VESTNIK UDMURTSKOGO UNIVERSITETA. MATEMATIKA. MEKHANIKA. KOMP'UTERNYE NAUKI
MATHEMATICS 2021. Vol. 31. Issue 4. Pp. 640–650.

MSC2020: 31C20, 31A05, 31A10

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INFINITE SCHRÖDINGER NETWORKS

Finite-difference models of partial differential equations such as Laplace or Poisson equations lead to a finite network. A discretized equation on an unbounded plane or space results in an infinite network. In an infinite network, Schrödinger operator (perturbed Laplace operator, q-Laplace) is defined to develop a discrete potential theory which has a model in the Schrödinger equation in the Euclidean spaces. The relation between Laplace operator Δ-theory and the Δ_q-theory is investigated. In the Δ_q-theory the Poisson equation is solved if the network is a tree and a canonical representation for non-negative q-superharmonic functions is obtained in general case.

Keywords: q-harmonic functions, q-superharmonic functions, Schrödinger network, hyperbolic Schrödinger network, parabolic Schrödinger network, integral representation.

DOI: 10.35634/vm210408

Introduction

The classical Schrödinger equation in the Euclidean space is $\Delta u(x) = \lambda u(x)$, $\lambda > 0$, in its simplest form. In a discretised form, finding solution to this equation is an eigenvalue problem when considered on a finite graph with a set of transition functions (as in the case of a finite dedicated network with known conductance on the branches). A suitable method [1] to consider this discrete problem is to make use of the Perron–Forbenius Theorem (see Gantmacher [2]).

A strikingly similar equation $\Delta u(x) = p(x)u(x)$ where $p(x) \geq 0$ is a C^1-function is considered in the classification theory of Riemannian manifolds [3]. Here the Dirichlet norm on a Riemannian manifold is widely used. Some of the questions not considered in this context are:

(i) Can we avoid the use of the Dirichlet norm and the Dirichlet principle so that the equation can be solved in a larger context?

(ii) Is it possible to relax the condition $p(x) \geq 0$ to include some negative values for $p(x)$?

(iii) If there is a positive solution $\phi(x)$, $\Delta\phi(x) = p(x)\phi(x)$, then $p(x) = \frac{\Delta\phi(x)}{\phi(x)}$. Is it possible to carry out the investigation if there exists a C^1-function $\xi(x) > 0$ such that $p(x) = \frac{\Delta\xi(x)}{\xi(x)}$?

As a prelude to investigate these questions by developing new techniques for proofs, we fall back on a discretised version of Riemannian manifolds: In an infinite graph with a countable number of vertices and a countable number of edges, provided with a set of transition indices (as in an infinite random walk with known transition probabilities) study the properties of the solutions to the equation $\Delta u(x) = q(x)u(x)$ where $q(x) \geq \frac{\Delta\xi(x)}{\xi(x)}$ for a function $\xi(x) > 0$. The methods we are using here to study functions on an infinite graph with a set of transition functions (named here an infinite network) have a reflection on the study of potential theory on locally compact spaces. Our investigation uses extensively the properties of superharmonic functions, potentials etc. defined on an infinite network by means of the discrete Laplace operators. Recall that such properties are variously introduced and studied extensively by different authors like Cartier [4], Yamasaki [5], Soardi [6], Cohen–Colonna–Singham [7], Soardi [8], Woess [9] and Anandam [10, 11], Hedenmalm–M. Rodriguez [12], Alyusof–Colonna [13], Simon [14], Colonna–Tjani [15,16]. For these works, the basic idea comes from Algebraic Geometry, Electric Networks, Random Walks or Classical and Axiomatic Potential Theory.

In a finite electrical network, the Schrödinger operator appears as a pertubation of the combinatorial Laplacian [17,18]. It introduces generalised versions of the classical condenser principle and the effective resistance. Abstracting this situation, let us take a finite graph X provided with a set of transition indices $t(x,y) \geq 0$ between the vertices. Suppose $q(x)$ is a real-valued function on X. Then using Perron–Forbenius Theorem (see Gantmacher [2]), it is shown that $q(x) = \frac{\Delta\xi(x)}{\xi(x)} + c$ where $\xi(x) > 0$ with $\sum\limits_x \xi(x) = 1$ and the constant 'c' is uniquely determined. Further study of functions defined on $\{X, t, q\}$ diversifies depending on whether $c = 0$ or $c > 0$ or $c < 0$. In a broad sense, these three cases correspond to the Laplace, Schrödinger, and Helmholtz operators respectively in the Euclidean spaces.

In this paper, we examine only the case $c > 0$ in an infinite network $\{X, t\}$, where X is an infinite graph and $\{t(x,y)\}$ is a set of transition indices. We fix a function $q(x)$ such that $q(x) \geq \frac{\Delta\xi(x)}{\xi(x)}$ for some function $\xi(x) > 0$. Remark that $q(x)$ can take negative values also. In this case, the Schrödinger operator Δ_q is given by $\Delta_q u(x) = \Delta u(x) - q(x)u(x)$. Then we develop a discrete Schrödinger potential theory based on the operator Δ_q. These results finally lead to an integral representation of positive Δ_q-*harmonic functions*, by using a Choquet theorem. We conclude by proving Δ_q-*harmonic extension theorem* and obtaining a solution to the Poisson equation, if the network X is actually an infinite tree.

§ 1. Preliminaries

Consider a graph X with countable infinite number of vertices and countable infinite number of edges (see Yamasaki [5]). We say two vertices x, y are neighbours if and only if there exists an edge joining these two vertices, denoted by $x \sim y$. For any pair of vertices x, y we associate a transition function $t(x,y)$ such that $t(x,y) \geq 0$ and $t(x,y) > 0$ if and only if $x \sim y$. We assume that $t(x,y)$ may be asymmetric, that is $t(x,y)$ and $t(y,x)$ may be different. We write $t(x) = \sum\limits_{(y \sim x)} t(x,y)$.

The following terms are defined in [10]:

- (Infinite Network) A pair $\{X, t\}$ is said to be an infinite network if it satisfies the following conditions:
  - (i) X is connected, that is, for any two distinct vertices $x \neq y$ there exists a path $\{x = x_0, x_1, \cdots, x_n = y\}$ connecting x and y.
  - (ii) X is locally finite, that is, for any vertex $x \in X$ has only finite number of neighbours.
  - (iii) X has no self-loops, that is, any x there is no edge connecting x to x.
- (Interior and Boundary of a set) We say a vertex x is an interior vertex of a subset F if and only if x and all its neighbours are in F. The set of all interior vertices of F is denoted by $\overset{\circ}{F}$ and the boundary of F by $\partial F = F \backslash \overset{\circ}{F}$.
- (Circled Sets) An arbitrary set F in X is said to be circled if every vertex in ∂F has at least one neighbour in $\overset{\circ}{F}$.
- (Laplacian operator) Let v be a real-valued function defined on F. For $x \in \overset{\circ}{F}$, the Laplacian of v at x is defined as

$$\Delta v(x) = \sum_{y \sim x} t(x,y)[v(y) - v(x)].$$

We say that v is harmonic, superharmonic or subharmonic at x if and only if $\Delta v(x) = 0$, $\Delta v(x) \leq 0$ or $\Delta v(x) \geq 0$ respectively. If $\Delta v(x) = 0$ (respectively $\Delta v(x) \leq 0$) for every $x \in \overset{\circ}{F}$, then v is said to be harmonic (respectively superharmonic) on F.

- A Schrödinger network $\{X, t, q\}$ is an infinite network $\{X, t\}$, in which $q(x)$ is a real-valued function on X such that $q(x) \geq \frac{\Delta \xi(x)}{\xi(x)}$, for some positive real-valued function $\xi(x)$ on X.

- (q-Laplacian) Let v be a real valued function defined on $\{X, t, q\}$. The q-Laplacian of v at $x \in X$ is defined by,

$$\begin{aligned}\Delta_q v(x) &= \Delta v(x) - q(x)v(x) \\ &= \sum_{y \sim x} t(x, y)[v(y) - v(x)] - q(x)v(x) \\ &= \sum_{y \sim x} t(x, y)v(y) - [t(x) + q(x)]v(x).\end{aligned}$$

  We say that v is *q-harmonic*, *q-superharmonic* or *q-subharmonic* at x if and only if $\Delta_q v(x) = 0$, $\Delta_q v(x) \leq 0$ or $\Delta_q v(x) \geq 0$ respectively.

Note:

1. $\xi(x)$ is a positive q-superharmonic function, that is, $\Delta_q \xi(x) = \Delta \xi(x) - q(x)\xi(x) \leq 0$.
2. $t(x) + q(x) > 0$ on X, for $q(x) \geq \frac{\Delta \xi(x)}{\xi(x)} = \sum_{y \sim x} \frac{t(x,y)\xi(y)}{\xi(x)} - t(x)$ so that $q(x) + t(x) > 0$.

- (g.q-h.m, greatest q-harmonic minorant) Let s be a q-superharmonic function and v is a q-subharmonic function on $F \subset X$ such that $s \geq v$. Then there exists a q-harmonic function h on F such that $s \geq h \geq v$. This function h can be chosen such that if there is another q-harmonic function h^* on F such that $s \geq h^* \geq v$ then $h \geq h^*$. This q-harmonic function h is called the greatest q-harmonic minorant, (g.q-h.m) of s on F.

- (q-potentials) A non-negative q-superharmonic function s defined on a subset F is said to be a q-potential if and only if the greatest q-harmonic minorant of s on F is 0.

Some of the propositions and theorems in the case of a Schrödinger network can be proved in a fashion similar to those of a usual infinite network, as given in [10]. Here we mention some of those results which we use in this paper.

Theorem 1 (Generalized q-Dirichlet Problem). *Let E be a subset of a Schrödinger network X and $F \subset \overset{\circ}{E}$. Suppose f is a function defined on $E \backslash F$ such that $u \geq f \geq v$ on $E \backslash F$ where u and v are defined on E, $u \geq v$ on E, $\Delta_q u \leq 0$ and $\Delta_q v \geq 0$ at each vertex in F. Then, there exists a function h on E such that $u \geq h \geq v$ on E, $h = f$ on $E \backslash F$, and $\Delta_q h(x) = 0$ at each $x \in F$, and the function h can be so chosen that if h_1 is another such q-harmonic function on E with these three properties, then $h_1 \leq h$. However, the function h is uniquely determined if F is a finite set.*

Theorem 2 (Riesz representation). *If $s \geq 0$ is a q-superharmonic function on E, then it can be represented as a sum of a q-potential p and a non-negative q-harmonic function h, $s = p + h$ on E; and this representation is unique.*

Theorem 3 (q-Domination Principle). *Let v and f be two functions on X. Let $\Delta_q v \leq 0$ on X and $A = \{x\colon \Delta_q f(x) < 0\}$. If $v \geq f$ on A, then $v \geq f$ on X.*

Remark 1. Let s be a q-superharmonic function on X. Suppose A is the smallest subset of X such that s is q-harmonic at each vertex of $X \backslash A$. Then A is referred to as the q-harmonic support of s in X.

Theorem 4. *Let $f(x)$ be a real-valued function on X. Suppose the family $\mathfrak{F}$ of q-superharmonic functions s majorizing f on X is non-empty. Then $Rf(x) = \inf_{s \in \mathfrak{F}} s(x)$ is q-superharmonic on X and q-harmonic at each vertex a where $f(x)$ is q-subharmonic.*

Theorem 5 (q-Balayage). *Consider an arbitrary subset F of X and let $w \geq 0$ be a q-superharmonic function on X. Then a q-superharmonic function $R_w^F > 0$ exists on X, such that $R_w^F \leq w$ on X, $R_w^F = w$ on F and R_w^F is q-harmonic at each vertex in $X \backslash F$.*

§ 2. The operator Δ^* and its relation to the operator Δ_q

Let $\{X, t, q\}$ be a Schrödinger network. Since we have taken $q(x) \geq \frac{\Delta\xi(x)}{\xi(x)}$ for some $\xi(x) > 0$ there exists a positive q-superharmonic function $\xi(x)$ on X (see [10, Theorem 4.1.9]). That is, $q(x) = \frac{\Delta\xi(x)}{\xi(x)}$, however now $-\Delta_q R_w^{\{a\}}(b) = \lambda\delta_a(b)$ where λ is a constant, so that $-\Delta_q R_w^{\{a\}}(a) = \lambda\delta_a(a) = \lambda$. Consequently $-\Delta_q R_w^a(b) = [\Delta_q R_w^a(a)]\delta_a(b)$. This shows that $-\Delta_q[G_a(b)] = \delta_a(b)$. This does not ensure the existence of a q-potential on X.

The following proposition gives the relation between the Laplacian and the q-Laplacian operators.

Proposition 1. *Let $u(x)$ be a real valued function on $\{X, t, q\}$. If $v(x) = \frac{u(x)}{\mu(x)}$, then*

$$\Delta_q u(x) = \sum_{y \sim x} t(x, y)\mu(y)[v(y) - v(x)].$$

P r o o f.

$$\begin{aligned}
\Delta_q u(x) &= \Delta u(x) - q(x)u(x) \\
&= \Delta u(x) - \frac{\Delta\mu(x)}{\mu(x)}u(x) \\
&= \frac{1}{\mu(x)}[\mu(x)\Delta u(x) - u(x)\Delta\mu(x)] \\
&= \frac{1}{\mu(x)}[\mu(x)\sum_{y \sim x} t(x, y)[u(y) - u(x)] - u(x)\sum_{y \sim x} t(x, y)[\mu(y) - \mu(x)]] \\
&= \frac{1}{\mu(x)}\sum_{y \sim x} t(x, y)[\mu(x)\mu(y)v(y) - \mu(x)\mu(y)v(x)] \\
&= \sum_{y \sim x} t(x, y)\mu(y)[v(y) - v(x)].
\end{aligned}$$

□

Now we consider a new set of transition functions $t^*(x, y) = t(x, y)\xi(y)$. Then $\{X, t^*(x, y)\}$ is an infinite network and Δ^* denotes the Laplacian of this network. From Proposition 1 we get $\Delta_q u(x) = \Delta^*[\frac{u(x)}{\xi(x)}]$ for any real valued function $u(x)$ on X. Consequently $u(x)$ is q-superharmonic (q-harmonic or q-subharmonic) in $\{X, t, q\}$ if and only if $\frac{u(x)}{\xi(x)}$ is superharmonic (harmonic or subharmonic respectively) in $\{X, t^*\}$. So the potential theoretic properties of the Schrödinger network $\{X, t, q\}$ can be deduced from those of the network $\{X, t^*\}$. It follows from the above narrative that, for any vertex a in X, it is not difficult to see that there exists a q-superharmonic function $s_a(x)$ on X such that $-\Delta_q[s_a(x)] = \delta_a(x)$. It is possible to give some more special properties of this function $s_a(x)$ depending on whether there exist positive q-potentials or not on X. Functions with these special properties will be named *q-Green potentials* or *q-logarithmic potentials*. In the classical potential theory, in $\mathbb{R}^3$ the basic kernel is the Newtonian potential $G_y(x) = \frac{1}{|x-y|}$. A generalisation of this kernel in measure spaces and topological spaces leads to a rich theory in random walks, Markov processes, elliptic differential equations etc. However in $\mathbb{R}^2$ the basic function is the logarithmic kernel $-\log|x-y|$ which is not a positive kernel. But this kernel is very useful in the study of function theory on Riemann surfaces and Riemannian manifolds. The two kernels, when a point is fixed, are superharmonic functions. Abstracting these in the context of a discrete potential theory on networks, we study hyperbolic and parabolic networks.

Theorem 6 (q-Green potentials). *If there are q-potentials on X, then for any vertex a, a unique q-potential $G_a(b) > 0$ exists on X, such that $\Delta_q G_a(b) = -\delta_a(b)$, for all $b \in X$.*

P r o o f. To construct this function, if $w > 0$ is a q-potential on X, then take

$$G_a(b) = \frac{R_w^{\{a\}}(b)}{(-\Delta_q)R_w^{\{a\}}(a)}$$

where $R_w^{\{a\}} > 0$ is a q-potential and $R_w^{\{a\}} \leq w$. Since a positive q-superharmonic function dominated by a q-potential is itself a q-potential (see Theorem 3), then $G_a^{(b)} > 0$ is a q-potential.

To prove the uniqueness, consider another q-potential $Q > 0$ such that

$$-\Delta_q Q(b) = \delta_a(b).$$

Then $Q(b) = G_a(b) + u(b)$, where u is a q-harmonic function on X. We note that $u \equiv 0$, by Riesz decomposition of a positive q-superharmonic function (see Theorem 2), as the unique sum of a q-potential and a non-negative q-harmonic function. □

An infinite network X is said to be parabolic if and only if there exists no positive potential in X. So from above Propositions 1 there are positive q-potentials in a Schrödinger network $\{X, t, q\}$ if and only if there are positive potentials on the network $\{X, t^*(x, y)\}$. Consequently the network $\{X, t, q\}$ is parabolic if and only if $\{X, t^*\}$ is parabolic.

Let us consider a parabolic Schrödinger network $\{X, t, q\}$. Fix the vertex $e \in X$. Then in the parabolic network $\{X, t^*\}$ there exists a function $H_e^*(x)$ on X such that $\Delta^*[H_e^*(x)] = 0$ if $x \neq e$ and $H_e^*(x) \geq 0$ outside a finite set A in X (Theorem 3.2.4 [10]). Set $H_e(x) = \mu(x)H_e^*(x)$. Note that $\Delta_q(H_e(x)) = \Delta_q(\mu(x)H_e^*(x)) = \Delta^*(H_e^*(x)) = 0$ and if $x \neq 0$ $H_e(x) \geq 0$ outside the finite set A. Fixing the function $H_e^*(x)$ on X some of the results in the parabolic Schrödinger network $\{X, t, q\}$ can be deduced from the corresponding results known in the parabolic network $\{X, t^*\}$. This function $H_e(x)$ has many important properties which are discrete analogous of the logarithmic function $\log|x|$ on $\mathbb{R}^2$.

§ 3. Integral representation of non-negative q-superharmonic functions on Schrödinger networks

Let $\{X, t, q\}$ be a hyperbolic Schrödinger network. In this section we give an integral representation of positive q-harmonic functions, using the Choquet Integral Representation Theorem as in [19]. Later we obtain a canonical representation for non-negative q-superharmonic functions on X.

Let H_q^+ be the set of all non-negative q-harmonic functions defined on $\{X, t, q\}$. H_q^+ is a convex cone which is a lattice for the natural order. Consider $\mathfrak{B}$ to be the base of H_q^+, consisting of the functions h such that for a fixed vertex $e \in X$, $h(e) = 1$, $h \in H_q^+$. By using the Harnack property [20, Lemma 3.1], we see that $\mathfrak{B}$ is compact. The extremal elements E of the base $\mathfrak{B}$ consist of the minimal q-harmonic functions. Take $H = H_q^+ - H_q^+$, we define the norm for $f, g \in H$, as $||f - g|| = \sup\limits_{x \in X} \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}$. In H, H_q^+ is a convex cone, which is a lattice, with a compact base $\mathfrak{B}$. By using the Choquet integral representation theorem from the lecture notes [20, 21] we have the following integral representation of a positive q-harmonic function.

Theorem 7. *Let X be a Schrödinger network. Then there exists a unique unitary measure $\nu \geq 0$ with support in the extremal set E in $\mathfrak{B}$ such that for any $f \in H_q^+$,*

$$f(x) = \int\limits_{h \in E} h(x) d\nu(h), \quad for\ x \in X.$$

We have just seen a representation for positive q-harmonic functions on a Schrödinger network $\{X, t, q\}$ as integrals. Let us now represent a q-potential as an infinite sum of q-potentials. For that recall that in a hyperbolic Schrödinger network, for any y in X there exists a q-potential $G_y^q(x)$ such that $\Delta_q[G_y^q(x)] = -\delta_y(x)$ for all x in X.

Theorem 8. *Let $p(x)$ be a q-potential on X. Then $p(x) = \sum\limits_{y \in X} G_y^q(x)[-\Delta_q p(y)]$.*

P r o o f. Write $p_n(x) = \sum\limits_{|y| \leq n} G_y^q(x)[-\Delta_q p(y)]$. Then as a finite sum of q-potentials, $p_n(x)$ is a q-potential. Now $p_n(x) \leq p(x)$, for

$$-\Delta_q p_n(x) = \sum_{|y| \leq n} [-\Delta_q p(y) \delta_y(x)] \leq -\Delta_q p(x),$$

hence, $-\Delta_q[p(x) - p_n(x)] \geq 0$. This means that there exists a q-superharmonic function $s(x)$ such that $p(x) - p_n(x) = s(x)$. Now $-s(x)$ is a q-subharmonic function dominated by the potential $p_n(x)$ so that $-s(x) \leq 0$; consequently, $p(x) \geq p_n(x)$.

Then take limits as $n \to \infty$ to arrive at

$$\sum_{y \in X} G_y^q(x)[-\Delta_q p(y)] = \lim_{n \to \infty} \sum_{|y| \leq n} G_y^q(x)[-\Delta_q p(y)] \leq p(x).$$

Now as the limit of q-potentials, the infinite sum represents a q-superharmonic function which is dominated by a q-potential. Hence, the infinite sum $u(x) = \sum\limits_{y \in X} G_y^q(x)[-\Delta_q p(y)]$ is a q-potential. Moreover,

$$-\Delta_q[u(x)] = \sum_{y \in X} \delta_y(x)[-\Delta_q p(y)] = -\Delta_q p(x).$$

Since $\Delta_q[p(x) - u(x)] = 0$, then there is a q-harmonic function $h(x)$ on X such that $p(x) = u(x) + h(x)$. Then $h \equiv 0$ by the uniqueness of representation of q-potentials. Thus, $p(x) = u(x) = \sum\limits_{|y| \leq n} G_y^q(x)[-\Delta_q p(y)]$. □

Let now $s(x) > 0$ be a q-superharmonic function on a Schrödinger network. Then $s(x)$ is the unique sum of a q-potential $p(x)$ and a non-negative q-harmonic function $h(x)$. Note that $-\Delta_q p(x) = -\Delta_q s(x)$. Let us now apply Theorem 7 and Theorem 8 to these two functions $p(x)$ and $h(x)$. Consequently we have proved the following theorem.

Theorem 9. *Let $s(x)$ be a non-negative q-superharmonic function on a Schrödinger network $\{X, t, q\}$. Then there exists a unique unitary Radon measure ν with the support in the set E of the minimal q-harmonic functions such that*

$$s(x) = \sum_{y \in X} G^q_y(x)[-\Delta_q s(y)] + \int h(x) d\nu(h), \quad h \in E.$$

§ 4. q-Poisson equation on infinite trees

In this section, we restrict ourselves to the consideration of functions defined on an infinite tree T. Recall that an infinite tree $\{T, t\}$ is an infinite network in which there is no closed path containing more than 2 vertices without terminal vertices (if a vertex has only one neighbour, then it is called a terminal vertex). Consequently, there is a unique path joining two given vertices. Let us fix a vertex e in T and measure distances from e; that is if x is any vertex in T and $\{e = x_0, x_1, x_2, \ldots, x_n = x\}$ is a unique path joining e to x, write $|x| = n$. Then remark that for any vertex x, $|x| = n \geq 1$, there is a unique predecessor vertex $\widetilde{x}$ such that $|\widetilde{x}| = n - 1$; and if $\{y_1, y_2, \ldots, y_m\}$ are the other neighbours of x then $|y_i| = n + 1$ for $i = 1, 2, \ldots, m$.

The reason for restricting our functions to an infinite tree without terminal vertices is that we are able to prove the crucial Lemma 1 only in this restricted case, not in the context of a general infinite network.

Lemma 1. *Suppose $u(x)$ is a q-harmonic function on $|x| \leq n$. Then there exists a q-harmonic function $v(x)$ on T such that $v(x) = u(x)$, for all $|x| \leq n$.*

P r o o f. Let x_0 be any vertex on the boundary of $|x| \leq n$, then $|x_0| = n$ and let $\{\widetilde{x}_0, y_1, \ldots, y_m\}$ be the neighbours of x_0 as given above.

Since $u(x)$ is q-harmonic on $|x| \leq n$, $\Delta_q u(x) = 0$, for all $|x| < n$. Choose a constant β, such that $\Delta u(x_0) = q(x_0)u(x_0)$; for this to happen we should have $\sum_{z \sim x_0} t(x_0, z)[u(z) - u(x_0)] = = q(x_0)u(x_0)$. That is, $\sum_{z \sim x_0} t(x_0, z)u(z) = [t(x_0) + q(x_0)]u(x_0)$. That is, $t(x_0, \widetilde{x}_0)u(\widetilde{x}_0) + \beta[t(x_0) - - t(x_0, \widetilde{x}_0)] = [t(x_0) + q(x_0)]u(x_0)$. Since $t(x_0) - t(x_0, \widetilde{x}_0) \neq 0$ (because x_0 is not a terminal vertex), the constant β is well-defined. By continuing this process at each vertex at the boundary of the set $\{x \colon |x| \leq n\}$, we will get a function $v(x)$ on $\{x \colon |x| \leq n + 1\}$, such that $u(x) = v(x)$ and $\Delta v(x) = q(x)v(x)$ at each vertex on $\{x \colon |x| \leq n\}$. Then $v(x)$ can be extended to a q-harmonic function on T, preserving the value $u(x)$ on $\{x \colon |x| \leq n\}$. □

Theorem 10 (q-Poisson equation on an infinite tree). *Suppose $f(x)$ is a real-valued function on an infinite tree T. Then there exists a function $u(x)$ such that $\Delta_q u(x) = f(x)$.*

P r o o f. By [10, Lemma 3.5.4] for any vertex a, there exists a q-superharmonic function $s_a(x)$ such that $\Delta_q s_a(x) = -\delta_a(x)$. Let $f_n(x)$ be the restriction of $f(x)$ to the set $n < |x| \leq n + 1$ and $f_n(x) = 0$ outside this set. Let

$$u_n(x) = \sum_{n < |a| \leq n+1} s_a(x) f_n^+(a)$$

which is a q-superharmonic function such that $-\Delta_q u_n(x) = f_n^+(x)$ on X. Now $u_n(x)$ is a q-harmonic function on $\{x \colon |x| \leq n\}$. Hence, by Lemma 1, there exists a q-harmonic function $h_n(x)$ on X such that $h_n(x) = u_n(x)$ on $\{x \colon |x| \leq n\}$. Write $v_n(x) = u_n(x) - h_n(x)$.

Let $v^+(x) = \sum_{n=1}^{\infty} v_n(x)$. This is a real-valued function since at any vertex, all but a finite number of terms in the infinite sum are zero and has the super mean-value property, hence, a q-superharmonic function. Also

$$-\Delta_q[v^+(x) + s_e(x)f^+(e)] = f^+(x)$$

for all x in X.

Similarly there exists a q-superharmonic function $[v^-(x) + s_e(x)f^-(e)]$ on T such that $-\Delta_q[v^-(x) + s_e(x)f^-(e)] = f^-(x)$ on T. Take now $u(x)$=$[v^-(x) - v^+(x) - s_e(x)f(e)]$ which is a difference of two q-superharmonic functions on T such that $\Delta_q u(x) = f(x)$. □

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Received 07.05.2021

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Citation: N. Nathiya, C. Amulya Smyrna. Infinite Schrödinger networks, *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki*, 2021, vol. 31, issue 4, pp. 640–650.

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Бесконечные сети Шрёдингера

Ключевые слова: q-гармонические функции, q-супергармонические функции, сеть Шрёдингера, гиперболическая сеть Шрёдингера, параболическая сеть Шрёдингера, интегральное представление.

УДК 517

DOI: 10.35634/vm210408

Конечно-разностные модели дифференциальных уравнений в частных производных, такие как уравнения Лапласа или Пуассона, приводят к конечной сети. Дискретизированное уравнение на неограниченном множестве на плоскости или в пространстве приводит к бесконечной сети. В бесконечной сети оператор Шрёдингера (возмущенный оператор Лапласа, q-оператор Лапласа) определяется для развития теории дискретного потенциала, которая имеет модель в уравнении Шрёдингера в евклидовых пространствах. Исследуется связь между Δ-теорией оператора Лапласа и Δ_q-теорией. В Δ_q-теории уравнение Пуассона решается, если сеть является деревом, и в общем случае получается каноническое представление для неотрицательных q-супергармонических функций.

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Поступила в редакцию 07.05.2021

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Цитирование: Н. Натия, Ч. Амуля Смырна. Бесконечные сети Шрёдингера // Вестник Удмуртского университета. Математика. Механика. Компьютерные науки. 2021. Т. 31. Вып. 4. С. 640–650.