧЕСКИЕ ОСНОВЫ И ЧИСЛЕННЫЕ МЕТОДЫ МОДЕЛИРОВАНИЯ

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К вопросу об определении ядра концевого вихря

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Дается обзор критериев, используемых при идентификации концевых вихрей, сходящих с несущих поверхностей летательного аппарата. В качестве основного метода идентификации вихря используется Q-критерий, в соответствии с которым ядро вихря ограничено поверхностью, на которой норма тензора завихренности равна норме тензора сдвиговых деформаций. При этом внутри ядра вихря должны выполняться следующие условия: (i) ненулевое значение нормы тензора завихренности, (ii) геометрия ядра вихря должна удовлетворять условию галилеевой инвариантности. На основе аналитических моделей вихря дается определение понятия центра двумерного вихря как точки, в которой Q-распределение принимает максимальное значение и много больше нормы тензора сдвиговых деформаций (для осесимметричного 2D-вихря норма тензора сдвиговых деформаций в центре вихря стремится к нулю). Поскольку необходимость существования оси вихря обсуждается в работах различных авторов и выглядит достаточно естественным требованием при анализе концевых вихрей, упомянутые выше условия (i), (ii) дополнены условием (iii): ядро вихря в трехмерном потоке должно содержать ось вихря. Анализируются течения, имеющие в 2D-сечениях осевую симметрию, а также форму ядра вихря, отличающуюся от окружности (в частности, эллиптического вида). Показывается, что в этом случае с использованием Q-распределения можно не только определить область ядра вихря, но и выделить ось ядра вихря. Для иллюстрации введенных понятий используются результаты численного моделирования обтекания крыла конечного размаха на базе решения осредненных по Рейнольдсу стационарных уравнений Навье – Стокса (RANS). Замыкание уравнений Навье – Стокса осуществлялось с использованием модели турбулентности k-ω.

Ключевые слова: концевой вихрь, ядро вихря, Q-критерий, ось вихря, численное моделирование

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1. Introduction

A tip vortex is most often interpreted to as a class of isolated (free) vortices in unbounded space, generated by lifting surfaces of aircraft: an airplane wing, rotor blades of a helicopter, a fuselage at an angle of attack, etc. Furthermore, the lifting surface can generate a single vortex or multiple vortices (e. g., vortices trailing behind a wing with deflected flaps).

Tip vortices are a source of induced drag and negatively affect aircraft performance. The ability to assess tip vortices of aircraft is essential for ensuring the safety and efficiency of flight operations [ICAO, 2012]. An appropriate choice of a suitable method for identification of vortical structures, generated by aircraft's lifting surfaces, continues to be an important and highly relevant task [Ahmad, Proctor, 2014; Epps, 2017].

Wing-tip vortices of an airplane is an example of a classical physical interpretation of a vortex. The Kutta – Joukowski theorem can be used to estimate the lift of a wing using the circulation of the tip vortex. Visualization and analysis of the vortical structures in wind tunnels allow localizing the spatial position of the vortex. The interest lies primarily in localizing the vortex core, because the tangential velocities reach peak values at the boundaries of the vortex core.

Currently, the analysis of the flow structure and performance of aircraft is done predominantly using numerical modeling. However, performing a high-fidelity numerical simulation requires substantial computational power. In certain cases, when it is necessary to carry out the numerical modeling of a vortical structure at large downstream distances from an aircraft, the required computational resources can become prohibitively large.

Isolated vortices (including tip vortices) are characterized by a near-conical shape of the vortex core. Vortical structures, with structures that are similar to tip vortices, can be encountered in semi-confined flow regions. Such examples include vortices, generated by extended ledges at low Reynolds numbers. This type of vortices is bounded in space (it is possible to identify the beginning and end of the vortex), and the cross section of the vortex can have a shape that is close to an ellipse. At the same time, tip vortices are a classic example of the term "vortex". The definitions (physical and mathematical) used in the literature can be tested during the analysis of the structure of the tip vortex. In this sense, a more general definition of the vortex, which goes beyond the definition of the tip vortex, should precede the analysis of the structure of the tip vortex encountered in a physical experiment or numerical modeling.

Intuitively, the vortex core can be characterized as a tube, whose surface consists of vortex lines [Lamb, 1945]. A more rigorous definition of the vortex core properties was given in [Jeong, Hussain, 1995]:

(i) A vortex core must have a net vorticity (hence, net circulation). Thus, potential flow regions are excluded from vortex cores, and a potential vortex is a vortex with zero cross-section.

(ii) The geometry of the identified vortex core should be Galilean invariant.

It is noted in [Haller, 2005], that there is no single universally accepted and correct method for identifying vortices. Currently, various Eulerian vortex identification methods are used. According to [Chong, Perry, Cantwell, 1990], a vortex is defined as a region of complex eigenvalues ∇u (Δ-criterion). In [Hunt, Wray, Moin, 1988], the vortex core is defined as a region, where the norm of the vorticity tensor $\|\boldsymbol{\Omega}\|$ is larger than the norm of the strain-rate tensor $\|\boldsymbol{S}\|$. The condition $Q = \frac{1}{2}\left(\|\boldsymbol{\Omega}\|^2 - \|\boldsymbol{S}\|^2\right) > 0$, introduced in [Hunt, Wray, Moin, 1988] for obtaining the vortex core region, is known as Q-criterion. Some works mention that in addition to $Q > 0$ condition, the vortex core region must be connected [Jeong, Hussain, 1995]. There are other Eulerian vortex core identification criteria, which are based on the analysis of the flow velocity field: λ-criterion [Zhou et al., 1999], λ_2-criterion [Jeong, Hussain, 1995], $\frac{\lambda_{cr}}{\lambda_i}$-criterion [Chakraborty, Balachandar, Adrian, 2005], etc. [Epps, 2017].

The above-mentioned methods define the vortex core as a localized three-dimensional (3D) spatial structure. A non-gradient method of Lagrangian coherent structures (LCS) is used in [Haller, 2005], which is based on analyzing the particle trajectories in the three-dimensional space.

There are also two-dimensional (2D) non-gradient vortex analysis methods: Γ_1 and Γ_2 criteria [Graftieaux, Michard, Grosjean, 2001; Huang, Green, 2015; Coletta et al., 2019], Cross-Sectional Lines (CSL) method [Vollmers, 2001; Bussière, Nobes, Koch, 2012], the winding angle (WA) method [Portela, 1999; Sadarjoen, 2000] and Corsiglia method [Corsiglia, Schwind, Chigier, 1973].

It is worth noting, that despite the term vortex ("eddy zone", "vortex tube") was used in [Hunt, Wray, Moin, 1988], subsequent works refer to the regions, identified by Q-criterion, as "vortical regions". The use of this term can be explained by two reasons. Firstly, the definition of Q-criterion suggests that the isosurface Q = const can bound a region, which upon closer inspection can reveal that it is not a vortex [Haller, 2005] (due to the presence of substantial shear deformations). Secondly, the vortex may disintegrate at significant downstream distances from the source of disturbances (for example, from the lifting surface). In such a case, the isosurface of the Q-distribution would not be a connected surface, but rather it would have the form of a vortical structure that consists of isolated subregions.

Another important question related to the study of vortices is finding the spatial position of the vortex axis, because a number of works (e. g., [Wu, Xiong, Yang, 2005; Wu, Ma, Zhou, 2006]) mention that vortex identification methods should be able to identify the vortex axis in three-dimensional flows. One of the first approaches to determining the vortex axis was based on the theory of critical points [Sujudi, Haimes, 1995]. The eigenvalues and eigenvectors of the velocity gradient tensor, calculated in the vicinity of the critical point, determine the local flow pattern near that point. A critical point is obtained as a point with one real eigenvalue and a complex conjugate pair. In this case, the critical point is assumed to be a point lying on the vortex axis.

Kolar [Kolar, 2007] mentions obtaining a vortex axis as one of the requirements for vortex identification methods. In [Kolar, 2007], a vortex identification method is proposed based on the triple decomposition of the fluid motion: rotational, shearing, straining deformations. The possibility of such a decomposition was proven for simplest model cases [Hoffman, 2021].

In [Gao et al., 2019], the vortex axis is considered as a line, where the vector product of a certain vector (Liutex) and the gradient of its magnitude is zero. The computational technique of this approach requires transforming the coordinate system for each vortex [Gao et al., 2019]. The main focus of using the Liutex vector is dedicated to constructing the vortex axis, whereas the vortex core region is obtained using an algorithm, that is not related to the construction of the Liutex vector.

Since the existence of a vortex axis seems to be a fairly natural requirement for the analysis of tip vortices, the above-mentioned conditions (i), (ii) can be supplemented by an additional condition:

(iii) The vortex core must contain an axis in three-dimensional flows.

It follows from condition (iii) that the vortex identification methods must be able to identify the vortex center in two-dimensional cross-sections. According to some existing analytic models, atmospheric vortical structures (e. g., tornados) can also be considered as axial vortices. Some works employ simple analytical models to assess the mutual influence of vortices (e. g., see [Kida, 1981]).

According to conditions (i)–(iii), the definition of the vortex core depends on the vortex identification method. The requirements for vortex identification methods are mentioned in [Kolar, 2007; Gao et al., 2019]. At the same time, a general definition of a vortex, without connection to a vortex identification method, is given in [Wu, Xiong, Yang, 2005]: *a generally applicable vortex definition should be able to identify the vortex axis and allow for arbitrary axial strain. What matters in the definition should be only the axial vorticity component or rotational motion of the fluid on a cross plane, compared to the strain rate on that plane*.

The center of a vortex can be determined using various vortex identification methods. For example, Γ_1-criterion defines a non-dimensional scalar function, that evaluates the position of the vortex center as the location where the average angle between the radius vector and the velocity vector approaches $\frac{\pi}{2}$. Γ_1-criterion is not Galilean invariant. The Galilean invariance of Γ_2-criterion is achieved by taking into account the local convection velocity of the integration domain [Graftieaux, Michard, Grosjean, 2001]. Although Γ_2-criterion is a reliable and robust method, which is well suited for PIV data analysis, the accuracy of this method depends significantly on the size of the integration region [Coletta et al., 2019].

The maximum vorticity magnitude can be used to obtain the position of the vortex center [Gerz, Holzapfel, Darracq, 2002; Cheng et al., 2019; Schauerhamer, Robinson, 2017]. However, this approach is unable to distinguish between the vortex core and the shear flow, especially if the background shear is comparable to the vorticity magnitude within the vortex [Jeong, Hussain, 1995]. Contour lines of vorticity (CLV) method defines the origin of the vortex as a point where the vorticity value reaches a local extremum [Vollmers, 2001]. The WA-method can also be used to locate the vortex center, but obtained results can depend on the adopted coordinate system [Vollmers, 2001].

The CSL method allows obtaining the center and the drift velocity of the vortex. The center of a vortex can be determined by connecting regions of opposite velocity directions (OVD), but it is a rather crude method [Vollmers, 2001].

The Corsiglia method is based on the assumption that the velocity, measured in the planes orthogonal to the vortex axis, represents the azimuthal velocity of the vortex [Corsiglia, Schwind, Chigier, 1973]. In [Gao et al., 2019], the vortex center is determined as an intersection point of the Liutex vector and a cross-section of the vortex.

The analysis of the existing literature suggests that Q-criterion is the most popular vortex identification method used in the analysis of the numerical and physical modeling results. Therefore, the questions related to the vortex identification will be discussed in this work using Q-criterion.

Some authors [Kolar, 2007; Gao et al., 2019] argue that Eulerian methods (including Q-criterion) do not define the vortex axis. Therefore, the main attention of this work is dedicated to discussing the possibility of using the Q-distribution to obtain the vortex axis.

In this work, the analytic vortex models (in particular, those used in the study of tip vortices generated by aircraft's lifting surfaces) are used to give a definition of the vortex core using Q-distribution. In addition, the definitions are given for the vortex center (for a two-dimensional vortex) and the vortex axis (for a three-dimensional case).

This work consists of several sections. The definition of Q-criterion is given for two- and three-dimensional flows in the second section. In the third section, the expression for the circumferential velocity distribution of an isolated vortex is determined for a case, when the streamlines in the vicinity of the vortex core center have the shape of a circle. The obtained expression for the circumferential velocity is compared with analytical vortex models. Then a more general velocity field is considered in the same section, when the shape of the streamlines is close to an elliptic form. The relationship between the values of the norm of the vorticity tensor $\|\boldsymbol{\Omega}\|$ and the norm of the rate of strain tensor $\|\boldsymbol{S}\|$ is determined as a function of an ellipse eccentricity. In the fourth section, the definition, given for a center of a two-dimensional vortex, is used to formulate the definition of the vortex axis in a three-dimensional flow. Furthermore, the definition of the vortex, given in [Wu, Xiong, Yang, 2005], is reformulated. In the fifth section, the numerical modeling results of the flow-field of a finite-span wing are used to illustrate the introduced concepts. The numerical modeling was performed using the VMB package (version of the HMB 2.0 package [Barakos et al., 2005], adapted for KNRTU-KAI), based on the solution of the Reynolds-averaged Navier – Stokes equations (RANS). The sixth section considers the identification of a pair of 2D vortices for the high-resolution and low-resolution discrete velocity fields. The visualization of the velocity fields was carried out in Tecplot.

2. *Q*-criterion

Q-crterion is one of the most popular vortex identification methods, which is based on local analysis of the velocity field with non-zero values of the norm of the vorticity tensor $\boldsymbol{\Omega}$. In addition, the norm of the rate of strain tensor $\boldsymbol{S}$ may also take a non-zero value. According to *Q*-criterion, and in accordance with the condition (i), the vortex core is considered to be a region where the Euclidian norm of the vorticity tensor $||\boldsymbol{\Omega}||$ exceeds the norm of the rate of the strain tensor $||\boldsymbol{S}||$ [Hunt, Wray, Moin, 1988; Haller, 2005]:

$$Q = \frac{1}{2}\left(||\boldsymbol{\Omega}||^2 - ||\boldsymbol{S}||^2\right) = \frac{1}{2}(\Omega_{ij}\Omega_{ji} - S_{ij}S_{ji}) > 0. \tag{1}$$

Here, $\Omega_{ij} = \frac{u_{i,j}-u_{j,i}}{2}$ and $S_{ij} = \frac{u_{i,j}+u_{j,i}}{2}$, where $u_{i,j}$ are the spatial derivatives of the velocity components. Criterion (1) for two-dimensional incompressible flows is also known as the Okubo – Weiss criterion [Okubo, 1970; Weiss, 1991].

During the analysis of wing-tip vortices, *Q*-criterion can be expressed in a non-dimensional form with respect to the wing chord length c and the free-stream velocity V_∞:

$$\overline{Q} = \frac{c^2}{2\pi^2 V_\infty^2} Q = \left(|\overline{\Omega}|^2 - |\overline{S}|^2\right) > 0. \tag{2}$$

Here,

$$|\overline{\Omega}|^2 = \frac{c^2}{V_\infty^2}\frac{||\boldsymbol{\Omega}||^2}{4\pi^2},$$
$$|\overline{S}|^2 = \frac{c^2}{V_\infty^2}\frac{||\boldsymbol{S}||^2}{4\pi^2}.$$

In cases when a two-dimensional velocity field is considered (at some fixed cross-section of the vortex), the values of $||\boldsymbol{\Omega}||$ and $||\boldsymbol{S}||$ are defined by expressions:

$$||\boldsymbol{\Omega}|| = \sqrt{2}|\Omega_{23}|, \tag{3}$$

$$||\boldsymbol{S}|| = \sqrt{2S_{23}S_{23} + S_{22}S_{22} + S_{33}S_{33}}, \tag{4}$$

where

$$\Omega_{23} = \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right), \tag{5}$$

$$S_{23} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \quad S_{22} = \frac{\partial v}{\partial y}, \quad S_{33} = \frac{\partial w}{\partial z}. \tag{6}$$

The geometry of the vortex core, identified using Q-criterion, is Galilean invariant [Haller, 2005]. The relatively straightforward algorithm of determining the vortex core boundary explains the popularity of Q-criterion as a reliable vortex core identification tool, used in a wide range of applications.

3. Using the Q-distribution to determine the center of the vortex core

Generally during the analysis of 2D vortical flow structures, the norms of the vorticity and strain-rate tensors take non-zero values for a velocity vector $\overline{U} = f(v, w)$: $\|\mathbf{\Omega}\| \neq 0$ and $\|\boldsymbol{S}\| \neq 0$. Consider an isolated vortex ($\|\mathbf{\Omega}\| \neq 0$), localized on a plane using condition (1) at the origin of the coordinate system ($y = 0$, $z = 0$). If in addition the following condition is satisfied in the neighborhood of the point of the localized vortex:

$$\|\boldsymbol{S}\|^2 = 0, \tag{7}$$

then the flow is purely rotational (there is no deformation component). In this case, it is natural to accept

$$v = w = 0, \tag{8}$$

where $r = \sqrt{y^2 + z^2}$ is the magnitude of the radius $\vec{r}$ vector in a Cartesian coordinate system. Condition (8) is consistent with the method [Sujudi, Haimes, 1995], that is based on the assumption that the vortex axis passes through the critical points of the flow.

For a steady flow of an incompressible fluid, the components are related to each other through the continuity equation:

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{9}$$

Note that in a more general case $\|\boldsymbol{S}\| \neq 0$, i. e. the flow may not be purely rotational at the origin of the coordinate system.

Most of existing analytical vortex models are three-dimensional, and the velocity of such a flow has three components: tangential, radial and axial [Vatistas, 1998]. Moreover, radial and axial components have an order of Re^{-1} relative to the tangential component (Re is the Reynolds number determined by the characteristic parameters of the flow).

In the simplest case, we will limit ourselves to considering a circular motion of the flow in the vicinity of the origin of the coordinate system (the center of the vortex): it is assumed that $\mathrm{Re} \gg 1$ and the influence of the radial and axial components on the norms of vorticity and strain-rate tensors can be neglected. For such a flow, the velocity vector $\overline{U}(v, w)$ is oriented orthogonally with respect to the radius vector $\overline{r}$, and the components of the velocity $\overline{U}$ are defined by

$$\begin{aligned} v(y, z) &= -U(y, z)\sin\theta, \\ w(y, z) &= U(y, z)\cos\theta, \end{aligned} \tag{10}$$

where $U(y, z)$ is a smooth function, $\cos\theta = \frac{y}{r}$, $\sin\theta = \frac{z}{r}$.

According to this representation, the streamlines of the flow have circular shapes. The scalar value of the velocity changes arbitrarily along a streamline for $r = \mathrm{const}$. In order to eliminate the singularity at the origin of the coordinate system in the representation (10), and by taking into account (8), it is assumed that

$$U(0, 0) = 0. \tag{11}$$

By substituting (10) into (9), it can be shown that the function $U(y, z)$ is axisymmetric:

$$U(y, z) = F(r), \tag{12}$$

where $F(r)$ is an arbitrary function.

In a more general case, the flow in a confined space does not necessarily has the property of the axial symmetry, at the same time the shape of the streamlines can be close to a circular shape in the vicinity of the vortex core. Such a flow is analyzed, in particular, in [Danaila, Kaplanski, Sazhin, 2015], where the vortex flow near a solid wall is considered.

By taking into account (11), the function $F(r)$ in the vicinity of the origin of the coordinate system ($r \approx 0$) can be represented in the form of a convergent power series [Маркушевич, 1950]:

$$F(r) = a_n r^n + a_{n+1} r^{n+1} + \ldots, \tag{13}$$

where a_n, a_{n+1}, ... are constants; $n \geqslant 1$ is an integer. A substitution of (13) into (3)–(6) gives the expressions

$$\begin{aligned} \|\boldsymbol{\Omega}\|^2 &= \frac{a_n^2}{2}(n+1)^2 r^{2(n-1)} + O_\Omega(r^m), \\ \|\boldsymbol{S}\|^2 &= \frac{a_n^2}{2}(n-1)^2 r^{2(n-1)} + O_S\left(r^k\right). \end{aligned} \tag{14}$$

Here $O_\Omega(r^m)$ and $O_S\left(r^k\right)$ are polynomials that contain the terms proportional to r^m and r^k, where m, $k \geqslant 1$ are integers.

It can be seen from (14) that for $r \to 0$, the condition $\|\boldsymbol{\Omega}\| \neq 0$ is satisfied only when $n = 1$. It represents the linear dependence of the velocity modulus on the radius. This also ensures that the strain-rate tensor is equal to zero, as described by condition (7).

Various 2D vortex models are consistent with dependence (12): Rankine, Lamb – Oseen [Holzäpfel et al., 2012], Proctor [Ahmad, Proctor, 2014], Burnham – Hallock [Burnham, Hallock, 2013; Proctor, 1998] models, etc. The Lamb – Oseen, Proctor, Burnham – Hallock models are used, in particular, for the analysis of aircraft tip vortices [Ahmad, Proctor, 2014]. A linear radial distribution of circumferential velocity can be used to model a tornado [Loper, 2020]. The Rankine model defines the tangential velocity $v_\theta(r)$ in the polar coordinate system as

$$\begin{aligned} v_\theta(r) &= \frac{\Gamma_0}{2\pi r_c}\frac{r}{r_c} \quad \text{for } r \leqslant r_c, \\ v_\theta(r) &= \frac{\Gamma_0}{2\pi r} \quad \text{for } r > r_c, \end{aligned} \tag{15}$$

where r is the distance from the center of the vortex; r_c is the vortex core radius; Γ_0 is the circulation value of the vortex.

By substituting (15) into (3)–(6), the following expressions can be obtained for the Rankine vortex:

$$\|\boldsymbol{\Omega}\| = \sqrt{2}|\Omega_{23}|_{\text{Rankine}} = \frac{1}{\sqrt{2}}\frac{\Gamma_0}{\pi r_c^2}, \quad \|\boldsymbol{S}\| = 0.$$

Burnham – Hallock model defines the dependence of the tangential velocity on the vortex radius by the expression

$$v_\theta(r) = \frac{\Gamma_0}{2\pi r}\frac{r^2}{r^2 + r_c^2}.$$

Similar to the Rankine vortex, the Burnham – Hallock model has a nearly linear dependence of $v_\theta(r)$ in the vicinity of the vortex center. An analogous statement is also true for other 2D models mentioned above. Therefore, it can be assumed that in the vicinity of the center of an axisymmetric vortex the following expression is satisfied

$$\|\boldsymbol{\Omega}\| \to \|\boldsymbol{\Omega}\|_{\max}, \quad \|\boldsymbol{S}\| \to 0 \quad \text{for } r \to 0, \tag{16}$$

where $r = 0$ corresponds to the center of the vortex. Conditions (2) and (16) can be used for obtaining the coordinate $\mathbf{x}_c$ of the center of a 2D vortex using Q-criterion [Степанов, Кусюмов, 2024]:

$$\text{if } \mathbf{x} \to \mathbf{x}_c, \text{ then } Q \to Q_{\max}, \ \|\boldsymbol{S}\| \to 0. \tag{17}$$

As shown in (15), the velocity $v_\theta(r)$ of the idealized Rankine vortex model has a linear dependence inside the vortex core, and there is an abrupt transition to the nonlinear part of $v_\theta(r)$ at the outer boundary of the core. Other known analytical vortex models employ nonlinear dependence of $v_\theta(r)$, with a near linear dependence of $v_\theta(r)$ in the vicinity of the center of the vortex. This type of $v_\theta(r)$ distribution allows not only to determine the position of the vortex center, but also to identify the vortex core using condition (1).

It should be noted that the vector field (10) considered above is axisymmetric, as a result there are no shear deformations in the vicinity of the vortex center. To summarize, consider the vicinity of a vortex center (as $r \to 0$) of a non-axisymmetric vector field [Kida, 1981] (in the non-dimensional form):

$$\begin{aligned} v(y, z) &= \alpha y - \gamma z, \\ w(y, z) &= -\alpha z + \gamma y, \end{aligned} \tag{18}$$

where α, γ are constants. For $\alpha \neq 0$, $\gamma \neq 0$ and $|\gamma| > |\alpha|$ the streamlines of the vector field (18) are close to having elliptical shapes. Figure 1 shows the velocity field and streamlines plotted in Tecplot using $\alpha = 0.3$ and $\gamma = 1$ values.

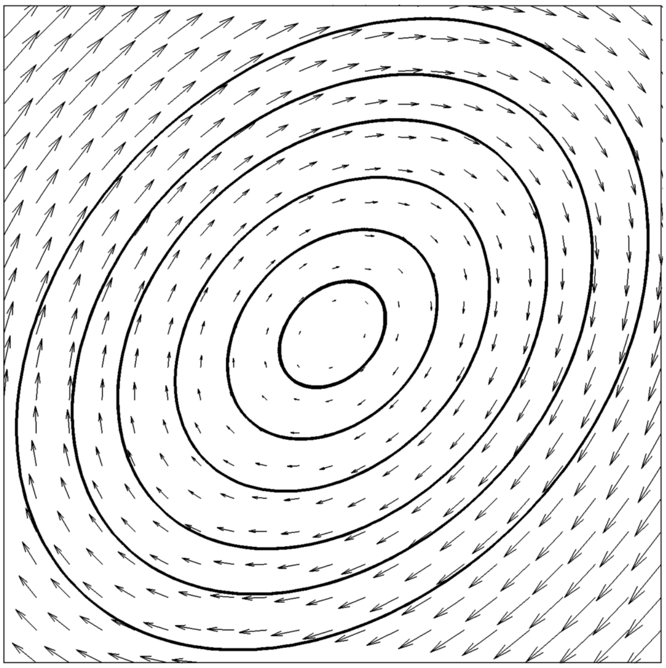

Figure 1. Streamlines for a non-axisymmetric vector field model

Substituting $v(y, z)$ and $w(y, z)$ from (18) into (3)–(6) yields expressions

$$\begin{aligned} \|\mathbf{\Omega}\|^2 &= 2\gamma^2, \\ \|\boldsymbol{S}\|^2 &= 2\alpha^2. \end{aligned} \tag{19}$$

The quantities γ and α in (19) define, respectively, the vorticity and shear deformation of the vector field in the vicinity of the considered vortex center. Furthermore, when $|\gamma| > |\alpha|$, the following relation holds

$$Q = \left(\gamma^2 - \alpha^2\right) > 0. \tag{20}$$

Since the streamlines have an elliptical shape, it is possible to define the eccentricity of the streamlines:

$$\varepsilon = \sqrt{1 - \frac{r_{\min}^2}{r_{\max}^2}},$$

where $r_{\min}$ and $r_{\max}$ are the semi-minor and semi-major axes of the ellipse. It can be shown that the eccentricity of the considered vector field is defined by the expression

$$\varepsilon = \sqrt{\frac{2\alpha}{\alpha + \gamma}}. \tag{21}$$

It can be seen from (21), that if $\gamma > 3.16\alpha$, then $\|\mathbf{\Omega}\|^2 > 10\|\boldsymbol{S}\|^2$ and the eccentricity $\varepsilon < 0.69$. Thus, it can be said that

$$\|\mathbf{\Omega}\|^2 \gg \|\boldsymbol{S}\|^2 \quad \text{for} \;\; \varepsilon < 0.5. \tag{22}$$

Therefore, for a non-axisymmetric vector field with moderate eccentricity, the square of the norm of the vorticity tensor is more than an order of magnitude greater than the square of the norm of the strain-rate tensor.

It should be noted that condition (22) is consistent with [Moore, Saffman, 1971], where the structure of a two-dimensional elliptical vortex in a uniform straining field was analyzed. It was shown [Moore, Saffman, 1971], that the stability of the elliptical shape of the vortex is preserved when $\frac{\alpha}{\gamma} < 0.15$ (which corresponds to $\varepsilon < 0.51$).

By using (20) and (21), it is possible to express the Q-value using the eccentricity of a streamline:

$$Q = \gamma^2 \frac{1 - \varepsilon^2}{\left(1 - \frac{\varepsilon^2}{2}\right)^2}. \tag{23}$$

The following expression can be obtained from (23) under condition (22):

$$Q > 0.98\gamma^2.$$

It is also possible to consider the influence of the eccentricity of a streamline on the value of Q^1-distribution [Степанов, Кусюмов, Баракос, 2022]:

$$Q^1 = \frac{1}{\sqrt{2}}(\|\mathbf{\Omega}\| - \|\boldsymbol{S}\|).$$

For the considered vector field

$$Q^1 = \gamma - \alpha = \gamma \frac{1-\varepsilon^2}{1-\frac{\varepsilon^2}{2}}$$

and under the condition $\varepsilon < 0.5$, the estimate is

$$Q^1 > 0.857\gamma.$$

Thus, for $\varepsilon < 0.5$ the value of Q differs from the square of the norm of the vorticity tensor by no more than 2 %. When a vector field is analyzed using Q^1-criterion, the condition $0.857\gamma < Q^1 < \gamma$ suggests that the streamlines in the vicinity of the vortex center have elliptical shapes.

Therefore, for 2D vector fields with elliptical stream functions and the eccentricity $\varepsilon < 0.5$, the condition (17), used to determine the vortex center, can be replaced by:

$$\text{if } \mathbf{x} \to \mathbf{x}_c, \text{ then } Q \to Q_{\max}, \ \|\boldsymbol{\Omega}\|^2 \gg \|\boldsymbol{S}\|^2. \tag{24}$$

In this work, conditions (17) and (24) are used to identify a vortex center in cross-sections of a vortex.

4. Definition of the vortex core and its axis using Q-distribution

Conditions (17), (24) can be used to find the location of the vortex center in various (2D) cross-sections of the tip vortex. It is necessary to give a general definition of the tip vortex core and vortex axis for 3D flows before considering a 2D cross-section of the vortex. In this work, the definitions of the vortex core and vortex axis are given using the Q-distribution.

We will refer to the *vortical region* as a region where $\|\boldsymbol{\Omega}\| \neq 0$. Let us define the *core* W_0 *of the vortical region* as a part of the vortical region, where the condition $Q \geqslant 0$ is satisfied for each point $\mathbf{x} = (x, y, z) \in W_0$, i. e.

$$W_0(\mathbf{x}) = \{\mathbf{x} \mid Q(\mathbf{x}) \geqslant 0\}.$$

Let us define the outer boundary (isosurface) $B_0(\mathbf{x})$ of the vortical core region W as:

$$B_0(\mathbf{x}) = \{\mathbf{x} \mid Q(\mathbf{x}) = 0\}. \tag{25}$$

Condition (25) is rarely used in practice when visualizing the surface of the vortical core region due to possible occurrence of a large number of artifacts (in the form of closed subregions). To eliminate (reduce the number of) such artifacts during the identification process of the vortical core region, a cut-off value is used:

$$W_A(\mathbf{x}) = \{\mathbf{x} \mid Q(\mathbf{x}) \geqslant A\},$$

here $A = \text{const} > 0$ ($A \approx 0$). The isosurface (possibly non-simply connected)

$$B_A(\mathbf{x}) = \{\mathbf{x} \mid Q(\mathbf{x}) = A\}$$

represents the outer boundary of the vortical core region W_A. (According to definition, a region is simply connected if every closed curve within it can be shrunk continuously to a point that is within the region [Arfken, Weber, Harris, 2013].)

For some constant values of A, the vortical core region W_A may consist of isolated non-intersecting (disjoint) regions W_C, i. e. $W_A = \bigcup W_C$. We will define each isolated simply connected region W_C as a *vortical core structure*.

Consider a vortical core structure W_C defined in the family of non-intersecting regions $W_A = \bigcup W_C$. The region W_C, containing an arbitrary point $\mathbf{x}_i$, will be denoted by $W_C(\mathbf{x}_i)$. Consider a plane $F(\mathbf{x}_i)$ that intersects $W_C(\mathbf{x}_i)$ so that $\mathbf{x}_i \in F(\mathbf{x}_i)$. It is clear that the boundary $B_C(\mathbf{x}_i) = B_C \cap F(\mathbf{x}_i)$ of the cross-section $F_C(\mathbf{x}_i)$ for considered $W_C(\mathbf{x}_i)$ is topologically equivalent to a circle (it is a simply connected closed curve). It is possible to find a point $\mathbf{x}_c \in F_C(\mathbf{x}_i)$ at the cross-section $F_C(\mathbf{x}_i)$, where the Q-distribution has the maximum value:

$$Q(\mathbf{x}_c) = \max\{Q(\mathbf{x}_j) \mid \forall \mathbf{x}_j \in F_C(\mathbf{x}_i)\}. \tag{26}$$

We will say that the simply connected curve $B_C(\mathbf{x}_i)$ bounds the core of a 2D vortex at the cross-section $F_C(\mathbf{x}_i)$ if an additional condition is imposed on (26), according to which the norm of the strain-rate tensor at the point $\mathbf{x}_c$ tends to zero:

$$Q(\mathbf{x}_c) = \max\{Q(\mathbf{x}_j) \mid \forall \mathbf{x}_j \in F_C(\mathbf{x}_i),\ \|\boldsymbol{S}(\mathbf{x}_j)\| \to 0\}. \tag{27}$$

Condition (27) is given for a 2D vortex with an axisymmetric circumferential velocity field and the circular shape of the vortex core. In real conditions, the cross-section of the vortical core structure may have a different shape (for example, close to an ellipse). Furthermore, the pre-processed data may contain various errors that distort the cross-sectional shape of the vortical core structure. Therefore, by taking into account (24), condition (27) can be replaced by the expression

$$Q(\mathbf{x}_c) = \max\left\{Q(\mathbf{x}_j) \mid \forall \mathbf{x}_j \in F_C(\mathbf{x}_i),\ \|\boldsymbol{\Omega}(\mathbf{x}_j)\|^2 \gg \|\boldsymbol{S}(\mathbf{x}_j)\|^2\right\}. \tag{28}$$

A family of points $\mathbf{x}_c$ at different cross-sections $F_C(\mathbf{x}_i)$ can be used to construct a smooth continuous curve L_C such that $\mathbf{x}_c \in L_C \subset W_C$. Let us define the curve L_C as the *vortex axis*. We will define the *vortex core* as the vortical core structure W_C that contains the vortex axis L_C.

It should be noted that the formulations of this work, related to the definitions of the core and axis of the vortex, are based on the application of the Q-distribution. Application of other vortex identification methods may result in a different family of geometries of the vortical structures, including the geometry of the isosurface and coordinates of the vortex axis. The quality of the computational grid of the numerical modeling can also affect significantly the identification accuracy of the vortex core parameters (in particular, see [Степанов, Кусюмов, Баракос, 2022]).

The formulation of the vortex, that was proposed in [Wu, Xiong, Yang, 2005], can be given using the definition of the vortex core. Based on the definition (28) for flows, when the shape of the streamlines at 2D cross-sections is close to an ellipse with a limited eccentricity of semi-axes, a simplified definition of a vortex core can be given as follows: *a vortex core must contain a vortex axis, comprised of the points of local maxima of Q-distribution, provided that* $\|\boldsymbol{\Omega}\|^2 \gg \|\boldsymbol{S}\|^2$.

5. Visualization of the results of numerical simulation of the wingtip vortex

The vortical region of the flow-field near a finite-span wing is an example when Q-criterion can be used to localize the vortical core region from the results of numerical modeling. Moreover, only a part of the vortical core region represents the tip vortex — the vortical core structure, containing the axis of the vortex.

Numerical modeling of the flow-field around a rectangular wing with an aspect ratio of 7.8 was carried out based on the solution of the Reynolds-averaged Navier–Stokes equations (RANS). The wing had a modified Göttingen-387 airfoil with a constant chord along its span [Степанов и др., 2019].

The computational domain was divided into 58 blocks. The grid contained over 4.5 million of cells (the number of cells was determined based on the results of the grid independence study). The numerical modeling was carried out for the angle of attack of 6°. Obtained integral aerodynamic characteristics were compared to experimental data presented in [Степанов и др., 2019], and the difference between the lift coefficient values was approximately 2 %. To resolve the boundary layer, the cell size near the wing surface was set to 10^{-5} of the wing chord length and the cell growth rate did not exceed 1.2. Closure of the Navier – Stokes equations was carried out using k-ω turbulence model [Menter, 1994]. Due to the symmetry of the flow-field at zero sideslip angle, the computational grid was constructed for one half of the wing. Some visualization results of the wing-tip vortices from the numerical modeling were also presented in [Степанов, Кусюмов, Баракос, 2022].

Figure 2 shows the isosurface B_A of the vortical core region W_A for the angle of attack of 6°. The flow in Fig. 2 is moving from right to left. The visualization of the isosurface B_A is obtained using $\overline{Q} = 0.001$ condition. The condition $\overline{Q} \geqslant 0.001$ is satisfied inside the region, bounded by the isosurface B_A.

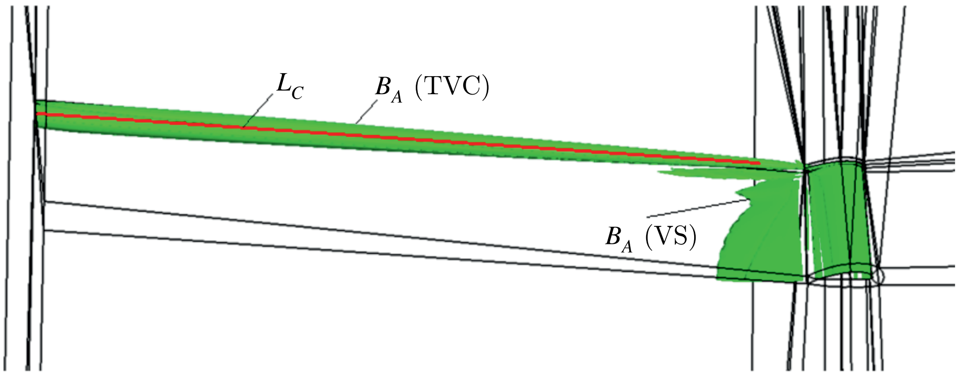

Figure 2. Visualization of the isosurface of the vortical core region W_A for $\overline{Q} = 0.001$

The vortical core region W_A in Fig. 2 consists of two regions: a streamwise conical region of the tip vortex core (TVC) near the wing tip, and the vortex sheet (VS) region downstream of the trailing edge of the wing. The analysis shows that the VS region is a vortical structure, characterized by high values of the strain-rate tensor and the absence of a vortex axis.

The conical region W_A contains the vortex axis — the curve L_C, comprised of the collection of points $\mathbf{x}_c$, where the condition (28) is satisfied. It should be noted that during the numerical modeling the vortex axis can be constructed for a finite set of points, determined by the distribution of the cells in the computational grid. The vortex axis is shown in Fig. 2 as a continuous smooth line using data interpolation.

Figure 3 shows the distribution of Q-values, normalized by the values of $Q_{\max}$ at the cross-sections F_C with coordinates $\overline{x} = 1.07$ and $\overline{x} = 2.14$. Here, $\overline{x} = \frac{x}{c}$ is the downstream distance from the trailing edge to the cross-section under consideration, non-dimensionalized with respect to the chord length c. The color-shaded region represents the cross-sectional boundary of the TVC. As shown in the figure, the cross-section of the vortex core does not have the shape of a circle. Hence, the location of the vortex center was obtained using condition (28). The radius of the vortex core in both cases is about 10 % of the wing chord length ($r_c \approx 0.1c$). The position of the vortex core center was localized as a point of $Q_{\max}$ (shown as white circles in Fig. 3).

The WA-method can also be used as an addition tool for finding the vortex center, which defines the vortex center as the center of the streamline spiral. As shown in Figs. 3, *a* and 3, *b*, the streamlines

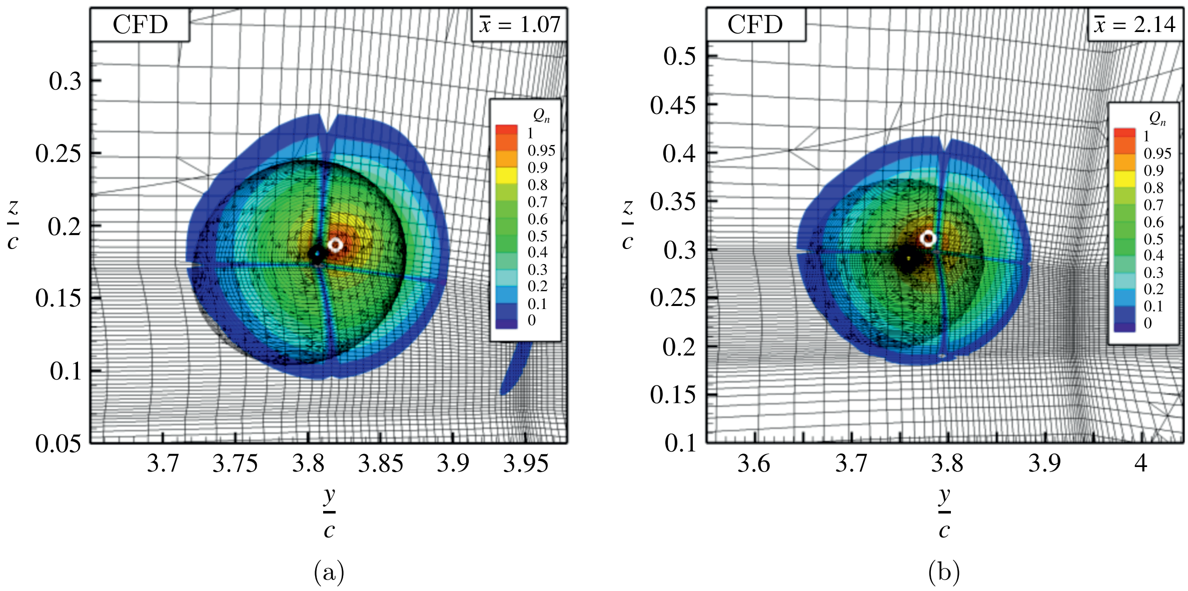

Figure 3. Location of the wing-tip vortex core centers using Q-distribution for various cross-sections: (a) $\overline{x} = 1.07$; (b) $\overline{x} = 2.14$

have similar diverging patterns, and each center of the spiral is close to the center of the vortex core, determined by the Q-distribution.

6. Identification of a pair of vortex cores

In § 5, the velocity field in cross-sections of a tip vortex core was obtained with a sufficiently high discretization (several hundred of cells), and condition (28) was used to construct the vortex axis. If the resolution of the velocity field in a cross-section of the vortex core is not high enough, it is possible that when the value of Q reaches $Q_{\max}$, instead of the condition $\|\boldsymbol{\Omega}\|^2 \gg \|\boldsymbol{S}\|^2$ the relation $\|\boldsymbol{\Omega}\|^2 > \|\boldsymbol{S}\|^2$ can take place. We will show that the use of a less stringent condition $\|\boldsymbol{\Omega}\|^2 > \|\boldsymbol{S}\|^2$ for low-resolution velocity fields also allows determining the position of the vortex center (with an accuracy corresponding to the resolution of the velocity field).

As an example, consider a model problem of the vortex structure defined by the cross-section of a paired vortex: a wing-tip vortex and a vortex generated by the flap. At high deflection angles of the flap, the vortex generated by the flap surface has a circulation comparable to the circulation of the tip vortex.

The mathematical model of the 2D flow under consideration can be represented by a pair of vortices, each having the same direction of rotation and the circumferential velocity described by the Burnham – Hallock model. It is assumed that the characteristics of the vortices are identical to the characteristics of the tip vortex (discussed above in § 5) at a cross-section located at a downstream distance of $\overline{x} = 2.14$ from the trailing edge of the wing. The size of the core radius of each vortex is $0.07c$, the distance between the vortices is $1.4c$. Each vortex has the circulation of $\Gamma_0 = 2.42$ m^2/s.

Figures 4 and 5 show the Q-distribution $\left(\text{in the non-dimensional form } Q_n = \frac{Q}{Q_{\max}}\right)$ and the streamlines, plotted in Tecplot software with different resolutions (with coordinates non-dimensionalized with respect to the chord length c). The cell dimensions in Fig. 4 are close to $0.002c \times 0.001c$ (the grid is not shown due to high cell density). In Fig. 5, the cell dimensions of the computational grid are close to $0.11c \times 0.05c$.

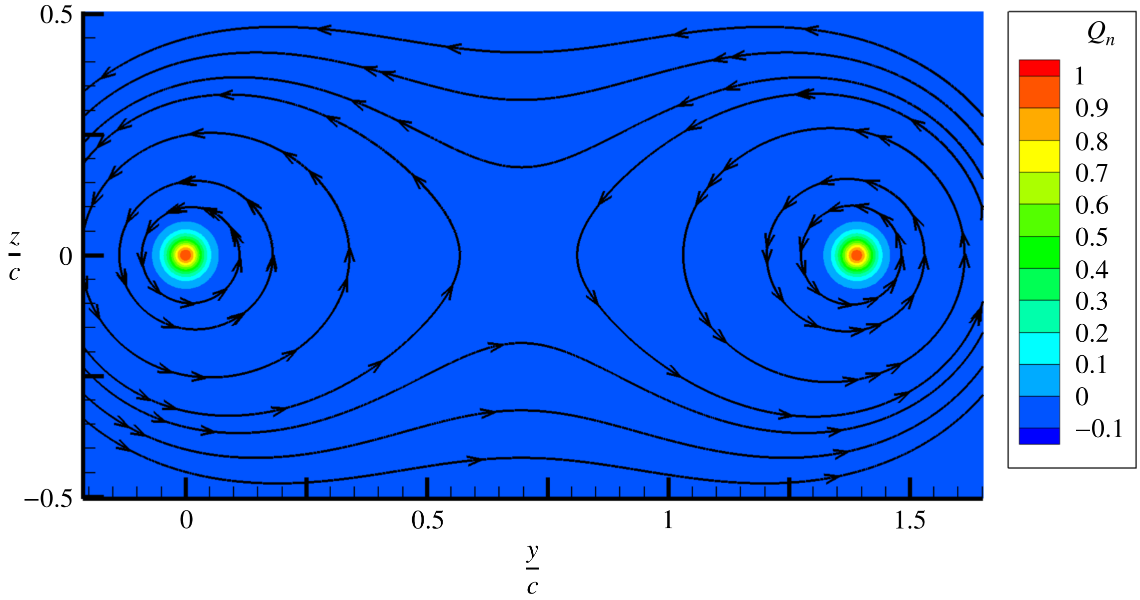

Figure 4. Q-distribution for the high-resolution grid

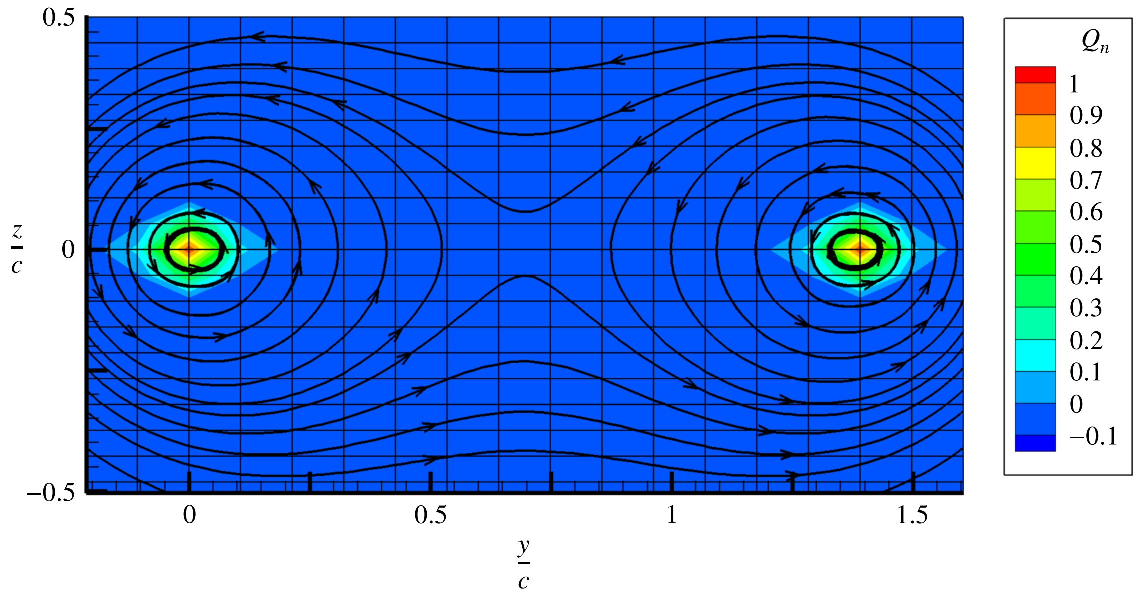

Figure 5. Q-distribution for the low-resolution grid

It can be seen from Figs. 4 and 5, that the mutual influence of the vortices leads to the deformation of the flow-field (in comparison with an isolated vortex), but the streamlines in the vicinity of the vortex cores are close to having a circular shape.

For the high-resolution grid (Fig. 4), the Q-distribution in the vicinity of the vortex core is close to being axisymmetric. The maximum value of Q is reached at the center of the vortex, where $\|\mathbf{\Omega}\|^2 = 10^5\|\mathbf{S}\|^2$, i. e. $\|\mathbf{\Omega}\|^2 \gg \|\mathbf{S}\|^2$ in accordance with condition (28). For the low-resolution grid (Fig. 5), the cell dimensions are close to the dimensions of the vortex core, and at the center of the vortex $\|\mathbf{\Omega}\|^2 = 7.57\|\mathbf{S}\|^2$, i. e. instead of $\|\mathbf{\Omega}\|^2 \gg \|\mathbf{S}\|^2$ in (28) the condition $\|\mathbf{\Omega}\|^2 > \|\mathbf{S}\|^2$ takes place.

As a result, the positions of the cores and the centers of the vortices for both grids are identified using the Q-distribution. In certain cases when the velocity vector field has a low resolution, the condition $||\boldsymbol{\Omega}||^2 \gg ||\boldsymbol{S}||^2$ in (28), that defines the location of the vortex center, can be replaced by a less stringent requirement: $||\boldsymbol{\Omega}||^2 > ||\boldsymbol{S}||^2$.

7. Conclusion

This work explores the identification of the vortical core structure using the Q-distribution, defined by the condition when the norm of the vorticity tensor $||\boldsymbol{\Omega}||$ is larger than the rate of the strain tensor $||\boldsymbol{S}||$. According to the formulations given in the literature, a vortex is defined as a vortical structure, that contains not only the vortex core, but the vortex axis as well. The definitions of the vortex center (for a two-dimensional case) and the vortex axis (for a three-dimensional case) are given based on the analytic models of an isolated axisymmetric vortex (in unbounded space), encountered in the aerodynamics of aircraft and used to describe certain classes of vortex flows such as tornadoes. Moreover, each point of the vortex axis is characterized by the maximum values of $||\boldsymbol{\Omega}||$, and by the values of $||\boldsymbol{S}||$ that tend to zero. A two-dimensional cross-section of the vortex core is also considered, particularly when the shape of the streamlines in the vicinity of the vortex center is close to an ellipse with a limited eccentricity. It is shown that in this case, $||\boldsymbol{S}||$ takes a non-zero value, but it is much smaller than $||\boldsymbol{\Omega}||$, i. e. $||\boldsymbol{\Omega}||^2 \gg ||\boldsymbol{S}||^2$. For vortical flows with elliptical shapes of the streamlines, the simplified definition of the vortex core can be given as follows: *a vortex core must contain a vortex axis, comprised of the points of local maxima of Q-distribution, provided that* $||\boldsymbol{\Omega}||^2 \gg ||\boldsymbol{S}||^2$.

The vortical structures can be generated by various lifting surfaces of aircraft, such as the wing or the rotor blade of a helicopter, extended wing flaps, etc. To illustrate the introduced concepts, the vortical region of the flow-field around a finite-span wing is analyzed based on the results of numerical modeling. It is shown that the vortical structure, generated by the wing, consists of the tip vortex and the vortex sheet region, characterized by high values of the strain-rate tensor and by the absence of the vortex axis. A model example is considered, that illustrates the application of Q-distribution for identification of the vortex cores in a cross-section of the vortex pair for high- and low-resolution (discretization) of the velocity field. It is demonstrated that in certain cases, when the cell size of the discrete velocity field is comparable to the size of the vortex core, a less stringent condition $||\boldsymbol{\Omega}|| > ||\boldsymbol{S}||$ can be used to find the center of the vortex core at a given cross-section.

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