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PSEUDO SEMI-PROJECTIVE MODULES AND ENDOMORPHISM RINGS

A module M is called pseudo semi-projective if, for all $\alpha, \beta \in \operatorname{End}_R(M)$ with $\operatorname{Im}(\alpha) = \operatorname{Im}(\beta)$, there holds $\alpha \operatorname{End}_R(M) = \beta \operatorname{End}_R(M)$. In this paper, we study some properties of pseudo semi-projective modules and their endomorphism rings. It is shown that a ring R is a semilocal ring if and only if each semiprimitive finitely generated right R-module is pseudo semi-projective. Moreover, we show that if M is a coretractable pseudo semi-projective module with finite hollow dimension, then $\operatorname{End}_R(M)$ is a semilocal ring and every maximal right ideal of $\operatorname{End}_R(M)$ has the form $\{s \in \operatorname{End}_R(M) | \operatorname{Im}(s) + \operatorname{Ker}(h) \neq M\}$ for some endomorphism h of M with $h(M)$ hollow.

Keywords: pseudo semi-projective module, hollow module, finite hollow dimension, perfect ring.

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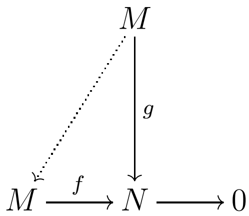
Introduction

Following [15], a module M is called *semi-projective* if, for any submodule N of M, every diagram with exact row



can be extended by a homomorphism $h\colon M \to M$ with $fh = g$. It is equivalent to $fS = \operatorname{Hom}(M, f(M))$ for every $f \in \operatorname{End}_R(M) = S$. One can check that M is semi-projective if and only if for all $\alpha, \beta \in \operatorname{End}_R(M)$ with $\operatorname{Im}(\alpha) \le \operatorname{Im}(\beta)$, there holds $\alpha \operatorname{End}_R(M) \le \beta \operatorname{End}_R(M)$. The endomorphism rings of semi-projective modules are studied. It is shown that if M is a finitely generated, semi-projective R-module satisfying DCC for M-cyclic submodules, then $\operatorname{End}_R(M)$ satisfies DCC for cyclic left ideals [15, 31.10]. Recently, some authors considered some generalizations of semi-projective modules and dual automorphism-invariant modules (see [1, 2, 5, 7, 8, 10–12, 14]).

A generalization of semi-projective modules is considered, namely, pseudo semi-projective modules. In [10], a right R-module M is called *pseudo semi-projective* if, for any endomorphism ε of M, every epimorphism $p\colon M \to \varepsilon(M)$ and every epimorphism $f\colon M \to \varepsilon(M)$, there exists an endomorphism h of M such that $ph = f$. A characterization of Artinian pseudo semi-injective modules is considered. It is shown that if M is an Artinian pseudo semi-injective module then $S = \operatorname{End}_R(M)$ is semiprimary (see [10, Theorem 3.10]). Moreover, the author [10] studied semiperfect rings and perfect rings via modules having pseudo semi-projective covers.

In this paper, we continue on pseudo semi-projective modules and their endomorphism rings. It is shown that a ring R is a semilocal ring if and only if each semiprimitive finitely generated right R-module is pseudo semi-projective (Theorem 1). Considering coretractable modules, we show that if M is a coretractable pseudo semi-projective module with $S = \operatorname{End}_R(M)$, then S is left perfect if and only if for any infinite sequence $s_1, s_2, \ldots \in S$, the chain $\operatorname{Im}(s_1) \ge \operatorname{Im}(s_1 s_2) \ge \ldots$ is stationary (Theorem 2). Moreover, if M is a coretractable pseudo semi-projective module with finite hollow dimension, then $\mathrm{End}_R(M)$ is a semilocal ring and every maximal right ideal of $\mathrm{End}_R(M)$ has the form $\{s \in \mathrm{End}_R(M) | \mathrm{Im}(s) + \mathrm{Ker}(h) \neq M\}$ for some endomorphism h of M with $h(M)$ hollow (Theorem 3).

§ 1. Notations and definitions

Throughout this article all rings are associative rings with unity and all modules are right unital modules over a ring. We denote by $|X|$ the cardinality of a set X. For a submodule N of M, we write $N \leq M$ ($N < M, N \ll M$) iff N is a submodule of M (respectively, a proper submodule, a small submodule). We denote by $J(R)$ the Jacobson radical of the ring R. For any term not defined here the reader is referred to [3] and [9].

§ 2. Some results of pseudo semi-projective modules

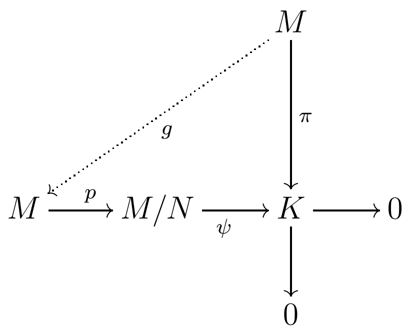
Following [10], a right R-module M is called *pseudo semi-projective* if, for any endomorphism ε of M, every epimorphism $p\colon M \to \varepsilon(M)$ and every epimorphism $f\colon M \to \varepsilon(M)$, there exists an endomorphism h of M such that $ph = f$, or equivalently if, for any endomorphism ε of M and every epimorphism f from M to $M/\mathrm{Ker}(\varepsilon)$, there exists an endomorphism h of M such that $\pi h = f$ with $\pi\colon M \to M/\mathrm{Ker}(\varepsilon)$ the natural projection.

Lemma 1 (see [10, Lemma 3.1]). *Let M be a right R-module and $S = \mathrm{End}_R(M)$. Then the following are equivalent:*

1) *M is pseudo semi-projective;*
2) *for all $\alpha, \beta \in S$ with $\mathrm{Im}(\alpha) = \mathrm{Im}(\beta)$, $\alpha S = \beta S$.*

Lemma 2. *Let N be a submodule of a pseudo semi-projective right R-module M. Then, N is a direct summand of M if and only if M/N is isomorphic to a direct summand of M.*

P r o o f. Assume that N is a direct summand of M. One can check that M/N is isomorphic to a direct summand of M. Now, assume that M/N is isomorphic to a direct summand of M. Let $\psi\colon M/N \to K$ be an isomorphism with $M = K \oplus K'$. Let $\pi\colon M \to K$ be the canonical projection, $\iota\colon K \to M$ be the inclusion map and $p\colon M \to M/N$ be the natural projection. We consider the following diagram



Note that K is an epimorphic image of M. Since M is pseudo semi-projective, $\psi p g = \pi$ for some endomorphism g of M or $pg = \psi^{-1}\pi$. Then, we have $pg\iota\psi = 1_{M/N}$. It shows that p is a splitting epimorphism, and so N is a direct summand of M. □

Corollary 1. *If $M = A \oplus B$ is a pseudo semi-projective module, then every epimorphism $A \to B$ splits.*

P r o o f. Assume that $M = A \oplus B$ is a pseudo semi-projective right R-module. Call $f\colon A \to B$ an epimorphism. Then, $A/\mathrm{Ker}(f)$ is isomorphic to B being a direct summand of M. From Lemma 2, it immediately infers that $\mathrm{Ker}(f)$ is a direct summand of M, and so it is a direct summand of A. We deduce that f splits. □

Proposition 1. *The following conditions are equivalent for a ring R:*

1) *R is semisimple Artinian;*
2) *each finitely generated right R-module is pseudo semi-projective;*
3) *each 2-generated right R-module is pseudo semi-projective.*

P r o o f. (1)⇒ (2) ⇒ (3) are obvious.

(3)⇒ (1) In order to prove the semisimplicity of R, we show that every simple right R-module is projective. Indeed, let M be a simple right R-module. Take $N = R \oplus M$. Then, N is a 2-generated right R-module, and so it is pseudo semi-projective. Note that M is an epimorphic image of R. Then, it follows, from Corollary 1, that M is isomorphic to a direct summand of R_R, and so it is projective. We deduce that R is semisimple Artinian. □

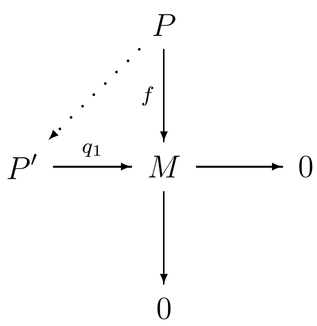
Recall that a module P is a *pseudo semi-projective cover* (resp., *projective cover*) of a right R-module M if, there exists an epimorphism $p\colon P \to M$ such that P is pseudo semi-projective (resp., *projective*) and $\mathrm{Ker}(p)$ is small in P [10].

Proposition 2. *Let $f\colon P \to M$ be an epimorphism from a right R-module M to a projective right R-module P. Then*

1) *$P \oplus M$ is pseudo semi-projective if and only if M is projective;*
2) *$P \oplus M$ has a pseudo semi-projective cover if and only if M has a projective cover.*

P r o o f. (1) is obvious by Corollary 1.

(2) If M has a projective cover, then $P \oplus M$ has a pseudo semi-projective cover. Assume that $P \oplus M$ has a pseudo semi-projective cover. We show that M is projective. Take $q\colon Q \to P \oplus M$ an epimorphism with small kernel and Q pseudo semi-projective. Call $\pi\colon P \oplus M \to P$ the canonical projection. Then, $\pi \circ q\colon Q \to P$ is an epimorphism. We have that P is projective and obtain that $\pi \circ q$ is a splitting epimorphism. Therefore, there exists a monomorphism $\beta\colon P \to Q$ such that $\pi \circ q \circ \beta = 1_P$, and so $Q = \mathrm{Im}(\beta) \oplus \mathrm{Ker}(\pi \circ q)$. Let $P' = \mathrm{Ker}(\pi \circ q)$ and $q_1 = q|_{P'}$. Then, we have $q_1(P') = q(P') = \mathrm{Ker}(\pi) = M$ which implies that $q_1\colon P' \to M$ is an epimorphism. One can check that $\mathrm{Ker}(q_1) = \mathrm{Ker}(q)$, and so $\mathrm{Ker}(q_1)$ is small in P'. Next, we show that P' is projective. We consider the following diagram



We have that P is projective and obtain that there is a homomorphism $g\colon P \to P'$ such that the above diagram is commutative, and so $q_1 \circ g = f$. Since $\mathrm{Ker}(q_1)$ is small in P', g is an epimorphism. On the other hand, $Q = \mathrm{Im}(\beta) \oplus P' \cong P \oplus P'$ is pseudo semi-projective. By Corollary 1, g splits and so P' is isomorphic to a direct summand of P. Thus, P' is projective. □

Recall that a module M is called *semiprimitive* if it's Jacobson radical is zero ([6]).

Next, we give the structure of rings via semiprimitive finitely generated modules accompanying with the pseudo semi-projectivity of modules.

Lemma 3. *If each semiprimitive finitely generated right R-module is pseudo semi-projective, then every quotient ring of R has this property.*

P r o o f. Let S be a quotient ring of R. Assume that M is a semiprimitive finitely generated right S-module. Then M is also a semiprimitive finitely generated right R-module. By the hypothesis, M is a pseudo semi-projective right R-module. It follows that M is a pseudo semi-projective right S-module. □

Lemma 4 (see [3, Proposition 10.15]). *The following conditions are equivalent for a right R-module M:*

1) *M is semiprimitive Artinian;*
2) *M is semiprimitive finitely cogenerated;*
3) *M is a semisimple finitely generated module.*

Corollary 2. *A semiprimitive Artinian module is pseudo semi-projective.*

The following result for semilocal rings via the pseudo semi-projectivity of modules is true.

Theorem 1. *The following conditions are equivalent for a ring R:*

1) *R is a semilocal ring (i. e., $R/J(R)$ is semisimple Artinian);*
2) *each semiprimitive finitely generated right R-module is pseudo semi-projective.*

P r o o f. (1) ⇒ (2). Assume that R is semilocal. From Corollary 2, we show that every semiprimitive finitely generated right R-module is Artinian. In order to complete the proof we will continue by induction on generated elements of M. Assume that M is generated by n elements. The case $n = 1$, we have M is a cyclic module. This means that $M \cong R/K$ for some right ideal K of R. By assumption, we have $J(R/K) = 0$ or $J(R)$ is contained in K, and so $R/K \cong (R/J(R))/(K/J(R))$. We have that $R/J(R)$ is a semilocal ring and obtain that $R/J(R)$ is semisimple Artinian, and so R/K is semisimple. It follows that R/K is Artinian. Suppose now that each semiprimitive right R-module generated by $n = k$ elements is Artinian. Call $M = m_1R + m_2R + \cdots + m_{k+1}R$ a semiprimitive finitely generated right R-module. We show that M is Artinian. Indeed, we have the following short exact sequence:

$$0 \to m_1R \to M \to M/m_1R \to 0.$$

The induction hypothesis can be applied to the modules m_1R and M/m_1R. It follows that m_1R and M/m_1R are Artinian modules, which implies that M is Artinian. Thus, it is shown that every semiprimitive finitely generated right R-module is Artinian. We deduce that every semiprimitive finitely generated right R-module is pseudo semi-projective.

(2) ⇒ (1) Let $\bar{R} = R/J(R)$. We show that every simple right $\bar{R}$-module is projective. Indeed, let S be an arbitrary simple right $\bar{R}$-module. Take $M = \bar{R}_{\bar{R}} \oplus S$. Then, M is a semiprimitive finitely generated $\bar{R}$-module. By (2) and Lemma 3, we have that M is pseudo semi-projective. Note that S is an epimorphic image of $\bar{R}_{\bar{R}}$. It follows, from Corollary 1, that S is isomorphic to a direct summand of $\bar{R}_{\bar{R}}$, and so S is projective. We deduce that $\bar{R}$ is a semilocal ring. □

Corollary 3. *The following conditions are equivalent for a ring R:*

1) *R is a semilocal ring;*
2) *each semiprimitive 2-generated right R-module is pseudo semi-projective.*

Let N and L be submodules of a right R-module M. N is called a *supplement* of L, if $N + L = M$ and $N \cap L \ll N$. Recall that a submodule U of the R-module M has *ample supplement* in M if, for every $V \leq M$ with $U + V = M$, there is a supplement V_0 of U with $V_0 \leq V$. M is called *supplemented (resp., amply supplemented)* if each of its submodules has a supplement (resp., ample supplement) in M (see [15]).

From Corollary 1, we have the following results.

Proposition 3. *For a ring R, the following statements are equivalent:*

1) *R is right perfect;*
2) *every pseudo semi-projective right R-module is amply supplemented;*
3) *every pseudo semi-projective right R-module is supplemented.*

Let M be a right R-module with $S = \operatorname{End}_R(M)$. We denote by

$$\nabla(S) = \{f \in S \mid \operatorname{Im}(f) \ll M\}$$

the set of all endomorphisms of M with small image. One can check that $\nabla(S)$ is the ideal of S.

Recall that an element $a \in R$ is said to be *regular* (in the sense of von Neumann) if there exists $x \in R$ such that $axa = a$. A ring R is called regular if every element of R is regular.

A right R-module M is said to be *coretractable* if $\operatorname{Hom}_R(K, M) \neq 0$ for every nonzero factor K of M.

Lemma 5 (McCoy's Lemma). *Let R be a ring and $a, c \in R$. If $b = a - aca$ is a regular element of R, then so is a.*

P r o o f. This is by definition. □

Lemma 6. *Let M be a coretractable pseudo semi-projective module with $S = \operatorname{End}_R(M)$. If $\alpha \notin \nabla(S)$, then $\operatorname{Im}(\alpha - \alpha\beta\alpha) < \operatorname{Im}(\alpha)$ for some $\beta \in S$.*

P r o o f. Assume that $\alpha \notin \nabla(S)$. Then, we have that $\operatorname{Im}(\alpha)$ is not a small submodule of M. It means that there exists a proper submodule A of M such that $A + \operatorname{Im}(\alpha) = M$. We have the natural isomorphism

$$M/(A \cap \operatorname{Im}(\alpha)) \cong M/\operatorname{Im}(\alpha) \oplus M/A.$$

Since M is coretractable, there exists a nonzero homomorphism $M/A \to M$. It follows that there is a nonzero endomorphism λ of M such that A is contained in $\operatorname{Ker}(\lambda)$. Then, we have $\operatorname{Im}(\alpha) + \operatorname{Ker}(\lambda) = M$, and so $(\lambda\alpha)(M) = \lambda(M)$. Since M is pseudo semi-projective, $(\lambda\alpha)S = \lambda S$ and so $\lambda = \lambda\alpha s$ for some $s \in S$. On the other hand, as λ is nonzero, there is $m \in M$ such that $\lambda(m)$ is nonzero. Call $y = \alpha s(m) \in \operatorname{Im}(\alpha)$. One can check that y and $\lambda(y)$ are nonzero. Next, we show that y is not in $\operatorname{Im}(\alpha - \alpha s\alpha)$. Indeed, suppose that $y = (\alpha - \alpha s\alpha)(x) \in \operatorname{Im}(\alpha - \alpha s\alpha)$ for some $x \in M$. Then, we have

$$\lambda(y) = \lambda(\alpha - \alpha s\alpha)(x) = (\lambda\alpha - \lambda\alpha s\alpha)(x) = (\lambda\alpha - \lambda\alpha)(x) = 0.$$

This is a contradiction, and so $y \in \operatorname{Im}(\alpha) \setminus \operatorname{Im}(\alpha - \alpha s\alpha)$. □

From the proof of [15, 22.2], we have the following result of the Jacobson radical of a pseudo semi-projective module.

Lemma 7. *Let M be a right R-module. If M is a pseudo semi-projective module with $S = \operatorname{End}_R(M)$, then $\nabla(S) \leq J(S)$.*

Theorem 2. *Let M be a coretractable pseudo semi-projective module with $S = \operatorname{End}_R(M)$. Then the following conditions are equivalent:*

1) *S is left perfect;*
2) *for any infinite sequence $\alpha_1, \alpha_2, \ldots \in S$, the chain $\operatorname{Im}(\alpha_1) \geq \operatorname{Im}(\alpha_1\alpha_2) \geq \ldots$ is stationary.*

P r o o f. (1) $\Rightarrow$ (2). Let $\alpha_1, \alpha_2, \ldots \in S$. We have that S is left perfect and obtain that S satisfies DCC on finitely generated right ideals. Then, the chain $\alpha_1 S \geq \alpha_1\alpha_2 S \geq \ldots$ terminates. Thus, there exists $n > 0$ such that $\alpha_1\alpha_2\ldots\alpha_n S = \alpha_1\alpha_2\ldots\alpha_k S$ for all $k > n$. It follows that $\alpha_1\alpha_2\ldots\alpha_n = \alpha_1\alpha_2\ldots\alpha_k f$ and $\alpha_1\alpha_2\ldots\alpha_k = \alpha_1\alpha_2\ldots\alpha_n g$ for some $f, g \in S$. Thus, $\alpha_1\alpha_2\ldots\alpha_n(M) = \alpha_1\alpha_2\ldots\alpha_k(M)$ for all $k > n$.

(2) $\Rightarrow$ (1). Firstly, we show that $S/\nabla(S)$ is a von Neumann regular ring. Let $a_1 \notin \nabla(S)$. Then by Lemma 6, there is $\gamma_1 \in S$ such that $\mathrm{Im}(\alpha_1 - \alpha_1\gamma_1\alpha_1) < \mathrm{Im}(\alpha_1)$. Put $\alpha_2 = \alpha_1 - \alpha_1\gamma_1\alpha_1$, and so $\mathrm{Im}(\alpha_2) < \mathrm{Im}(\alpha_1)$. If $\alpha_2 \in \nabla(S)$, then we have $\bar{\alpha}_1 = \bar{\alpha}_1\bar{\gamma}_1\bar{\alpha}_1$, i. e., $\bar{\alpha}_1$ is a regular element of $S/\nabla(S)$ (where $\bar{s} = s + \nabla(S)$ for all $s \in S$). If $\alpha_2 \notin \nabla(S)$, there exists $\alpha_3 \in S$ such that $\mathrm{Im}(\alpha_3) < \mathrm{Im}(\alpha_2)$ with $\alpha_3 = \alpha_2 - \alpha_2\gamma_2\alpha_2$ for some $\gamma_2 \in S$ by the preceding proof. Repeating the above-mentioned process, we get a strictly ascending chain $\mathrm{Im}(\alpha_1) > \mathrm{Im}(\alpha_2) > \ldots$, where $\alpha_{i+1} = \alpha_i - \alpha_i\gamma_i\alpha_i$ for some $\gamma_i \in S$, $i = 1, 2, \ldots$. Let

$$\beta_1 = \alpha_1,\ \beta_2 = 1 - \gamma_1\alpha_1,\ \ldots,\ \beta_{i+1} = 1 - \gamma_i\alpha_i,\ \ldots,$$

then

$$\alpha_1 = \beta_1,\ \alpha_2 = \beta_1\beta_2,\ \ldots,\ \alpha_{i+1} = \beta_1\beta_2\ldots\beta_{i+1},\ \ldots,$$

and we have the following strictly ascending chain $\mathrm{Im}(\beta_1) > \mathrm{Im}(\beta_1\beta_2) > \ldots$, which contradicts the hypothesis. Hence there exists a positive integer m such that $\alpha_{m+1} \in \nabla(S)$, i. e., $\alpha_m - \alpha_m\gamma_m\alpha_m \in \nabla(S)$. This shows that $\bar{\alpha}_m$ is a regular element of $S/\nabla(S)$, and hence $\bar{\alpha}_{m-1}, \bar{\alpha}_{m-2}, \ldots, \bar{\alpha}_1$ are regular elements of $S/\nabla(S)$ by Lemma 5, i. e., $S/\nabla(S)$ is von Neumann regular.

Now, we show that $J(S)$ is left T-nilpotent. In fact, if for any sequence $\alpha_1, \alpha_2, \ldots$ from $J(S)$, the chain $\mathrm{Im}(\alpha_1) \geq \mathrm{Im}(\alpha_1\alpha_2) \geq \ldots$ is stationary. Thus, there exists n such that $\alpha_1\alpha_2\ldots\alpha_n(M) = \alpha_1\alpha_2\ldots\alpha_k(M)$ for all $k > n$. We have that M is pseudo semi-projective and obtain that $\alpha_1\alpha_2\ldots\alpha_n S = \alpha_1\alpha_2\ldots\alpha_k S$ for all $k > n$. Then, $\alpha_1\alpha_2\ldots\alpha_n(1 - \alpha_{n+1}s) = 0$ for some $s \in S$, and so $\alpha_1\alpha_2\ldots\alpha_n = 0$ (since $1 - \alpha_{n+1}s$ is unit). It means that $J(S)$ is left T-nilpotent. We have that $\nabla(S) \leq J(S)$ and obtain that $\nabla(S)$ is also left T-nilpotent.

Next, we prove that $S/\nabla(S)$ contains no infinite sets of non-zero orthogonal idempotents. Indeed, let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k, \ldots$ be a countably infinite set of non-zero orthogonal idempotents in $S/\nabla(S)$. Then, there exist non-zero orthogonal idempotents $e_1, e_2, \ldots, e_k, \ldots$ in S such that $\varepsilon_i = e_i + \nabla(S)$, $i = 1, 2, \ldots$, by [3, Proposition 27.1]. Put $\alpha_i = 1 - (e_1 + e_2 + \ldots + e_i)$, $i = 1, 2, \ldots$. Then $\alpha_{i+1} = \alpha_i - \alpha_i e_{i+1}\alpha_i$. One can check that $e_{i+1}\alpha_{i+1} = 0$ and $e_{i+1}\alpha_i = e_{i+1} \neq 0$. Take $m \in M$ with $e_{i+1}(m) \neq 0$. Call $y = \alpha_i(m)$, and so y is nonzero in $\mathrm{Im}(\alpha_i)$. Suppose that $y \in \mathrm{Im}(\alpha_{i+1})$, $y = \alpha_{i+1}(t)$ for some $t \in M$. Then, we have

$$e_{i+1}\alpha_i(m) = e_{i+1}(y) = e_{i+1}\alpha_{i+1}(t) = 0.$$

Thus, $e_{i+1}(m) = e_{i+1}\alpha_i(m) = 0$, a contradiction. It means that we have the strict sequence $\mathrm{Im}(\alpha_i) > \mathrm{Im}(\alpha_{i+1})$, $i = 1, 2, \ldots$. Let $\beta_i = 1 - e_i$, $i = 1, 2, \ldots$. Then $\alpha_i = \beta_1\beta_2\ldots\beta_i$ and $\mathrm{Im}(\beta_1\beta_2\ldots\beta_i) > \mathrm{Im}(\beta_1\beta_2\ldots\beta_{i+1})$, $i = 1, 2, \ldots$. We obtain the following strictly ascending chain $\mathrm{Im}(\beta_1) > \mathrm{Im}(\beta_1\beta_2) > \ldots$, a contradiction. Hence $S/\nabla(S)$ contains no infinite sets of non-zero orthogonal idempotents. We deduce that $S/\nabla(S)$ is semisimple. Thus, $S/J(S) \cong$ $\cong [S/\nabla(S)]/[J(S)/\nabla(S)]$ is semisimple. It means that S is left perfect. □

Corollary 4. *Let R_R be a coretractable module. If for any infinite sequence $r_1, r_2, \ldots$ in R, the chain $r_1R \geq r_1r_2R \geq \ldots$ is stationary, then R is left perfect.*

Note that if M has DCC on the submodules of the form IM, where I is a right ideal of $\mathrm{End}_R(M)$, $\nabla(S)$ is nilpotent. Thus, we have the following corollary.

Corollary 5. *Let M be a coretractable pseudo semi-projective module with $S = \mathrm{End}_R(M)$. If M has DCC on the submodules of the form IM, where I is a right ideal of S, then S is semiprimary.*

Next, we characterize left perfect rings via the pseudo semi-projectivity of modules without the coretractability.

A submodule N of M is called *M-cyclic* if, it is an epimorphic image of an endomorphism of M.

Proposition 4. *Let M be a pseudo semi-projective R-module satisfying DCC for M-cyclic submodules. Then $\mathrm{End}_R(M)$ is left perfect.*

P r o o f. Take $S = \mathrm{End}_R(M)$. We consider a descending chain of cyclic right ideals $f_1S \geq f_2S \geq \ldots \geq \ldots$ yielding a descending chain of M-cyclic submodules $f_1(M) \geq f_2(M) \geq \geq \ldots \geq \ldots$. By the hypothesis, there is n such that $f_n(M) = f_{n+k}(M)$ for all $k \geq 0$. Since M is semi-projective, $f_nS = f_{n+k}S$ for all $k \geq 0$ by Lemma 1. Thus, S is left perfect. □

Corollary 6. *If M is a semi-projective R-module satisfying DCC for M-cyclic submodules, then $\mathrm{End}_R(M)$ is left perfect.*

§ 3. On maximal ideals

Recall that a module M is called *quasi-projective* if every homomorphism from M to each quotient module of M can be lifted to an endomorphism of M. One can check that every quasi-projective module is pseudo semi-projective. The following example shows that the converse is not true in the general case.

Example 1 (see [5, Example 5.1]). Let $R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 & 0 \\ 0 & 0 & \mathbb{Z}_2 \end{bmatrix}$. Since R is a finite-dimensional algebra over $\mathbb{Z}_2$, the functors

$$\mathrm{Hom}_{\mathbb{Z}_2}(-, \mathbb{Z}_2)\colon \text{ Mod-R} \to \text{R-Mod}$$

and

$$\mathrm{Hom}_{\mathbb{Z}_2}(-, \mathbb{Z}_2)\colon \text{ R-Mod} \to \text{Mod-R}$$

establish a contravariant equivalence between the subcategories of left and right finitely generated modules over R. Then, $\mathrm{Hom}_{\mathbb{Z}_2}(M, \mathbb{Z}_2)$ is a pseudo semi-projective left R-module and it is not quasi-projective.

Let M be a right R-module with $S = \mathrm{End}_R(M)$. A nonzero module M is said to be *hollow* if every proper submodule is small in M. An element h in S is called a *right hollow* element of S if h is nonzero and $\mathrm{Im}(h)$ is a hollow submodule of M.

Let h be a right hollow element of S. We call

$$\mathcal{V}_h = \{s \in S \mid \mathrm{Im}(s) + \mathrm{Ker}(h) \neq M\}.$$

One can check that $\mathcal{V}_h$ is a proper right ideal of S.

Let α be an endomorphism of M with $S = \mathrm{End}_R(M)$. We denote by

$$r_S(\alpha) = \{s \in S \mid \alpha s = 0\}$$

the annihilator of α in S. If α is a right hollow element of S, then $r_S(\alpha)$ is a right ideal of S contained in $\mathcal{V}_\alpha$.

Lemma 8. *Assume that M is a pseudo semi-projective module. If h is a right hollow element of S, $\mathcal{V}_h$ is the unique maximal right ideal of S containing $r_S(h)$.*

P r o o f. Take s an element of S and $s \notin \mathcal{V}_h$. From the definition of $\mathcal{V}_h$, it infers that $\mathrm{Im}(s) + \mathrm{Ker}(h) = M$. Then, $hs(M) = h(M)$. By Lemma 1, we have that $hsS = hS$ and obtain that $h = hsk$ for some k in S. It follows that $S = r_S(h) + sS \leq \mathcal{V}_h + sS$, and so $S = \mathcal{V}_h + sS$. It's shown that $\mathcal{V}_h$ is maximal in S. It remains to show that $\mathcal{V}_h$ is the unique right ideal of S containing $r_S(h)$. Indeed, let I be another maximal ideal of S containing $r_S(h)$ and $I \neq \mathcal{V}_h$. Then, there exists an element $\alpha \in I \setminus \mathcal{V}_h$. It follows that $\mathrm{Im}(\alpha) + \mathrm{Ker}(h) = M$. By the process of proof above, we have $S = \alpha S + r_S(h) \leq I$ and so $S = I$, a contradiction. □

A family $\{M_\lambda\}_\Lambda$ of proper submodules of M is called *coindependent* if, for any $\lambda \in \Lambda$ and any finite subset $I \subseteq \Lambda \setminus \{\lambda\}$, $M_\lambda + \bigcap_{i \in F} M_i = M$.

Lemma 9 (see [13, Lemma 3.5]). *Assume that M has coindependent submodules $M_1, M_2, \ldots, M_k$ such that $\bigcap_{i=1}^{k} M_i \ll M$ and M/M_i is hollow for every $1 \leq i \leq k$. If M has a submodule L such that $L + M_i \neq M$ for every $1 \leq i \leq k$, then L is small in M.*

Lemma 10. *Let M be a pseudo semi-projective right R-module with $S = \mathrm{End}_R(M)$ and $\{\varphi_i\}_{i=1}^{k}$ be a family of nonzero elements of S with $\{\mathrm{Ker}(\varphi_1), \mathrm{Ker}(\varphi_2), \ldots, \mathrm{Ker}(\varphi_k)\}$ a finite coindependent family in M and $\{\mathrm{Im}(\varphi_1), \mathrm{Im}(\varphi_2), \ldots, \mathrm{Im}(\varphi_k)\}$ hollow modules. If I is a maximal right ideal of S which is not of the form $\mathcal{V}_h$ for some right hollow element h of S, then there is an endomorphism $\psi \in I$ such that*

$$[\mathrm{Im}(1-\psi) + \bigcap_{i=1}^{k} \mathrm{Ker}(\varphi_i)] / \bigcap_{i=1}^{k} \mathrm{Ker}(\varphi_i) \ll M / \bigcap_{i=1}^{k} \mathrm{Ker}(\varphi_i)$$

P r o o f. Take $W = \bigcap_{i=1}^{k} \mathrm{Ker}(\varphi_i)$. Let $\alpha \in I \setminus \mathcal{V}_{\varphi_1}$ and so $M = \mathrm{Im}(\alpha) + \mathrm{Ker}(\varphi_1)$. Then $\varphi_1(M) = (\varphi_1\alpha)(M)$. From Lemma 1, it immediately infers that $\varphi_1 S = (\varphi_1\alpha)S$. Thus, $\varphi_1 = (\varphi_1\alpha)s_1 = \varphi_1(\alpha s_1)$ for some $s_1 \in S$. Call $\psi_1 = \alpha s_1 \in I$, and so $\varphi_1(1-\psi_1) = 0$. This implies that $\mathrm{Im}(1-\psi_1) + \mathrm{Ker}(\varphi_1) = \mathrm{Ker}(\varphi_1) \neq M$. Suppose that $\mathrm{Im}(1-\psi_1) + \mathrm{Ker}(\varphi_j) \neq M$ for all $2 \leq j \leq k$. We have $\{\mathrm{Ker}(\varphi_1), \mathrm{Ker}(\varphi_2), \ldots, \mathrm{Ker}(\varphi_k)\}$ is a finite coindependent family in M and obtain that there is an isomorphism $\phi\colon M/W \to \bigoplus_{i=1}^{k} M/\mathrm{Ker}(\varphi_i)$ defined by

$$\phi(m + W) = (m + \mathrm{Ker}(\varphi_1), m + \mathrm{Ker}(\varphi_2), \ldots, m + \mathrm{Ker}(\varphi_k)).$$

One can check that $\phi^{-1}[\bigoplus_{i=1}^{k} \frac{\mathrm{Im}(1-\psi_1) + \mathrm{Ker}(\varphi_i)}{\mathrm{Ker}(\varphi_i)}] = \frac{\mathrm{Im}(1-\psi_1) + W}{W}$. Since every $M/\mathrm{Ker}(\varphi_j) \cong \mathrm{Im}(\varphi_j)$ is hollow, $(\mathrm{Im}(1-\psi_1) + W)/W \ll M/W$. Without loss of generality, we now assume that $\mathrm{Im}(1-\psi_1) + \mathrm{Ker}(\varphi_2) = M$. Then $\varphi_2(1-\psi_1)(M) = \varphi_2(M)$. Since $\varphi_2(M)$ is hollow, $\varphi_2(1-\psi_1)(M)$ is hollow. Thus $\varphi_2(1-\psi_1)$ is a right hollow element of S. Since $I \neq \mathcal{V}_{\varphi_2(1-\psi_1)}$ and $\mathcal{V}_{\varphi_2(1-\psi_1)}$ is a maximal right ideal of S, we take $h \in I \setminus \mathcal{V}_{\varphi_2(1-\psi_1)}$. By using the above argument, we can find $s_2 \in S$ such that $\varphi_2(1-\psi_1) = \varphi_2(1-\psi_1)hs_2$, and so $\varphi_2(1-(\psi_1 + (1-\psi_1)hs_2) = 0$. Put $\psi_2 = \psi_1 + (1-\psi_1)hs_2$. Then, we have $\varphi_i(1-\psi_2) = 0$ for all $i = 1, 2$. Continuing this process, we eventually get a $\psi \in I$ such that $\varphi_i(1-\psi) = 0$ for all $i = 1, 2, \ldots, k$. Thus, $\mathrm{Im}(1-\psi) \leq W$. We deduce that $[\mathrm{Im}(1-\psi) + W]/W \ll M/W$. □

If M has coindependent submodules $\{M_1, M_2, \ldots, M_k\}$ such that $\bigcap\limits_{i=1}^{k} M_i \ll M$ and M/M_i is hollow for every $1 \leq i \leq k$, M is said to have *hollow dimension* k, denoting this by $\operatorname{hdim}(M) = k$.

Theorem 3. *Let M be a coretractable pseudo semi-projective module having finite hollow dimension with $S = \operatorname{End}_R(M)$. Then*

1) *if I is a maximal right ideal, then $I = \mathcal{V}_h$ for some right hollow element $h \in S$;*
2) *S is semilocal.*

P r o o f. Assume that M has finite hollow dimension, there exists a coindependent family $\{N_1, N_2, \ldots, N_n\}$ of submodules of M, where $M/N_1, M/N_2, \ldots, M/N_n$ are hollow, $\bigcap\limits_{i=1}^{n} N_i \ll M$ and an isomorphism $M/(\bigcap\limits_{i=1}^{n} N_i) \cong \bigoplus_{i=1}^{n}(M/N_i)$. Take $\pi_j \colon M \to M/M_j$ the natural projections for all $j = 1, 2, \ldots, n$. We have that M is coretractable, there is a nonzero homomorphism $f_j \colon M/N_j \to M$. Then, we have the homomorphisms $h_j = f_j\pi_j \in S$ for all $j = 1, 2, \ldots, n$. One can check that $N_j \leq \operatorname{Ker}(h_j)$ for all $j = 1, 2, \ldots, n$. We deduce that $M/\operatorname{Ker}(h_j)$ is hollow and the family $\{\operatorname{Ker}(h_1), \operatorname{Ker}(h_2), \ldots, \operatorname{Ker}(h_n)\}$ is coindependent. Take $W = \bigcap_{i=1}^{n} \operatorname{Ker}(h_i)$, and so $\bigcap\limits_{i=1}^{n} N_i \leq W$. We have that $M/(\bigcap\limits_{i=1}^{n} \operatorname{Ker}(h_i)) \cong \bigoplus\limits_{i=1}^{n} M/\operatorname{Ker}(h_i)$ and obtain that $\operatorname{hdim}(M/(\bigcap\limits_{i=1}^{n} \operatorname{Ker}(h_i))) = n = \operatorname{hdim}(M)$. Thus, $W \ll M$ by [4, 5.4(2)].

(1) Suppose that I is a maximal right ideal of S with $I \neq \mathcal{V}_h$ for every right hollow element h of S. Then by Lemma 10, there is a homomorphism φ in I such that $[\operatorname{Im}(1-\varphi) + W]/W \ll\ll M/W$. We have that $W \ll M$ and obtain that $\operatorname{Im}(1-\varphi) \ll M$. From Lemma 7, it immediately infers that $1 - \varphi \in J(S) \leq I$, and so $1 \in I$, a contradiction.

(2) We have $J(S) \leq \bigcap\limits_{i=1}^{n} \mathcal{V}_{h_i}$. If $f \in \bigcap\limits_{i=1}^{n} \mathcal{V}_{h_i}$, $\operatorname{Im}(f) + \operatorname{Ker}(h_j) \neq M$ for each $j = 1, 2, \ldots, n$. It follows that $\operatorname{Im}(f) \ll M$ by Lemma 9, and so $f \in J(S)$ by Lemma 7. Thus, $J(S) = \bigcap\limits_{i=1}^{n} \mathcal{V}_{h_i}$. We deduce that S is semilocal. □

Corollary 7. *Let R be a coretractable ring with finite hollow dimension. If I is a maximal right ideal of R, $I = \mathcal{V}_h$ for some right hollow element $h \in R$.*

Example 2. (1) Let R be the ring of integers $\mathbb{Z}$. Take $M = \mathbb{Z}$. Then M is pseudo semi-projective with infinite hollow dimension. Note that $\operatorname{End}_R(M)$ contains no hollow elements. Thus the statements (1) and (2) of Theorem 3 are not satisfied. This shows that the hypothesis "M has finite hollow dimension" in Theorem 3 is not superfluous.

(2) Let R be a nonlocal commutative domain with finitely many maximal ideals. Then, every nonzero element h in R is not hollow. So $\operatorname{End}_R(R)$ contains no hollow elements. Thus the statements (1) and (2) of Theorem 3 are not satisfied. Note that R is pseudo semi-projective with finite hollow dimension. But R is not coretractable because $\operatorname{Hom}(R/J(R), R) = 0$. This example shows that Theorem 3 is not true if M is not coretractable.

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Псевдополупроективные модули и кольца эндоморфизмов

Ключевые слова: псевдополупроективный модуль, пустотелый модуль, конечная размерность пустоты, совершенное кольцо.

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Модуль M называется псевдополупроективным, если для всех $\alpha, \beta \in \operatorname{End}_R(M)$ таких, что $\operatorname{Im}(\alpha) = \operatorname{Im}(\beta)$, выполнено $\alpha \operatorname{End}_R(M) = \beta \operatorname{End}_R(M)$. В данной работе мы изучаем некоторые свойства псевдополупроективных модулей и их колец эндоморфизмов. Показано, что кольцо R является полулокальным тогда и только тогда, когда каждый полупримитивный конечно порожденный правый R-модуль является псевдополупроективным. Кроме того, мы показываем, что если M — коретрактабельный псевдополупроективный модуль с конечной размерностью пустоты, то $\operatorname{End}_R(M)$ — полулокальное кольцо и каждый максимальный правый идеал $\operatorname{End}_R(M)$ имеет вид $\{s \in \operatorname{End}_R(M) | \operatorname{Im}(s) + \operatorname{Ker}(h) \neq M\}$ для некоторого эндоморфизма h модуля M, где $h(M)$ пустотелый.

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