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A MEAN FIELD TYPE DIFFERENTIAL INCLUSION WITH UPPER SEMICONTINUOUS RIGHT-HAND SIDE

Mean field type differential inclusions appear within the theory of mean field type control through the convexification of a right-hand side. We study the case when the right-hand side of a differential inclusion depends on the state of an agent and the distribution of agents in an upper semicontinuous way. The main result of the paper is the existence and the stability of the solution of a mean field type differential inclusion. Furthermore, we show that the value function of the mean field type optimal control problem depends on an initial state and a parameter semicontinuously.

Keywords: mean field type differential inclusions, mean field type optimal control, stability analysis.

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Introduction

The mean field type control theory tries to examine the behavior of systems consisting of many identical agents who play cooperatively. The approach of mean field type control implies the study of the limiting system where the number of agents tends to infinity. In this case, the probability that describes the distribution of agents plays the role of a state variable. Note that the space of probabilities is not a Banach space. It is barely metric and endowed with the so called Wasserstein distance [2].

The mean field type control theory borrows a lot from the finite dimensional control theory. In particular, in the finite dimensional case, it is productive to replace the original control system with a differential inclusion. This tool is used within the viability theory [3] and in the analysis of Hamilton–Jacobi equations [10]. Recall that differential inclusions are usually studied under the assumption of semicontinuity of the right-hand sides. This corresponds to the discontinuous dependence of the capabilities on a phase variable.

The notion of a differential inclusion in the space of probability measures was previously introduced and studied in [4–9]. In particular, the differential inclusions in the space of probability measures were used to study the dynamic programming and properties of the value function of the mean field type optimal control problem. However, the existing literature deals only with the case of the right-hand side depending continuously on the state of an agent.

The paper is concerned with the mean field type differential inclusions which are a specific type of the differential inclusions in the space of probability measures. The mean field type differential inclusions appear naturally within the theory of mean field type optimal control. The paper aims to lift the assumption of continuity of the velocity field for each agent and prove the existence of a solution for the mean field type differential inclusion with only upper semicontinuous right-hand side. Additionally, we study the stability of solutions of mean field type differential inclusion w. r. t. perturbations of the right-hand side and initial distribution of agents. The main result of the paper applied to the mean field type optimal control problem allows one to prove the existence of the value function of the mean field type optimal control problem and its lower semicontinuos dependence on an initial distribution of agents and a parameter.

The paper is organized as follows. The general notation and the concept of mean field type differential inclusion are introduced in Section 1. The existence theorem for the mean field type differential inclusion with the upper semicontinuous right-hand side is derived in Section 2. The next section is concerned with the stability analysis of the solutions of the mean field type differential inclusions. Finally, in Section 4, we apply the obtained result to the mean field type optimal control and prove that its value function depends on an initial state and a parameter in a lower semicontinuous way.

§ 1. Problem statement

A mean field type differential inclusion describes the dynamics of the distribution of agents under assumption that each agent obeys the differential inclusion

$$\frac{d}{dt}x(t) \in F(t, x(t), m(t)). \tag{1.1}$$

Here t is time, $x(t)$ is the state of an agent, while $m(t)$ is a probability on the state space that describes the distribution of all agents. Formally integrating (1.1) w. r. t. the measure variable, we arrive at the following mean field type differential inclusion

$$\frac{d}{dt}m(t) \in \langle F(t, \cdot, m(t)), \nabla\rangle m(t). \tag{1.2}$$

Notice that this system appears by the convexification of the right-hand side of the mean field type control system with the dynamics of each agent

$$\frac{d}{dt}x(t) = f(t, x(t), m(t), u(t, x(t))), \quad u(t, x) \in U(x, m(t)).$$

To introduce the definition of the solution as well as the standing assumption we need some notation.

First, recall that, if X is a metric space with a distance ρ_X, then the Borel σ-algebra $\mathcal{B}(X)$ is the minimal σ algebra containing all open subsets of X. Further, a Borel probability on X is a probability defined on $\mathcal{B}(X)$. The set of Borel probabilities is denoted by $\mathcal{P}(X)$.

If $x_0 \in X$, $R > 0$, then we denote by $\mathbb{B}_R(x_0)$ the ball of radius R centered in x_0. If, additionally, X is a Banach space, x_0 is its origin, we will omit the argument.

If $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ are two measurable spaces, $\mathbb{P}$ is a probability on $\mathcal{F}$, while $\xi\colon \Omega \to \Omega'$ is $\mathcal{F}/\mathcal{F}'$-measurable mapping, then $\xi\sharp\mathbb{P}$ is the probability on $\mathcal{F}'$ defined by the rule: for $\Upsilon' \in \mathcal{F}'$,

$$(\xi\sharp\mathbb{P})(\Upsilon') = \mathbb{P}(\xi^{-1}(\Upsilon')).$$

The probability $\xi\sharp\mathbb{P}$ is called a push-forward of $\mathbb{P}$ through ξ.

Now let us consider the space of probabilities on X with a finite p-th moment. Let the set $\mathcal{P}^p(X)$ be equal to the set of all probabilities on X such that, for some (equivalently, any $x_* \in X$)

$$\varsigma_p^p(m) \triangleq \int_X \rho_X^p(x, x_*)m(dx) < \infty.$$

The set $\mathcal{P}^p(X)$ is endowed by the Kantorovich–Rubinstein metric (also called in the literature the Wasserstein metric). If $m_1, m_2 \in \mathcal{P}^p(X)$, then

$$W_p(m_1, m_2) \triangleq \left[\inf_{\pi\in\Pi(m_1,m_2)} \int_{X\times X} \rho_X^p(x_1, x_2)\pi(d(x_1, x_2))\right]^{1/p}.$$

Here $\Pi(m_1, m_2)$ is the set of plans between m_1, m_2, i. e., $\pi \in \Pi(m_1, m_2)$ if π is a probability on $X \times X$ such that $\mathrm{p}^i\sharp\pi = m_i$. Hereinafter, $\mathrm{p}^i(x_1, x_2) \triangleq x_i$.

Further, we denote the set of curves defined on $[s, r]$ with values in $\mathbb{R}^d$ by $\Gamma_{s,r}$, i. e.,

$$\Gamma_{s,r} \triangleq C([s, r]; \mathbb{R}^d).$$

If $s = 0$, $r = T$ (here T is a fixed final time), then we will omit the subindices. Let $t \in [s, r]$, then $e_t \colon \Gamma_{s,r} \to \mathbb{R}^d$ stand for the evaluation operator, i. e., if $\gamma \in \Gamma_{s,r}$,

$$e_t(\gamma) = \gamma(t).$$

If $C' > 0$, then $\mathrm{Lip}(C')$ stands for the set of curves in Γ which are Lipschitz continuous with the constant C'.

Let, for each n, $[0, T] \ni t \mapsto m_n(t) \in \mathcal{P}^p(\mathbb{R}^d)$ be a flow of probabilities. We say that the sequence of flows of probabilities $\{m_n(\cdot)\}$ converges to $m(\cdot)$, if

$$\sup_{t\in[0,T]} W_p(m_n(t), m(t)) \to 0 \text{ as } n \to \infty.$$

The main feature of the paper is the assumption on the right-hand side of (1.2). We assume that

(A1) for each $t \in [0, T]$, $x \in \mathbb{R}^d$, $m \in \mathcal{P}^p(\mathbb{R}^d)$, $F(t, x, m)$ is compact and convex subset of $\mathbb{R}^d$;

(A2) the mapping $(t, x, m) \mapsto F(t, x, m)$ is upper semicontinuous;

(A3) there exists a constant $C_0 > 0$ such that, for each $t \in [0, T]$, $x \in \mathbb{R}^d$, $m \in \mathcal{P}^p(\mathbb{R}^d)$, $w \in F(t, x, m)$,

$$\|w\| \leq C_0(1 + \|x\| + \varsigma_p(m)).$$

In the paper, we use the following definition of the solution of (1.2).

Definition 1. We say that a flow of probabilities $m(\cdot)$ is a solution of (1.2) if there exists a probability $\chi \in \mathcal{P}^p(\Gamma)$ such that

- $m(t) = e_t \sharp \chi$;
- χ-a.e. $\gamma \in \Gamma$ satisfy

$$\frac{d}{dt}\gamma(t) \in F(t, \gamma(t), m(t)). \tag{1.3}$$

In this case, we say that χ determines the flow of probabilities $m(\cdot)$.

Recall that $x(\cdot)$ is a solution of the differential inclusion

$$\frac{d}{dt}x(t) \in G(t, x(t)) \tag{1.4}$$

if $x(\cdot)$ is absolutely continuous and satisfies (1.4) for a. e. $t \in [0, T]$.

§ 2. Existence of solution to the mean field type differential inclusion

In this section, we prove the existence theorem for a differential inclusion.

Theorem 1. *Let $m_0 \in \mathcal{P}^p(\mathbb{R}^d)$, then there exists at least one solution of* (1.2).

To prove this theorem, we will use the famous Peano method.

Let N be a natural number, for each $n \in \{0, \dots, N\}$, we put $t_n^N \triangleq Tn/N$. If $y \in \mathbb{R}^d$, let $f(y)$ be a measurable selector of the set-valued mapping $F(0, y, m_0)$. Its existence follows from [1, Theorem 18.13]. On $[0, t_1^N]$ we put

$$x^N(t, y) \triangleq y + f(y)t.$$

Further, set $m^N(t) \triangleq x^N(t, \cdot)\sharp m_0$. Now assume that the curves $x^N(t, y)$ and the flow of probabilities $m_n^N(t)$ are constructed for all $y \in \mathbb{R}^d$, $t \in [0, t_n^N]$. For each $t \in [t_{n-1}^N, t_n^N]$, let $f^N(t, y) \in F(t, x^N(t, y), m^N(t))$. Without loss of generality, one can assume that f^N is measurable [1, Theorem 18.13]. If $t \in [t_n^N, t_{n+1}^N]$, put

$$x^N(t, y) \triangleq x^N(t_n^N, y) + \int_{t_n^N}^t f^N(\tau - T/N, y)d\tau, \quad m^N(t) \triangleq x^N(t, \cdot)\sharp m_0.$$

Let us prove the following properties of the constructed family of curves.

Lemma 1. *There exist constants C_1, C_2 such that, for each $t \in [0, T]$,*

$$\varsigma_p(m^N(t)) \leq C_1(1 + \varsigma_p(m_0)).$$

$$\|x^N(t, y)\| \leq C_2(1 + \varsigma_p(m_0) + \|y\|))$$

P r o o f. We shall use induction that on each time interval $[0, t_n^N]$ the following inequalities hold true:

$$\varsigma_p(m^N(t)) \leq \varsigma_p(m_0)e^{2C_0t} + e^{C_0t}. \tag{2.1}$$

$$\|x^N(t, y)\| \leq \|y\|e^{C_0t} + \varsigma_p(m_0)e^{2C_0t} + e^{2C_0t}. \tag{2.2}$$

First, we have that for each $y \in \mathbb{R}^d$, $f(y) \leq C_0(1 + \|y\| + \varsigma_p(m_0))$. Thus, for $t \in [0, t_1^N]$,

$$\|x^N(t, y)\| \leq \|y\| + C_0(1 + \|y\| + \varsigma_p(m_0))t. \tag{2.3}$$

Using the definition of the measure $m^N(t)$, we have that

$$\varsigma_p(m^N(t)) \leq \varsigma_p(m_0) + C_0(1 + 2\varsigma_p(m_0))t.$$

Therefore,

$$\varsigma_p(m^N(t)) \leq \varsigma_p(m_0)e^{2C_0t} + C_0t.$$

Further, this and (2.3) imply that

$$\|x^N(t, y)\| \leq \|y\|e^{C_0t} + tC_0(1 + \varsigma_p(m_0)).$$

This proves (2.1) and (2.2) on $[0, t_1^N]$.

Now assume that (2.1) and (2.2) are fulfilled on $[0, t_n^N]$. We shall prove that these estimates hold true for $t \in [t_n^N, t_{n+1}^N]$. Indeed, if $\tau \in [t_n^N, t_{n+1}^N]$,

$$\|f^N(\tau - T/N, y)\| \leq C_0(1 + \|x^N(\tau - T/N)\| + \varsigma_p(m^N(\tau - T/N))).$$

Therefore, for $t \in [t_n^N, t_{n+1}^N]$,

$$\varsigma_p(m^N(t)) \leq \varsigma_p(m^N(t_n^N)) + C_0 \int_{t_n^N}^t (1 + 2\varsigma_p(m^N(\tau - Tn/N)))d\tau.$$

This implies the following inequality on $[t_n^N, t_{n+1}^N]$:

$$\varsigma_p(m^N(t)) \leq \varsigma_p(m_0)e^{2C_0 t_n^N}(1 + 2C_0(t - t_n^N)) + e^{C_0 t_n^N}(1 + C_0(t - t_n^N)).$$

Using estimate (2.1) on $[0, t_n^N]$, we conclude that this inequality is valid for $t \in [t_n^N, t_{n+1}^N]$.

Now, notice that, for $t \in [t_n^N, t_{n+1}^N]$,

$$\|x^N(t)\| \leq \|x_n^N(t_n^N)\| + C_0 \int_{t_n^N}^t (1 + \|x^N(\tau - T/N)\| + \varsigma_p(m^N(\tau - T/N)))d\tau.$$

Using the assumption that (2.1), (2.2) hold on $[0, t_n^N]$, we have that

$$\|x^N(t)\| \leq \|y\|e^{C_0 t_n^N}(1+C_0(t - t_n^N)) + \\ + \varsigma_p(m_0)e^{2C_0 t_n^N}(1 + 2C_0(t - t_n^N)) + e^{2C_0 t_n^N}(1 + 2C_0(t - t_n^N)).$$

This gives (2.2) on $[t_n^N, t_{n+1}^N]$.

Inequalities (2.1), (2.2) imply the statement of the lemma. □

This lemma and the construction of the curves x^N imply the following.

Corollary 1. *The curves $x^N(\cdot, y)$ are uniformly Lipchitz continuous whenever $y \in \mathbb{B}_R$.*

Now let us introduce the probabilities χ^N on Γ

$$\chi^N \triangleq X^N \sharp m_0. \tag{2.4}$$

Hereinafter, X^N is an operator that assigns to y the whole trajectory $x^N(\cdot, y)$.

Lemma 2. *Each probability $\chi^N \in \mathcal{P}^p(\Gamma)$. The sequence of measures $\{\chi^N\}_{N=1}^\infty$ is tight. Moreover, measures $\{\chi^N\}$ have a uniformly integrable p-th moment.*

P r o o f. The inclusion $\chi^N \in \mathcal{P}^p(\Gamma)$ directly follows from (2.4) and Lemma 1.

To prove the tightness of the sequence $\{\chi^N\}$, we consider the set $K_R \triangleq \{X^N(y) \colon y \in \mathbb{B}_R\}$. By Lemma 1, Corollary 1 and the Arzela–Ascoli theorem, the set K_R is compact. Further,

$$\chi^N(K_R) = m_0(\mathbb{B}_R).$$

This gives the tightness.

Further, we have that

$$\int_{\|\gamma\| \geq R} \|\gamma\|^p \chi^N(d\gamma) = \int_{\{y: \|x^N(\cdot, y)\| \geq R\}} \|x^N(\cdot, y)\|^p m_0(dy).$$

Now let us use Lemma 1. We have that

$$\int_{\{y: \|x^N(\cdot, y)\| \geq R\}} \|x^N(\cdot, y)\|^p m_0(dy) \leq 2^{p-1} \int_{\mathbb{R}^d \setminus \mathbb{B}_{R/C_2 - T - \varsigma_p(m_0)}} (1 + \|y\|^p + \varsigma_p^p(m_0))m_0(dy).$$

The right-hand side of this equality tends to zero when $R \to \infty$ due to the fact that $m_0 \in \mathcal{P}^p(\mathbb{R}^d)$. This gives the uniform integrability of the p-th moment of χ^N. □

Lemma 3. *The following estimate is valid:*

$$W_p(m^N(s), m^N(r)) \leq C_2|s-r|(1+2\varsigma_p(m_0)).$$

P r o o f. Without loss of generality we assume that $r > s$. Since $m^N(s) = e_s\sharp\chi^N$, $m^N(r) = e_r\sharp\chi^N$, we have that

$$W_p(m^N(s), m^N(r)) \leq \Big[\int_\Gamma \Big\| \int_s^r \gamma(t)dt\Big\|^p \chi^N(d\gamma)\Big]^{1/p}.$$

By the Minkowski's integral inequality,

$$W_p(m^N(s), m^N(r)) \leq \int_s^r \Big[\int_\Gamma \|\gamma(t)\|^p \chi^N(d\gamma)\Big]^{1/p} dt.$$

Using the definition of the measure χ^N, we conclude that

$$W_p(m^N(s), m^N(r)) \leq \int_s^r \Big[\int_{\mathbb{R}^d} \|x^N(t,y)\|^p m_0(dy)\Big]^{1/p} dt.$$

This and Lemma 1 give that

$$W_p(m^N(s), m^N(r)) \leq C_2(r-s)(1+2\varsigma_p(m_0)).$$ □

P r o o f o f T h e o r e m 1. By [2, Proposition 7.1.5] and Lemma 2, we have that the sequence $\{\chi^N\}$ is relatively compact. This means that there exists a subsequence $\{\chi^{N_l}\}_{l=1}^\infty$ and a measure $\chi \in \mathcal{P}^p(\Gamma)$ such that $W_p(\chi^{N_l}, \chi) \to 0$ as $l \to \infty$. Put

$$m(t) \triangleq e_t\sharp\chi.$$

By definition we have that $m(0) = m_0$. Let us prove that $m(\cdot)$ solves (1.2). To this end, choose $\gamma \in \mathrm{supp}(\chi)$. Due to [2, Proposition 5.1.8], there exists a sequence $\{\gamma^{N_l}\}$ that converges to γ such that $\gamma^{N_l} \in \mathrm{supp}(\chi^{N_l})$. Using the definition of χ^N (see (2.4)), one can assume that $\gamma^{N_l} = x^{N_l}(\cdot, y^{N_l})$, where the sequence $\{y^{N_l}\}$ converges to $y = \gamma(0)$.

The definition of x^{N_l} implies that, for a. e. $t \in [0,T]$,

$$\frac{d}{dt}x^{N_l}(t, y^{N_l}) \in F(t-T/N_l, x^{N_l}(t-T/N_l, y^{N_l}), m^{N_l}(t-T/N_l)).$$

By [11, Theorem 1.5], we have that, for a.e. $t \in [0,T]$,

$$\frac{d}{dt}x(t,y) \in \bigcap_{L=1}^\infty \overline{\mathrm{co}} \bigcup_{l=L}^\infty F(t-T/N_l, x^{N_l}(t-T/N_l, y^{N_l}), m^{N_l}(t-T/N_l)). \tag{2.5}$$

Now notice that the convergence of $\{y^{N-l}\}$ to y implies the boundness of this sequence. Thus, $x^{N_l}(\cdot, y^{N_l})$ are uniformly Lipschitz continuous. Moreover, recall (see Lemma 3) that $m^N(t-T/N) \to m(t)$ as $N \to \infty$. Therefore, the right-hand side in (2.5) converges to $F(t, \gamma(t), m(t))$. This gives the fact χ-a. e. $\gamma \in \Gamma$ satisfies (1.2). □

§ 3. Stability analysis

In this section, we study the convergence of the solution of mean field type differential inclusions. To this end, we consider a family of differential inclusions

$$\frac{d}{dt}m_n(t) \in \langle F_n(t,\cdot,m_n(t)), \nabla\rangle m_n(t). \tag{3.1}$$

Additionally each differential inclusion is endowed with the initial condition $m_n(0) = m_{0,n}$. We assume that each multifunction F_n satisfies conditions (A1)–(A3).

Definition 2. We say that a multivalued velocity field $F^*\colon [0,T]\times\mathbb{R}^d\times\mathcal{P}^p(\mathbb{R}^d)$ is an upper limit of multivalued velocity fields F_n provided that

$$F^*(t,x,m) = \bigcap_{n\in\mathbb{N},\delta>0} \overline{\mathrm{co}}\Bigg[\bigcup_{l\geq n,\|x'-x\|\leq\delta,W_p(m',m)\leq\delta} F_n(t,x',m')\Bigg].$$

In this case, the mapping F^* is denoted by

$$\operatorname*{Limsup}_{n\to\infty} F_n.$$

The inclusion

$$\operatorname*{Limsup}_{n\to\infty} F_n \subset F$$

means that, for each t, x, m, one has

$$\Big[\operatorname*{Limsup}_{n\to\infty} F_n\Big](t,x,m) \subset F(t,x,m).$$

Theorem 2. *Assume that* $\operatorname{Limsup}_{n\to\infty} F_n \subset F$*, while* $W_p(m_{0,n}, m_0) \to 0$ *as* $n\to\infty$*. Let, for each natural* n*,* $m_n(\cdot)$ *be a solution of* (3.1) *with initial condition* $m_n(0) = m_{0,n}$*. Then,* $\{m_n(\cdot)\}_{n=1}^{\infty}$ *is relatively compact. If it converges to* $m(\cdot)$*, then* $m(\cdot)$ *solves* (1.2) *with initial condition* $m(0) = m_0$*.*

Lemma 4. *If* $m(\cdot)$ *is a solution of* (1.2) *with initial condition* $m(0) = m_0$*, then there exists a constant* C_3 *computed by* T *and* C_0 *such that*

$$\varsigma_p(m(t)) \leq C_3(1 + \varsigma_p(m_0)). \tag{3.2}$$

Additionally, if $\gamma\in\Gamma$ *satisfies* (1.3)*, then*

$$\|\gamma(t)\| \leq C_4(1 + \varsigma_p(m_0) + \|\gamma(0)\|).$$

Here C_4 *is a constant determined only by* T *and* C_0*.*

P r o o f. Let χ determine the flow of probabilities $m(\cdot)$. Since χ-a. e. γ satisfies (1.3), one has that

$$\|\gamma(t)\| \leq \|\gamma(0)\| + C_0\int_0^t (1 + \|\gamma(\tau)\| + \varsigma_p(m(\tau)))d\tau. \tag{3.3}$$

Therefore,

$$\varsigma_p(m(t)) \leq \varsigma_p(m_0) + C_0\Big[\int_\Gamma\Big(\int_0^t (1 + \|\gamma(\tau)\| + \varsigma_p(m(\tau)))d\tau\Big)^p \chi(d\gamma)\Big]^{1/p}.$$

Using Minkowski's integral inequality, we have that

$$\varsigma_p(m(t)) \leq \varsigma_p(m_0) + C_0 \int_0^t (1 + 2\varsigma_p(m(\tau)))d\tau.$$

This and Gronwall's inequality imply the first statement of the lemma.

To prove the second one, let us substitute (3.2) into (3.3). We have that

$$\|\gamma(t)\| \leq \|\gamma(0)\| + C_3(1 + \varsigma_p(m_0))t + C_0 \int_0^t (1 + \|\gamma(\tau)\|)d\tau.$$

As above, we use Gronwall's inequality and obtain the second statement of the lemma. □

Corollary 2. *Assume that $m(\cdot)$ is a solution of (1.2) with initial condition $m(0) = m_0$, while $\chi \in \mathcal{P}(\Gamma)$ determines this solution. Then, for χ-a. e. $\gamma \in \Gamma$, one has*

$$\|\gamma(s) - \gamma(r)\| \leq C_5|s - r|(1 + \varsigma_p(m_0) + \|\gamma(0)\|).$$

Here C_5 is a constant determined only by T and C_0.

This corollary directly follows from Lemma 4 and Assimption (A3).

Lemma 5. *Assume that*

- *F_n satisfy conditions (A1)–(A3) with the same constant C_0;*
- *$\{m_{0,n}\}_{n=1}^{\infty} \subset \mathcal{P}^p(\mathbb{R}^d)$ are such that $\varsigma_p(m_{0,n}) \leq C'$;*
- *for each natural n, $m_n(\cdot)$ solves mean field type differential inclusion (3.1) with the initial condition $m_n(0) = m_{0,n}$;*
- *$\chi_n \in \mathcal{P}(\Gamma)$ determines $m_n(\cdot)$.*

Then, the probabilities $\{\chi_n\}$ are relatively compact.

P r o o f. First, we will prove that the measures $\{\chi_n\}$ are tight. To this end, consider the set of curves

$$K_R^1 \triangleq \{\gamma \in \Gamma \colon \|\gamma(0)\| \leq R, \quad \gamma \in \mathrm{Lip}(C_5(1 + C' + R))\}.$$

Notice that K_R^1 is compact for every $R > 0$. By Corollary 2, we have that

$$\chi_n(\Gamma \setminus K_R^1) = m_n(\Gamma \setminus \mathbb{B}_R).$$

By the Chebyshev inequality, we conclude that

$$m_n(\Gamma \setminus \mathbb{B}_R) \leq \frac{C_6}{R}.$$

This gives the tightness of $\{m_n\}$.

Now let us prove the fact that the probabilities χ_n have uniformly integrable p-th moment. We have that, for each n,

$$\int_{\|\gamma\| \geq R} \|\gamma\|^p \chi_n(d\gamma) \leq C_4^p \int_{\|\gamma(0)\| \geq R/C_4 - C_4\varsigma_p(m_0)} (1 + \varsigma_p(m_0) + \|\gamma(0)\|)^p \chi_n(d\gamma)$$
$$\leq 2^{p-1} C_4^p \int_{\|y\| \geq R/C_4 - C_4\varsigma_p(m_0)} (1 + \varsigma_p^p(m_0) + \|y\|^p) m_0(d\gamma).$$

The right-hand side of this inequality tends to zero when $R \to \infty$ due to the fact that $m_0 \in \mathcal{P}^p(\mathbb{R}^d)$. This proves the uniform integrability of p-th moments of χ_n.

By [2, Proposition 7.1.5], the tightness of $\{\chi_n\}$ together with the uniform integrability of p-th moment is equivalent to the relative compactness of $\{\chi_n\}$. □

P r o o f o f T h e o r e m 2. Let $\chi_n \in \mathcal{P}^(\Gamma)$ determine $m_n(\cdot)$. By Lemma 5, the probabilities χ_n are relatively compact. Therefore, there exists a sequence $\{\chi_n n_k\}_{k=1}^\infty$ and a probability χ such that $\{\chi_n n_k\}$ converges to χ. Put $m(t) \triangleq e_t \sharp \chi$. Since

$$W_p(m_{n_k}(t), m(t)) = W_p(e_t \sharp \chi_{n_k}, \chi) \leq W_p(\chi_{n_k}, \chi).$$

This gives the relative compactness of flows of probabilities $m_n(\cdot)$.

Now let us prove the second part of the theorem. We assume that $\{m_n(\cdot)\}_{n=1}^\infty$ converges to $m(\cdot)$. As above, for each n, let $\chi_n \in \mathcal{P}^p(\Gamma)$ determine the flow of probabilities $m_n(\cdot)$. Without loss of generality, we assume that $\{\chi_n\}_{n=1}^\infty$ converges to some probability χ. Further,

$$m(t) = e_t \sharp \chi.$$

It remains to prove that, χ-a.e. $\gamma \in \Gamma$ satisfies (1.2). Notice that χ_n-a.e. γ_n satisfies (3.1). Further, by [2, Proposition 5.1.8] for each $\gamma \in \operatorname{supp}(\chi_n)$, there exists a sequence $\{\gamma_n\}_{n=1}^\infty \subset \Gamma$ that converges to γ and satisfies the inclusions $\gamma_n \in \operatorname{supp}(\chi_n)$ for each n. Theorem 1.5 of [11] gives that

$$\frac{d}{dt}\gamma(t) \in \bigcap_{L=1}^\infty \overline{\mathrm{co}} \bigcup_{l=L}^\infty F_n(t, \gamma_n(t), m_n(t)).$$

Using the facts that $\|\gamma_n(t) - \gamma(t)\|, W_p(m_n(t), m(t)) \to 0$ as $n \to \infty$, while F comprises the upper limit of $\{F_n\}$, we obtain the second statement of the theorem. □

§ 4. Application to the mean field type optimal control problems

Let us consider the following mean field type optimal control problem:

$$\text{minimize } \sigma(m(T)) \tag{4.1}$$

subject to $m(\cdot)$ satisfying

$$\frac{d}{dt}m(t) \in \langle F(t, \cdot, m(t)), \nabla\rangle m(t), \quad m(0) = m_0. \tag{4.2}$$

Proposition 1. *Assume that F satisfies conditions (A1)–(A3), while $\sigma\colon \mathcal{P}^p(\mathbb{R}^d) \to \mathbb{R}$ is lower semicontinuous. Then, there exists a solution of* (4.1)*,* (4.2)*.*

P r o o f. Due to Theorems 1, 2, the set of solutions of (4.2) is nonempty compact. Using the fact the function σ is lower semicontinuous, we conclude that, if $m_n(\cdot)$ is a minimizing sequence, then its limit is a solution of mean field optimal control problem (4.1), (4.2). □

One may introduce the value function

$$\mathrm{Val}(m_0) \triangleq \min\{\sigma(m(T))\colon m(\cdot) \text{ satisfying (4.2)}\}.$$

Further, let us consider the sequence of mean field optimal control problems

$$\text{minimize } \sigma_n(m(T)) \tag{4.3}$$

subject to $m(\cdot)$ satisfying

$$\frac{d}{dt}m(t) \in \langle F_n(t, \cdot, m(t)), \nabla\rangle m(t), \quad m(0) = m_0. \tag{4.4}$$

We denote the value function of this problem by Val_n.

Proposition 2. *Assume that*

- *F and F_n satisfy (A1)–(A3);*
- *σ and σ_n are lower semicontinuous;*
- *F comprises the upper limit of F_n;*
- *σ is lower semicontinuous.*

Then, for every $\{m_{0,n}\}_{n=1}^{\infty}$ converging to m_0, the following inequality holds:

$$\mathrm{Val}(m_0) \leq \liminf_{n\to\infty} \mathrm{Val}_n(m_{0,n}).$$

P r o o f. Let $m_n(\cdot)$ be an optimal solution for (4.3), (4.4) with $m(0) = m_{0,n}$, i. e., we assume that

$$\mathrm{Val}_n(m_{0,n}) = \sigma_n(m_n(T)).$$

Using Theorem 2, without loss of generality, we assume that $m_n(\cdot)$ conveges to $m(\cdot)$ that satisfies (4.2). Therefore, due to assumption that σ is lower semicontinuous, one has

$$\mathrm{Val}(m_0) \leq \sigma(m(T)) \leq \liminf_{n\to\infty} \sigma_n(m_n(T)) = \liminf_{n\to\infty} \mathrm{Val}_n(m_{0,n}). \qquad \square$$

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Дифференциальные включения типа среднего поля с полунепрерывной правой частью

Ключевые слова: дифференциальные включения типа среднего поля, оптимальное управление средним полем, стабильность.

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Дифференциальные включения типа среднего поля возникают в рамках теории управления средним полем при овыпуклении правой части. Мы исследуем случай, когда правая часть дифференциального включения зависит от положения агента и от распределения всех агентов полунепрерывно. Основной результат статьи состоит в доказательстве существования и стабильности решений дифференциальных включений типа среднего поля. Также мы показываем полунепрерывную снизу зависимость функции цены задачи оптимального управления средним полем от начального состояния и параметра.

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