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**NEW HADAMARD-TYPE INEQUALITIES VIA $(S, M_1, M_2)$-CONVEX FUNCTIONS**

The article introduces a new concept of convexity of a function: $(s, m_1, m_2)$-convex functions. This class of functions combines a number of convexity types found in the literature. Some properties of $(s, m_1, m_2)$-convexities are established and simple examples of functions belonging to this class are given. On the basis of the proved identity, new integral inequalities of the Hadamard type are obtained in terms of the fractional integral operator. It is shown that these results give us, in particular, generalizations of a number of results available in the literature.

*Keywords*: convex function, Hadamard type inequality, Riemann–Liouville fractional integral, Hölder inequality, power mean inequality.

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## Introduction

In current decades, many mathematician have worked on inequalities in the area of mathematics especially mathematical analysis, mathematical physics and functional analysis. No one can refuse from its importance and significance and with the upcoming times this field is going on to robustness and widespread. Inequalities have many uses in probability and statistical problems (see, for example, [7, 15, 17–20, 23]). Due to paramount background, convex analysis and inequalities have become an absorbing field for the attention of mathematicians and readers, see [2, 10, 11, 14] and references therein.

**Definition 1.** Function $\psi\colon\ [a,b] \to \mathbb{R}$, is called *convex*, if inequality

$$\psi\left(tx + (1-t)\,y\right) \leq t\psi\left(x\right) + (1-t)\,\psi\left(y\right)$$

is true $\forall x, y \in [a,b]$ and $t \in [0,1]$.

Famous Hermite–Hadamard integral inequality:

$$\psi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \psi(x)\,dx \leq \frac{\psi(a)+\psi(b)}{2}.$$

Here, the function of real variables $\psi$ is convex on the interval $[a,b]$. If $\psi$ is concave then both inequalities hold in the reversed direction.

The following definition along with some new properties was given by J. Park in [19].

**Definition 2** (see [19]). The function $\psi\colon I \subset [0,\infty) \to \mathbb{R}$ is called $(s,m)$-*convex in the second sense on* $I$ if for some fixed $s \in (0,1]$ and $m \in [0,1]$ the inequality

$$\psi(tx + m(1-t)y) \leq t^s\psi(x) + m(1-t)^s\psi(y)$$

holds $\forall x, y \in I$ and $t \in [0,1]$.

**Remark 1.** In the definition, if we choose $s = 1$, then we get an $m$-convex function introduced in [22], and for $m = 1$, we get an $s$-convex function of the second sense, introduced in [6].

H. Kadakal in [12, 13] gave the following definitions.

**Definition 3** (see [12]). For $m_1$, $m_2 \in (0,1]$, a function $\psi\colon [0,b] \to \mathbb{R}$ is said to be $(m_1, m_2)$-*convex*, if the inequality

$$\psi(m_1 tx + m_2(1-t)y) \leq m_1 t\psi(x) + m_2(1-t)\psi(y)$$

is true $\forall x, y \in [0,b]$, $t \in [0,1]$.

**Definition 4** (see [13]). For $\alpha$, $m_1$, $m_2 \in (0,1]$, a function $\psi\colon [0,b] \to \mathbb{R}$ is said to be $(\alpha, m_1, m_2)$-convex, if the inequality

$$\psi(m_1 tx + m_2(1-t)y) \leq m_1 t^{\alpha}\psi(x) + m_2(1-t^{\alpha})\psi(y)$$

is true $\forall x, y \in [0,b]$, $t \in [0,1]$.

In many studies on convex analysis, various operators of fractional integration play an important role (see, for example, [1–3, 21] and references therein).

Below we give the definition of the operator of the Riemann–Liouville fractional integral, which is known in the literature.

**Definition 5.** Let $\psi \in L[a,b]$. *The Riemann–Liouville integrals* $J_{a+}^{\alpha}\psi$ *and* $J_{b-}^{\alpha}\psi$ *of order* $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha}\psi\ (x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\psi(t)\,dt, \quad x > a,$$

and

$$J_{b-}^{\alpha}\psi\ (x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\psi(t)\,dt, \quad x < b,$$

respectively where $\Gamma(\alpha) = \int\limits_0^{\infty} e^{-u}u^{\alpha-1}du$. Here is $J_{a+}^{0}\psi(x) = J_{b-}^{0}\psi(x) = \psi(x)$. In the case of $\alpha = 1$ the fractional integral reduces to the classical integral.

Nápoles et al. in [16] give a comprehensive review of a wide range of different concepts of convexity which are in practice for work in recent years. Based on these concepts, various relationships between them are was shown.

The aim of the article is primarily to define a new convexity class and present its important features. Secondly, obtaining new generalized Hadamard-type midpoint inequalities by proving a new identity and using classical inequalities.

## § 1. New convexity and its properties

Firstly, we introduce the definition of a $(s, m_1, m_2)$-convex function.

**Definition 6.** For $m_1$, $m_2 \in (0,1]$ and some $s \in [-1,1]$, a function $\psi\colon [0,b] \to \mathbb{R}$ is said to be $(s, m_1, m_2)$-*convex in the second sense*, if the inequality

$$\psi(m_1 tx + m_2(1-t)y) \leq m_1 t^{s}\psi(x) + m_2(1-t)^{s}\psi(y)$$

is true $\forall x, y \in [0,b]$, $t \in [0,1]$.

We will denote by $K_{m_1,m_2}^{s}(b)$ the class of all $(s, m_1, m_2)$-convex functions on the interval $[0,b]$.

It is not difficult to prove the validity of the following propositions.

**Proposition 1.** *If* $\psi \in K^s_{m_1;m_2}(b)$*, then* $\psi(0) \leq 0$ *for any* $t \in [0,1]$.

P r o o f. We have

$$\psi(0) = \psi(m_1 \cdot 0 + m_2(1-t) \cdot 0) \leq m_1 t^s \psi(0) + m_2(1-t)^s \psi(0)$$

or

$$\psi(0) \cdot [1 - (m_1 t^s + m_2(1-t)^s))] \leq 0.$$

In view of the fact that the expression in the square brackets for any $t$ is non-negative, then $\psi(0) \leq 0$. □

**Proposition 2.** *Any* $(m_1, m_2)$*-convex function is* $(s, m_1, m_2)$*-convex in the second sense.*

P r o o f. Indeed, for $(m_1, m_2)$-convex functions we have

$$\psi(m_1 tx + m_2(1-t)y) \leq m_1 t\psi(x) + m_2(1-t)\psi(y).$$

Since $t \leq t^s$ and $1 - t \leq (1-t)^s$ for all $s \in [-1, 1]$, then we can write

$$m_1 t\psi(x) + m_2(1-t)\psi(y) \leq m_1 t^s \psi(x) + m_2(1-t)^s \psi(y)$$

and, then,

$$\psi(m_1 tx + m_2(1-t)y) \leq m_1 t^s \psi(x) + m_2(1-t)^s \psi(y).$$

The proof is completed. □

**Example 1.** The function $\psi(x) = x^2$ on any segment $[a, b]$ with $a \geq 0$ is a $(s, m_1, m_2)$-convex.

P r o o f. Firstly, show that $\psi(x) = x^2$ is a $(m_1, m_2)$-convex:

$$\begin{aligned}&[m_1 tx + m_2(1-t)y]^2 \leq m_1 tx^2 + m_2(1-t)y^2 \\ &\Longrightarrow m_1 t(1 - m_1 t)x^2 - 2m_1 m_2 t(1-t)xy + m_2(1-t)(1 - m_2(1-t))y^2 \geq 0.\end{aligned}$$

The resulting quadratic form is positive definite with respect to $(x, y)$. Calculate the discriminant of the form:

$$\begin{aligned}\Delta &= [m_1 m_2 t(1-t)]^2 - m_1 t(1 - m_1 t)m_2(1-t)\,[1 - m_2(1-t)] \\ &= [m_1 m_2 t(1-t)]^2 - m_1 m_2 t(1-t)(1 - m_1 t)\,[1 - m_2(1-t)] \\ &= m_1 m_2 t(1-t)\,[m_1 t + m_2(1-t) - 1]\,.\end{aligned}$$

Because $(m_1, m_2) \in (0, 1]^2$, then $m_1 t + m_2(1-t) \in [m_1, m_2]$, i. e., $m_1 t + m_2(1-t) - 1 \leq 0$ and $\Delta \leq 0$. Thus, from Proposition 2 it follows that the function is $(s, m_1, m_2)$-convex. The proof is complete. □

**Remark 2.** Since $(1-t) < (1-t)^s$, any $(m_1, m_2)$-convex function on the interval $[0, b]$ is $(s, m_1, m_2)$-convex. For example, the function $\psi(x) = x^3$ on any interval $[0, b]$ is $(s, m_1, m_2)$-convex.

**Remark 3.** It follows from Definition 6 that:

$(i1)$ if $m_1 = m_2 = 1$ and $s = 1$, then $\psi$ is a classical convex function on $[0, b]$;

$(i2)$ if $s = 1$, then $\psi$ is an $(m_1, m_2)$-convex function on $[0, b]$;

$(i3)$ if $m_1 = s = 1$, $m_2 = m$, then $\psi$ is an $m$-convex function on $[0, b]$;

$(i4)$ if $m_1 = m_2 = 1$, then $\psi$ is an $s$-convex function in the second sense on $[0, b]$;

$(i5)$ if $m_1 = 1$, $m_2 = m$, then $\psi$ is an $(s, m)$-convex function on $[0, b]$.

Now, we will study some of its algebraic properties.

**Theorem 1.** *Let $0 \le x < y < \infty$ and $\psi,\ g\colon [x, y] \to \mathbb{R}$. If $\psi$ and $g$ are two $(s, m_1, m_2)$-convex functions, then:*

$(i)$ *$\psi + g$ is an $(s, m_1, m_2)$-convex function;*

$(ii)$ *for a nonnegative real number $c$, $c\psi$ is an $(s, m_1, m_2)$-convex function.*

P r o o f.
$(i)$ Let $\psi$ and $g$ be an $(s, m_1, m_2)$-convex function, then

$$\begin{aligned}
&(\psi + g)\left(m_1tx + m_2(1-t)y\right)\\
&= \psi(m_1tx + m_2(1-t)y) + g(m_1tx + m_2(1-t)y)\\
&\le m_1t^s\psi(x) + m_2(1-t)^s\psi(y) + m_1t^sg(x) + m_2(1-t)^sg(y)\\
&= m_1t^s\left[\psi(x) + g(x)\right] + m_2(1-t)^s\left[\psi(y) + g(y)\right]\\
&= m_1t^s(\psi + g)(x) + m_2(1-t)^s(\psi + g)(y).
\end{aligned}$$

$(ii)$ Let $\psi$ be an $(s, m_1, m_2)$-convex function, then

$$\begin{aligned}
c\psi\Big(m_1tx + m_2(1-t)y\Big) &= c\Big(\psi(m_1tx + m_2(1-t)y)\Big)\\
&\le c\Big(m_1t^s\psi(x) + m_2(1-t)^s\psi(y)\Big)\\
&= m_1t^sc(\psi(x)) + m_2(1-t)^sc(\psi(y))\\
&= m_1t^s(c\psi)(x) + m_2(1-t)^s(c\psi)(y),
\end{aligned}$$

which completes the proof. □

**Theorem 2.** *Let $\psi\colon I \to J$ be an $(m_1, m_2)$-convex function and $g\colon J \to \mathbb{R}$ is a $(s, m_1, m_2)$-convex and non-decreasing function. Then the function $g \circ \psi\colon I \to \mathbb{R}$ is an $(s, m_1, m_2)$-convex function.*

P r o o f. For all $x, y \in I$, and $t \in [0, 1]$, we have

$$\begin{aligned}
(g \circ \psi)\left(m_1tx + m_2(1-t)y\right) &= g\left(\psi(m_1tx + m_2(1-t)y)\right)\\
&\le g\left(m_1t\psi(x) + m_2(1-t)\psi(y)\right)\\
&\le m_1t^sg(\psi(x)) + m_2(1-t)^sg(\psi(y))\\
&= m_1t^s(g \circ \psi)(x) + m_2(1-t)^s(g \circ \psi)(y),
\end{aligned}$$

which completes the proof. □

## § 2. Main results

At the beginning, for a function from the class $K^s_{m_1,m_2}(I)$, we obtain an analogue of the inequality for various classes of convexities, which is available in the literature.

**Theorem 3.** *Let $\psi\colon I=[a,b]\to\mathbb{R}$ and $\psi\in K^s_{m_1,m_2}(I)$. If $a\geq 0$, $s\in(-1,1]$, and $\psi\in L[a,b]$, then the inequality is valid:*

$$\frac{1}{b-a}\int_a^b \psi\left(x\right)dx \leq \min\left\{\frac{m_1\psi\left(\frac{a}{m_1}\right)+m_2\psi\left(\frac{b}{m_2}\right)}{s+1}, \frac{m_2\psi\left(\frac{a}{m_2}\right)+m_1\psi\left(\frac{b}{m_1}\right)}{s+1}\right\}. \tag{2.1}$$

P r o o f. It's obvious that

$$\frac{1}{b-a}\int_a^b \psi\left(x\right)dx = \int_0^1 \psi\left(ta+(1-t)b\right)dt = \int_0^1 \psi\left(m_1 t\frac{a}{m_1}+m_2(1-t)\frac{b}{m_2}\right)dt.$$

Since $\psi\in K^s_{m_1,m_2}$, we get

$$\frac{1}{b-a}\int_a^b \psi(x)\,dx \leq \int_0^1\left[m_1\psi\left(\frac{a}{m_1}\right)t^s+m_2\psi\left(\frac{b}{m_2}\right)(1-t)^s\right]dt = \frac{m_1\psi\left(\frac{a}{m_1}\right)+m_2\psi\left(\frac{b}{m_2}\right)}{s+1}.$$

Similarly, we get

$$\frac{1}{b-a}\int_a^b \psi\left(x\right)dx = \int_0^1 \psi\left((1-t)a+tb\right)dt = \int_0^1 \psi\left(m_2\left(1-t\right)\frac{a}{m_2}+m_1 t\frac{b}{m_1}\right)dt$$
$$\leq \frac{m_2\psi\left(\frac{a}{m_2}\right)+m_1\psi\left(\frac{b}{m_1}\right)}{s+1}.$$

From the last inequalities, we get (2.1). Proof is completed. □

**Remark 4.** From Theorem 3,

$(i1)$ if $m_1=m_2=1$ and $s=1$, then we get the right-hand side Hermite–Hadamard inequality for an ordinary convex function

$$\frac{1}{b-a}\int_a^b \psi\left(x\right)dx \leq \frac{\psi\left(a\right)+\psi\left(b\right)}{2}.$$

This inequality is the right-hand side Hadamard inequality for an ordinary convex function.

$(i2)$ if $s=1$, then we get

$$\frac{1}{b-a}\int_a^b \psi\left(x\right)dx \leq \min\left\{\frac{m_1\psi\left(\frac{a}{m_1}\right)+m_2\psi\left(\frac{b}{m_2}\right)}{2}, \frac{m_2\psi\left(\frac{a}{m_2}\right)+m_1\psi\left(\frac{b}{m_1}\right)}{2}\right\}.$$

This inequality was obtained in [12] for the $(m_1,m_2)$-convex function.

$(i3)$ if $m_1=s=1$, $m_2=m$ then, we get

$$\frac{1}{b-a}\int_a^b \psi\left(x\right)dx \leq \min\left\{\frac{m\psi\left(\frac{a}{m}\right)+\psi\left(b\right)}{2}, \frac{\psi\left(a\right)+m\psi\left(\frac{b}{m}\right)}{2}\right\}.$$

This inequality was obtained in [8] for the $m$-convex function.

$(i4)$ if $m_1 = m_2 = 1$, then we get

$$\frac{1}{b-a}\int_a^b \psi(x)\,dx \le \frac{\psi(a)+\psi(b)}{s+1}.$$

This inequality was obtained in [9] for the $s$-convex functions in the second sense.

$(i5)$ if $m_1 = 1$, $m_2 = m$, we get

$$\frac{1}{b-a}\int_a^b \psi(x)\,dx \le \min\left\{\frac{m\psi\left(\frac{a}{m}\right)+\psi(b)}{s+1}, \frac{\psi(a)+m\psi\left(\frac{b}{m}\right)}{s+1}\right\}.$$

This inequality was obtained in [5] for the $(s, m)$-convex functions in the second sense.

Now, using the introduced new convexity class and identity, and the classical Hölder inequality and power mean, we obtain some inequalities.

**Lemma 1.** *Let $0 \le m_1 a < m_2 b < \infty$ and $\psi\colon I = [m_1 a, m_2 b] \to \mathbb{R}$ be a twice differentiable function on $(m_1 a, m_2 b)$. If $\psi'' \in L[m_1 a, m_2 b]$, then $\forall \alpha > 1$ we have*

$$\frac{2^{\alpha-2}\Gamma(\alpha)}{(m_2 b - m_1 a)^{\alpha-1}}\left[J^{\alpha-1}_{\left(\frac{m_1 a+m_2 b}{2}\right)^+}\psi(m_2 b) + J^{\alpha-1}_{\left(\frac{m_1 a+m_2 b}{2}\right)^-}\psi(m_1 a)\right] - \psi\left(\frac{m_1 a+m_2 b}{2}\right) = \frac{(m_2 b - m_1 a)^2}{\alpha 2^{2-\alpha}}\left(I_1 + I_2\right), \tag{2.2}$$

*where*

$$I_1 = \int_0^{1/2} t^\alpha \psi''(m_1 a t + m_2(1-t)b)\,dt \text{ and } I_2 = \int_{1/2}^1 (1-t)^\alpha \psi''(m_1 a t + m_2(1-t)b)\,dt.$$

P r o o f. We integrate the integrals $I_1$ and $I_2$ twice by parts; as a result, we obtain

$$I_1 = -\frac{1}{(m_2 b - m_1 a)2^\alpha}\psi'\left(\frac{a m_1 + m_2 b}{2}\right) - \frac{\alpha}{(m_2 b - m_1 a^2)2^{\alpha-1}}\psi\left(\frac{a m_1 + m_2 b}{2}\right) + \frac{\alpha(\alpha-1)}{(m_2 b - m_1 a)^2}\int_0^{1/2} t^{\alpha-2}\psi(m_1 a t + m_2(1-t)b)\,dt \tag{2.3}$$

and

$$I_2 = \frac{1}{(m_2 b - m_1 a)2^\alpha}\psi'\left(\frac{a m_1 + m_2 b}{2}\right) - \frac{\alpha}{(m_2 b - m_1 a)^2 2^{\alpha-1}}\psi\left(\frac{a m_1 + m_2 b}{2}\right) + \frac{\alpha(\alpha-1)}{(m_2 b - m_1 a)^2}\int_{1/2}^1 (1-t)^{\alpha-2}\psi(m_1 a t + m_2(1-t)b)\,dt. \tag{2.4}$$

If we make the change of variables in the integrals in expressions (2.3) and (2.4): $a m_1 t + m_2(1-t)b = z$, as a result, we obtain

$$I_1 + I_2 = -\frac{2\alpha}{(m_2 b - m_1 a)^2 2^{\alpha-1}}\psi\left(\frac{a m_1 + m_2 b}{2}\right) + \frac{\alpha(\alpha-1)(\Gamma(\alpha-1))}{(m_2 b - m_1 a)^{\alpha+1}} \times \left[J^{\alpha-1}_{\left(\frac{m_1 a+m_2 b}{2}\right)^+}\psi(m_2 b) + J^{\alpha-1}_{\left(\frac{m_1 a+m_2 b}{2}\right)^-}\psi(a m_1)\right]. \tag{2.5}$$

If we multiply both sides of (2.5) by the value $\frac{(m_2 b - m_1 a)^2}{\alpha 2^\alpha - 1}$, we complete the proof. □

**Remark 5.** If we take $m_1 = 1$, $m_2 = m$, then from (2.2) we obtain the equality known in [4, see Lemma 2.1].

**Remark 6.** In proving the theorems, we need the following inequality:

$$\begin{aligned}&\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(m_2b-m_1a)^{\alpha-1}}\left[J^{\alpha-1}_{\left(\frac{m_1a+m_2b}{2}\right)^+}\psi(m_2b)+J^{\alpha-1}_{\left(\frac{m_1a+m_2b}{2}\right)^-}\psi(am_1)\right]\right.\\&\left.-\psi\left(\frac{m_1a+m_2b}{2}\right)\right|\le\frac{(bm_2-am_1)^2}{\alpha 2^{2-\alpha}}(|I_1|+|I_2|).\end{aligned}\tag{2.6}$$

This inequality was obtained by using the properties of the module from (2.2).

**Theorem 4.** *Let $a$, $b$ and $b^*\in\mathbb{R}$, with $0\le m_1a<m_2b<b^*$, and $\psi\colon I=[0,b^*]\to\mathbb{R}$ be twice differentiable function on $I^\circ$. If $\psi''\in L[m_1a,m_2b]$ and $\psi''\in K^s_{m_1,m_2}[m_1a,m_2b]$, then $\forall\alpha>1$ the inequality*

$$\begin{aligned}&\frac{2^{\alpha-2}\Gamma(\alpha)}{(m_2b-m_1a)^{\alpha-1}}\left|\left[J^{\alpha-1}_{\left(\frac{m_1a+m_2b}{2}\right)^+}\psi(m_2b)+J^{\alpha-1}_{\left(\frac{m_1a+m_2b}{2}\right)^-}\psi(am_1)\right]-\psi\left(\frac{m_1a+m_2b}{2}\right)\right|\\&\le\frac{(bm_2-am_1)^2}{\alpha 2^{2-\alpha}}\left[\frac{1}{(s+\alpha+1)\,2^{\alpha+s+1}}+B_{\frac{1}{2}}(1+\alpha,1+s)\right][m_1|\psi''(a)|+m_2|\psi''(b)|]\end{aligned}\tag{2.7}$$

*is valid, where $B_x(u,v)=\int_0^x t^{u-1}(1-t)^{v-1}dt$ is incomplete beta function.*

P r o o f. Since the function $|\psi''|$ is a $(s,m_1,m_2)$-convex, then for $|I_1|$ in inequality (2.6), we can write

$$\begin{aligned}|I_1|&\le m_1|\psi''(a)|\int_0^{\frac{1}{2}}t^{s+\alpha}dt+m_2|\psi''(b)|\int_0^{\frac{1}{2}}t^\alpha(1-t)^s dt\\&=\frac{1}{(s+\alpha+1)\,2^{s+\alpha+1}}m_1|\psi''(a)|+B_{\frac{1}{2}}(\alpha+1,s+1)\,m_2|\psi''(b)|\end{aligned}\tag{2.8}$$

and likewise

$$|I_2|\le B_{\frac{1}{2}}(\alpha+1,s+1)\,m_1|\psi''(a)|+\frac{1}{(s+\alpha+1)\,2^{s+\alpha+1}}m_2|\psi''(b)|.\tag{2.9}$$

By adding (2.8) and (2.9), we get

$$|I_1|+|I_2|\le\left[\frac{1}{(\alpha+s+1)\,2^{s+\alpha+1}}+B_{\frac{1}{2}}(1+\alpha,1+s)\right][m_1|\psi''(a)|+m_2|\psi''(b)|].\tag{2.10}$$

By multiplying both sides (2.10) by $\frac{(bm_2-am_1)^2}{\alpha 2^{2-\alpha}}$, taking into account (2.6), we get inequality (2.7). The proof is completed. □

**Corollary 1.** *If $s=m_1=m_2=1$ and $\alpha=2$, then from (2.7) for an ordinary convex function we obtain*

$$\left|\frac{1}{b-a}\int_a^b\psi(x)dx-\psi\left(\frac{a+b}{2}\right)\right|\le\frac{(b-a)^2}{48}[|\psi''(a)|+|\psi''(b)|].$$

This inequality was obtained in [20] (see Proposition 1) and in [4](see Corollary 2.1).

**Corollary 2.** *If $s=m_1=m_2=1$, then from (2.7) for an ordinary convex function we obtain*

$$\begin{aligned}&\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(b-a)^{\alpha-1}}\left[J^{\alpha-1}_{\left(\frac{a+b}{2}\right)^+}\psi(b)+J^{\alpha-1}_{\left(\frac{a+b}{2}\right)^-}\psi(a)\right]-\psi\left(\frac{a+b}{2}\right)\right|\\&\le\frac{(b-a)^2}{8\alpha(\alpha+1)}[|\psi''(a)|+|\psi''(b)|].\end{aligned}$$

This inequality was obtained in [3] (see Theorem 4, for $n = 1$).

**Corollary 3.** *If $s = 1$, then from (2.7) for an $(m_1, m_2)$-convex function we obtain*

$$\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(m_2b-m_1a)^{\alpha-1}}\left[J^{\alpha-1}_{\left(\frac{m_1a+m_2b}{2}\right)^+}\psi(m_2b)+J^{\alpha-1}_{\left(\frac{m_1a+m_2b}{2}\right)^-}\psi(am_1)\right]\right.$$
$$\left.-\psi\left(\frac{m_1a+m_2b}{2}\right)\right|\leq\frac{(bm_2-am_1)^2}{8\alpha(\alpha+1)}\left[m_1|\psi''(a)|+m_2|\psi''(b)|\right].$$

**Theorem 5.** *Let $a$, $b$ and $b^* \in \mathbb{R}$, with $0 \leq m_1a < m_2b < b^*$ and $\psi\colon I = [0, b^*] \to \mathbb{R}$ be twice differentiable function on $I^\circ$. If $\psi'' \in L[m_1a, m_2b]$ and $|\psi''|^q \in K^s_{m_1,m_2}[m_1a, m_2b]$, then $\forall\alpha > 1$ and $q \geq 1$ the inequality*

$$\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(m_2b-m_1a)^{\alpha-1}}\left[J^{\alpha-1}_{\left(\frac{m_1a+m_2b}{2}\right)^+}\psi(m_2b)+J^{\alpha-1}_{\left(\frac{m_1a+m_2b}{2}\right)^-}\psi(am_1)\right]\right.\tag{2.11}$$
$$\left.-\psi\left(\frac{m_1a+m_2b}{2}\right)\right|\leq\frac{(bm_2-am_1)^2}{\alpha 2^{2-\alpha}}\cdot\left(\frac{1}{(1+\alpha)\,2^{1+\alpha}}\right)^{1-\frac{1}{q}}\cdot E$$

*is valid, where*

$$E=\left[\frac{m_1|\psi''(a)|^q}{(s+\alpha+1)\,2^{s+\alpha+1}}+B_{\frac{1}{2}}\left(1+\alpha,1+s\right)m_2\,|\psi''(b)|^q\right]^{\frac{1}{q}}$$
$$+\left[B_{\frac{1}{2}}\left(1+\alpha,1+s\right)m_1|\psi''(a)|^q+\frac{m_2|\psi''(b)|^q}{(s+\alpha+1)\,2^{s+\alpha+1}}\right]^{\frac{1}{q}}.$$

P r o o f. From inequality (2.6) for $|I_1|$, taking into account that the function $|\psi''|^q$ is a $(s, m_1, m_2)$-convex, using the known power mean inequality, we can write

$$|I_1|\leq\int_0^{\frac{1}{2}}t^\alpha|\psi''(m_1at+m_2(1-t)b|\,dt$$
$$\leq\left(\int_0^{\frac{1}{2}}t^\alpha dt\right)^{1-\frac{1}{q}}\left[m_1|\psi''(a)|^q\int_0^{\frac{1}{2}}t^{\alpha+s}dt+m_2|\psi''(b)|^q\int_0^{\frac{1}{2}}t^\alpha\left(1-t\right)^s dt\right]^{\frac{1}{q}}\tag{2.12}$$
$$\leq\left(\frac{2^{-\alpha-1}}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\frac{m_1|\psi''(a)|^q}{(s+\alpha+1)\,2^{s+\alpha+1}}+B_{\frac{1}{2}}\left(1+\alpha,1+s\right)m_2\,|\psi''(b)|^q\right]^{\frac{1}{q}}.$$

Similarly, for the $|I_2|$, we get

$$|I_2|\leq\left(\frac{1}{(1+\alpha)\,2^{1+\alpha}}\right)^{1-\frac{1}{q}}\left[B_{\frac{1}{2}}\left(1+\alpha,1+s\right)m_1|\psi''(a)|^q+\frac{m_2|\psi''(b)|^q}{(\alpha+s+1)\,2^{\alpha+s+1}}\right]^{\frac{1}{q}}.\tag{2.13}$$

By adding inequalities (2.12) and (2.13), we get

$$|I_1|+|I_2|\leq\left(\frac{1}{(\alpha+1)\,2^{\alpha+1}}\right)^{1-\frac{1}{q}}\cdot E.$$

By multiplying both sides of the last inequality by $\frac{(bm_2-am_1)^2}{\alpha 2^{2-\alpha}}$ and taking into account (2.6), we get inequality (2.11). The proof is completed. □

**Remark 7.** If we choose $s = m_1 = m_2 = 1$ and $\alpha = 2$, then from Theorem 5 we will have the inequality for convex functions that was obtained in [20] and [4] (see Proposition 5, Corollary 2.2, respectively).

**Corollary 4.** *If $s = m_1 = m_2 = 1$, then from (2.11) for an ordinary convex function we obtain*

$$\begin{aligned}&\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(b-a)^{\alpha-1}}\left[J^{\alpha-1}_{\left(\frac{a+b}{2}\right)^+}\psi(b)+J^{\alpha-1}_{\left(\frac{a+b}{2}\right)^-}\psi(a)\right]-\psi\left(\frac{a+b}{2}\right)\right|\\&\le\frac{(b-a)^2}{8\alpha(\alpha+1)}\cdot\left(\frac{\alpha+1}{2\alpha+4}\right)^{\frac{1}{q}}\cdot E,\end{aligned}$$

*where*

$$E=\left[|\psi''(a)|^q+\frac{\alpha+3}{\alpha+1}|\psi''(b)|^q\right]^{\frac{1}{q}}+\left[\frac{\alpha+3}{\alpha+1}|\psi''(a)|^q+|\psi''(b)|^q\right]^{\frac{1}{q}}.$$

This inequality was obtained in [3] (see Theorem 5 for $n = 1$).

**Corollary 5.** *If $s = 1$, then from (2.11) for an $(m_1, m_2)$-convex functions we obtain*

$$\begin{aligned}&\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(m_2b-m_1a)^{\alpha-1}}\left[J^{\alpha-1}_{\left(\frac{m_1a+m_2b}{2}\right)^+}\psi(m_2b)+J^{\alpha-1}_{\left(\frac{m_1a+m_2b}{2}\right)^-}\psi(am_1)\right]\right.\\&\left.-\psi\left(\frac{m_1a+m_2b}{2}\right)\right|\le\frac{(bm_2-am_1)^2}{8\alpha(\alpha+1)}\cdot\left(\frac{\alpha+1}{2\alpha+4}\right)^{\frac{1}{q}}\cdot E,\end{aligned}$$

*where*

$$E=\left[m_1|\psi''(a)|^q+\frac{\alpha+3}{\alpha+1}m_2|\psi''(b)|^q\right]^{\frac{1}{q}}+\left[\frac{\alpha+3}{\alpha+1}m_1|\psi''(a)|^q+m_2|\psi''(b)|^q\right]^{\frac{1}{q}}.$$

**Theorem 6.** *Let $a$, $b$ and $b^* \in \mathbb{R}$, with $0 \le m_1a < m_2b < b^*$, and $\psi\colon I = [0, b^*] \to \mathbb{R}$ be twice differentiable function on $I^\circ$. If $\psi'' \in L[m_1a, m_2b]$ and $|\psi''|^q \in K^s_{m_1,m_2}[m_1a, m_2b]$, then $\forall\alpha > 1$, $q > 1$ and $p > 1$, such that $\frac{1}{q}+\frac{1}{p} = 1$, the inequality*

$$\begin{aligned}&\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(m_2b-m_1a)^{\alpha-1}}\left[J^{\alpha-1}_{\left(\frac{m_1a+m_2b}{2}\right)^+}\psi(m_2b)+J^{\alpha-1}_{\left(\frac{m_1a+m_2b}{2}\right)^-}\psi(am_1)\right]\right.\\&\left.-\psi\left(\frac{m_1a+m_2b}{2}\right)\right|\le\frac{(bm_2-am_1)^2}{\alpha 2^{\frac{5q-\alpha q-3}{q}}}\cdot W\end{aligned}\tag{2.14}$$

*is valid, where*

$$\begin{aligned}W&=\left[\mu\cdot m_1|\psi''(a)|^q+B_{\frac{1}{2}}(2+\alpha q-q,s+1)m_2|\psi''(b)|^q\right]^{\frac{1}{q}}\\&\quad+\left[B_{\frac{1}{2}}(2+\alpha q-q,s+1)m_1|\psi''(a)|^q+\mu\cdot m_2|\psi''(b)|^q\right]^{\frac{1}{q}},\\\mu&=\frac{1}{(\alpha q-q+s+2)\,2^{\alpha q-q+s+2}}.\end{aligned}$$

P r o o f. From inequality (2.6) for $|I_1|$, taking into account that the function $|\psi''|^q$ is a $(s,m_1,m_2)$-convex, using the known Hölder inequality, we can write

$$|I_1| = \left|\int_0^{\frac{1}{2}} t^{\frac{1}{p}} t^{\frac{1}{q}} t^{\alpha-1}\psi''(m_1 at + m_2(1-t)b\,dt\right| \le \left(\int_0^{\frac{1}{2}} t^{\frac{1}{p}^p}\,dt\right)^{\frac{1}{p}}$$
$$\times\left[m_1|\psi''(a)|^q\int_0^{\frac{1}{2}} t^{\alpha q-q+s+1}dt + m_2|\psi''(b)|^q\int_0^{\frac{1}{2}} t^{\alpha q-q+1}(1-t)^s\,dt\right]^{\frac{1}{q}}$$

or

$$|I_1| \le 2^{\frac{-3}{p}}\left[\frac{2^{-(\alpha q-q+s+2)}m_1|\psi''(a)|^q}{\alpha q-q+s+2} + B_{\frac{1}{2}}((\alpha-1)q+2,1+s)m_2|\psi''(b)|^q\right]^{\frac{1}{q}}. \tag{2.15}$$

Similarly, from $|I_2|$, we get the inequalities

$$|I_2| \le 2^{\frac{-3}{p}}\left[B_{\frac{1}{2}}((\alpha-1)+2,s+1)m_1|\psi''(a)|^q + \frac{2^{-((\alpha-1)q+s+2)}m_2|\psi''(b)|^q}{(\alpha-1)q+s+2}\right]^{\frac{1}{q}}. \tag{2.16}$$

By adding inequalities (2.15) and (2.16), we get

$$|I_1| + |I_2| \le \frac{W}{2^{\frac{3}{p}}}.$$

By multiplying last inequality by $\frac{(bm_2-am_1)^2}{\alpha 2^{2-\alpha}}$ and taking into account (2.6), we get (2.14). The proof is completed. □

**Corollary 6.** *If $s=m_1=m_2=1$, then from (2.14) for the ordinary convex function we obtain*

$$\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(b-a)^{\alpha-1}}\left[J^{\alpha-1}_{\left(\frac{a+b}{2}\right)^+}\psi(b) + J^{\alpha-1}_{\left(\frac{a+b}{2}\right)^-}\psi(a)\right] - \psi\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^2}{\alpha 2^{\frac{5q-\alpha q-3}{q}}}\cdot\mu^{\frac{1}{q}}\cdot W,$$

*where*

$$W = \left[|\psi''(a)|^q + \left(1+\frac{2}{2+\alpha q-q}\right)|\psi''(b)|^q\right]^{\frac{1}{q}} + \left[\left(1+\frac{2}{2+\alpha q-q}\right)|\psi''(a)|^q + |\psi''(b)|^q\right]^{\frac{1}{q}},$$
$$\mu = \frac{1}{(\alpha q-q+3)\,2^{\alpha q-q+3}}.$$

This inequality was obtained in [3] (see Theorem 6, for $n=1$).

**Corollary 7.** *If $s=1$, then from (2.14) for $(m_1,m_2)$-convex function we obtain*

$$\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(m_2b-m_1a)^{\alpha-1}}\left[J^{\alpha-1}_{\left(\frac{m_1a+m_2b}{2}\right)^+}\psi(m_2b) + J^{\alpha-1}_{\left(\frac{m_1a+m_2b}{2}\right)^-}\psi(am_1)\right]\right.$$
$$\left. - \psi\left(\frac{m_1a+m_2b}{2}\right)\right| \le \frac{(bm_2-am_1)^2}{\alpha 2^{\frac{5q-\alpha q-3}{q}}}\cdot W,$$

*where*

$$W = \left[\mu\cdot m_1|\psi''(a)|^q + B_{\frac{1}{2}}(2+\alpha q-q,2)m_2|\psi''(b)|^q\right]^{\frac{1}{q}}$$
$$+\left[B_{\frac{1}{2}}(2+\alpha q-q,2)m_1|\psi''(a)|^q + \mu\cdot m_2|\psi''(b)|^q\right]^{\frac{1}{q}},$$
$$\mu = \frac{1}{(\alpha q-q+3)\,2^{\alpha q-q+3}}.$$

**Remark 8.** If we choose $m_1 = 1$, $m_2 = m$, then from Theorem 6 for an $(s, m)$-convex function we obtain the inequality proved in [4] (see Theorem 2.3, for $s = 1$).

**Corollary 8.** *If $m_1 = m_2 = 1$, then from (2.14) for an $s$-convex function we obtain*

$$\left| \frac{2^{\alpha-2}\Gamma(\alpha)}{(b-a)^{\alpha-1}} \left[ J^{\alpha-1}_{\left(\frac{a+b}{2}\right)^+}\psi(b) + J^{\alpha-1}_{\left(\frac{a+b}{2}\right)^-}\psi(a) \right] - \psi\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{\alpha 2^{\frac{5q-\alpha q-3}{q}}} \cdot W,$$

*where*

$$W = \left[ \mu \left|\psi''(a)\right|^q + B_{\frac{1}{2}}(2+\alpha q - q, s+1)|\psi''(b)|^q \right]^{\frac{1}{q}} + \left[ B_{\frac{1}{2}}(2+\alpha q - q, s+1)|\psi''(a)|^q + \mu|\psi''(b)|^q \right]^{\frac{1}{q}},$$

$$\mu = \frac{1}{(\alpha q - q + 3)2^{\alpha q - q + 3}}.$$

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***Б. Р. Байрактаров, С. И. Батт, Ш. Шаокат, Х. Э. Наполес Вальдес***

**Новые неравенства типа Адамара для $(s, m_1, m_2)$-выпуклых функций**

*Ключевые слова:* выпуклая функция, неравенство типа Адамара, дробный интеграл Римана–Лиувилля, неравенство Гёльдера, неравенство о средних.

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В статье вводится новое понятие выпуклости функции: $(s, m_1, m_2)$-выпуклые функции. Этот класс функций объединяет несколько типов выпуклости, встречающихся в литературе. Установлены некоторые свойства $(s, m_1, m_2)$-выпуклости и приведены простые примеры функций, принадлежащих этому классу. На основе доказанного тождества получены новые интегральные неравенства типа Адамара в терминах оператора дробного интегрирования. Показано, что эти результаты дают, в частности, обобщение ряда имеющихся в литературе результатов.

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