MATHEMATICS 2021. Vol. 31. Issue 4. Pp. 578–596.

MSC2020: 03E99, 91F99

© *S. Acharjee, D. A. Molodtsov*

SOFT RATIONAL LINE INTEGRAL

Soft set theory is a new area of mathematics that deals with uncertainties. Applications of soft set theory are widely spread in various areas of science and social science viz. decision making, computer science, pattern recognition, artificial intelligence, etc. The importance of soft set-theoretical versions of mathematical analysis has been felt in several areas of computer science. This paper suggests some concepts of a soft gradient of a function and a soft integral, an analogue of a line integral in classical analysis. The fundamental properties of soft gradients are established. A necessary and sufficient condition is found so that a set can be a subset of the soft gradient of some function. The inclusion of a soft gradient in a soft integral is proved. Semi-additivity and positive uniformity of a soft integral are established. Estimates are obtained for a soft integral and the size of its segment. Semi-additivity with respect to the upper limit of integration is proved. Moreover, this paper enriches the theoretical development of a soft rational line integral and associated areas for better functionality in terms of computing systems.

Keywords: soft rational analysis, soft gradient, soft integral, soft set.

DOI: 10.35634/vm210404

Introduction

In mathematics, the concept of real numbers is one of the most important concepts. The whole mathematical analysis is built on the basis of this concept: the concepts of limit, continuity, derivatives, integrals, ordinary differential equations, and partial differential equations. The set of real numbers is also the basis for complex analysis, probability theory, and many other sciences. The achievements of science based on real numbers are hard to overestimate. Almost all physical models of the world are built on real numbers. However, it can be noted that all practical calculations are not related to real numbers, but to rational numbers. Irrational numbers are basically inaccessible to practical use since they are described by an infinite non-periodic sequence of numbers. For example, the simplest elementary functions, such as the square root or the trigonometric function sine, cannot be used in calculations. Instead, their rational approximations are taken. Real numbers are closely related to the concept of limit. In practice, calculation of limits is often impossible. This is due to the fact that we cannot proceed to measure any small time interval, and also cannot measure the coordinates of a body with any accuracy. If the measurement accuracy is too high, the problem of identifying the body itself arises. The boundaries of the body become too blurry. In addition, the modern physical picture of the world is based on the quantum hypothesis, which does not correspond to the classical analysis. It turns out that classical analysis does not correspond to practical calculations and requires a theory that operates on similar concepts (continuity, derivative, integral), but uses only rational numbers.

The main difficulty in constructing an analysis on the set of rational numbers is based on the fact that the set of rational numbers is not a complete space with respect to the natural metric i.e. with respect to the module of difference. The boundaries of bounded sets may not exist; the limits of fundamental sequences may also not exist. All these circumstances significantly complicate the construction of classical analysis on incomplete space.

Some requirements for new analysis on rational numbers can be formulated. The requirements are as follows:

1) a new analysis should only deal with rational numbers;

2) a new analysis should abandon the use of limits;

3) a new analysis should be constructed based on soft set theory that has interpretations similar to interpretations of continuity, derivative, and integral.

These requirements, of course, are not quite formal; rather, they express some trends and wishes. They will acquire a very strict form below with the introduction of the corresponding formal concepts. Similar attempts to build a new analysis were made as part of a soft analysis [1–11], but using real numbers. In [8], the beginning of such an analysis was built only on rational numbers. This work continues the development of this direction.

§ 1. One-dimensional tools of rational analysis

The main tool for working in rational analysis is the set of rational numbers. The set of rational numbers is denoted by $\mathbf{Q}$ and the set of real numbers by $\mathbf{R}$. Therefore, we introduce the necessary operations and relations for the rational subsets.

Definition 1. Let A, B and C be subsets of $\mathbf{Q}$, and $a, b, c \in \mathbf{Q}$ be rational numbers. Then, arithmetic operations with subsets are defined as given below:

(i) $A + B = \{x \in \mathbf{Q} \mid \exists a \in A, b \in B$ such that $x = a + b\}$;

(ii) $A - B = \{x \in \mathbf{Q} \mid \exists a \in A, b \in B$ such that $x = a - b\}$;

(iii) $A \bullet B = \{x \in \mathbf{Q} \mid \exists a \in A, b \in B$ such that $x = ab\}$;

(iv) $\frac{A}{B} = \{x \in \mathbf{Q} \mid \exists a \in A, b \in B \setminus \{0\}$ such that $x = \frac{a}{b}\}$.

Definition 2. Let A be a subset of $\mathbf{Q}$. Then, the *non-negative* and *non-positive parts* of A are defined as given below:

(i) $A^{\oplus} = \{x \in A \mid x \geq 0\}$;

(ii) $A^{\ominus} = \{x \in A \mid x \leq 0\}$.

For rational numbers, in addition to the standard relations $\leq, \geq, <, >$, we shall use perturbed relations, which are defined as in the following Definition 3.

Definition 3. Let $a, b, c \in \mathbf{Q}$ be rational numbers and $\delta \in \mathbf{Q}$. Then,

(i) $a >^{\delta} b \Leftrightarrow a > b - \delta$;

(ii) $a \geq^{\delta} b \Leftrightarrow a \geq b - \delta$;

(iii) $a <^{\delta} b \Leftrightarrow a < b + \delta$;

(iv) $a \leq^{\delta} b \Leftrightarrow a \leq b + \delta$.

Now, let us consider $\mathcal{R}$ as a relation on the set $\mathbf{Q}$, that is $\mathcal{R} \subseteq \mathbf{Q} \times \mathbf{Q}$, and Q as a relation for the power set $\mathbf{Q}$, that is, $Q \subseteq 2^{\mathbf{Q}} \times 2^{\mathbf{Q}}$. We derive some relations for subsets of the set $\mathbf{Q}$.

Definition 4. Let $a, b \in \mathbf{Q}$ and $A, B \subseteq \mathbf{Q}$. Then,

(i) $a\mathcal{R}^{\leftarrow}b \Leftrightarrow b\mathcal{R}a$;

(ii) $AQ^{\leftarrow}B \Leftrightarrow BQA$;

(iii) $A\textbf{All}(\mathcal{R})B \Leftrightarrow a\mathcal{R}b\ \forall a \in A, \forall b \in B$;

(iv) $A\textbf{Exist}(\mathcal{R})B \Leftrightarrow \forall b \in B,\ \exists a \in A$ such that $a\mathcal{R}b$.

By a rational segment or simply a segment we mean the set of rational numbers, that are defined in the following Definition 5.

Definition 5. Let, A, B be the subsets of $\mathbf{Q}$. Then,

(i) $[A, B] = \{x \in \mathbf{Q} \mid A\textbf{All}(\leq)\{x\} \wedge \{x\}\textbf{All}(\leq)B\}$;

(ii) let, A, B be the subsets of $\mathbf{Q}$, then, the *rational interval* is defined as

$$(A, B) = \{x \in \mathbf{Q} \mid A\textbf{All}(<)\{x\} \wedge \{x\}\textbf{All}(<)B\}.$$

Definition 6. Let, A, B be the subsets of $\mathbf{Q}$. Naturally, we can consider the *half-intervals* of the form $(A, B]$ and $[A, B)$ as given below:

(i) $(A, B] = \{x \in \mathbf{Q} \mid A\textbf{All}(<)\{x\} \wedge \{x\}\textbf{All}(\leq)B\}$;

(ii) $[A, B) = \{x \in \mathbf{Q} \mid A\textbf{All}(\leq)\{x\} \wedge \{x\}\textbf{All}(<)B\}$.

Definition 7. Let A be the subset of $\mathbf{Q}$. Then,

(i) $[A, \infty) = \{x \in \mathbf{Q} \mid A\textbf{All}(\leq)\{x\}\}$;

(ii) $(A, \infty) = \{x \in \mathbf{Q} \mid A\textbf{All}(<)\{x\}\}$;

(iii) $(-\infty, A) = \{x \in \mathbf{Q} \mid \{x\}\textbf{All}(<)A\}$;

(iv) $(-\infty, A] = \{x \in \mathbf{Q} \mid \{x\}\textbf{All}(\leq)A\}$.

For rational subsets, we introduce the boundaries of these subsets that are defined in the following Definition 8.

Definition 8. Let A be the subset of $\mathbf{Q}$. Then,

(i) the *upper bounds* of the set A is $\mathrm{Up}(A) = \{b \in \mathbf{Q} \mid \forall a \in A, a \leq b\}$;

(ii) the *lower bounds* of the set A is $\mathrm{Down}(A) = \{b \in \mathbf{Q} \mid \forall a \in A, a \geq b\}$;

(iii) the *maximum element* of the set A is $\mathrm{Max}(A) = A \cap \mathrm{Up}(A)$;

(iv) the *minimal element* of the set A is $\mathrm{Min}(A) = A \cap \mathrm{Down}(A)$;

(v) a *subset* A for which $\mathrm{Up}(A) \neq \emptyset$ and $\mathrm{Down}(A) \neq \emptyset$ will be called *bounded*;

(vi) a *subset* A for which $\mathrm{Up}(A) \neq \emptyset$ will be called *bounded from above*;

(vii) a *subset* A for which $\mathrm{Down}(A) \neq \emptyset$ will be called *bounded from below*.

A bounded set $A \subset \mathbf{Q}$ will be called an interval if for any two numbers $a, b \in A$, $a \leq b$, the inclusion $[a, b] \subseteq A$ holds.

Theorem 1. *If $A \subset \mathbf{Q}$ is interval and bounded, then*

$$(\mathrm{Down}(A), \mathrm{Up}(A)) \subseteq A \subseteq [\mathrm{Down}(A), \mathrm{Up}(A)].$$

§ 2. The multidimensional tools of rational analysis

For simplicity, we consider the set $\mathbf{Q}^3$ as a three dimensional rational space. Traditionally, we denote an element of $\mathbf{Q}^3$ by $r = (x, y, z) = (r_x, r_y, r_z)$. The main tools when working with rational vectors are the subsets of rational vectors and families of these types of subsets. Therefore, we introduce the necessary operations and relations for rational vectors and the subsets of rational vectors.

Definition 9. Let $u, v \in \mathbf{Q}^3$ be rational vectors. Then,

(i) $u + v = (u_x + v_x, u_y + v_y, u_z + v_z)$;

(ii) $u - v = (u_x - v_x, u_y - v_y, u_z - v_z)$;

(iii) $tu = (tu_x, tu_y, tu_z)$, $t \in \mathbf{Q}$;

(iv) $\langle u, v\rangle = u_x v_x + u_y v_y + u_z v_z$.

Now, we define the arithmetic operations with subsets of rational vectors.

Definition 10. Let U, V be subsets of $\mathbf{Q}^3$; $u, v \in \mathbf{Q}^3$ be rational vectors and $t \in \mathbf{Q}$. Then,

(i) $U + V = \{r \in \mathbf{Q}^3 \mid \exists u \in U\ \exists v \in V$ such that $r = u + v\}$;

(ii) $U - V = \{r \in \mathbf{Q}^3 \mid \exists u \in U\ \exists v \in V$ such that $r = u - v\}$;

(iii) $tU = \{r \in \mathbf{Q}^3 \mid \exists u \in U$ such that $r = tu\}$.

Let $\mathcal{P}$ be a relation on the set $\mathbf{Q}^3$ and $\mathcal{S}$ be a relation for power set of $\mathbf{Q}^3$. We have the following derived relations for the subsets of $\mathbf{Q}^3$.

Definition 11. Let, U, V be subsets of $\mathbf{Q}^3$, and $u, v \in \mathbf{Q}^3$ be rational vectors. Then,

(i) $u\mathcal{P}^{\leftarrow}v \Leftrightarrow v\mathcal{P}u$;

(ii) $U\mathcal{S}^{\leftarrow}V \Leftrightarrow V\mathcal{P}U$;

(iii) $U\mathbf{All}(\mathcal{P})V \Leftrightarrow \forall u \in U \forall v \in V$ such that $u\mathcal{P}v$;

(iv) $U\mathbf{Exist}(\mathcal{P})V \Leftrightarrow \forall v \in V \exists u \in U$ such that $u\mathcal{P}v$.

§ 3. Vicinity mappings

In this section we introduce vicinity mappings and discuss some results.

Definition 12. Let H be a set. The mappings of the form $\tau\colon H \to 2^H$ will be called *point-set mappings*. The set of all such mappings is denoted by $F(H)$. A subset of the set H for which the τ images of each point of this subset is not empty, will be called a *domain* of this map and it will be denoted by $\mathrm{Dom}(H, \tau) = \{r \in H \mid \tau(r) \neq \emptyset\}$.

Definition 13. If the image of a point-set mapping of the form $\tau\colon H \to 2^H$ for any argument has cardinality no more than unity, then such a mapping will be called a *function*. We denote a set of functions by $\Phi(H)$.

Definition 14. The mappings of the form $\tau\colon 2^H \to 2^H$ will be called *set-set mappings*. The notation for the domain remains unchanged $\mathrm{Dom}(H, \tau) = \{U \subseteq H \mid \tau(U) \neq \emptyset\}$.

For each point-set mapping $\tau\colon H \to 2^H$, we define a set-set mapping $\tau^1\colon 2^H \to 2^H$ using the formula $\tau^1(U) = \bigcup_{r\in U} \tau(r)$ if $U \neq \emptyset$, and $\tau^1(\emptyset) = \emptyset$. If there is no confusion, then we write $\tau^1(U) = \tau(U)$. We note the monotonicity property for such extensions of mappings. If $U, V \subseteq H$ and $U \subseteq V$, then $\tau^1(U) \subseteq \tau^1(V)$.

Any point-set mapping $\tau\colon H \to 2^H$ can be considered as a vicinity mapping. Essentially, vicinity mapping is simply an equivalent name for point-set mapping. The mapping value $\tau(r)$ is interpreted as the set of points (or vectors) τ-close to the point r.

Example 1. Some of the vicinity mappings are $\omega_{abs}[\varepsilon]$, $\omega_{sp}[\alpha] \in F(\mathbf{Q}^3)$, where

$$\omega_{abs}[\varepsilon](r) = \{v \in \mathbf{Q}^3 \mid \|r_x - v_x\| \leq \varepsilon_x(r), \quad \|r_y - v_y\| \leq \varepsilon_y(r), \quad \|r_z - v_z\| \leq \varepsilon_z(r)\},$$

and

$$\omega_{sp}[\alpha](r) = \{v \in \mathbf{Q}^3 \mid (r_x - v_x)^2 + (r_y - v_y)^2 + (r_z - v_z)^2 \leq \alpha^2\}, \qquad \alpha \in \mathbf{Q}.$$

Definition 15. For two vicinity mappings $\tau, \theta \in F(H)$, we say that τ is *narrower* than θ or θ is *wider* that τ, if for any $r \in H$ we have $\tau(r) \subseteq \theta(r)$. We denote this as $\tau \subseteq \theta$ or $\theta \supseteq \tau$.

Definition 16. The mapping $\tau^{\leftarrow}\colon H \to 2^H$ for the mapping $\tau\colon H \to 2^H$ is determined by the formula $\tau^{\leftarrow}(v) = \{u \in H \mid v \in \tau(u)\}$. It is easy to see that the condition $\theta \supseteq \tau$ implies $\theta^{\leftarrow} \supseteq \tau^{\leftarrow}$.

In addition to vicinity mappings, families of vicinity mappings, which are naturally called soft vicinity mappings, will be used as our requirements.

Definition 17. By *soft vicinity mapping*, we mean a parameterized family of vicinity mappings of the form $\tau\colon L \to F(H)$, where L is a set of parameters of an arbitrary structure.

Example 2. A few soft vicinity mappings are $\theta[\alpha, \beta, .]$, $\theta^+[\alpha, .]$, $\theta^-[\beta, .]\colon L \to F(\mathbf{Q})$, where

$$\begin{gathered}
\theta[\alpha, \beta, r](t) = [t - \beta(r), t + \alpha(r)], \qquad L = \Phi(\mathbf{Q}^3) \times \Phi(\mathbf{Q}^3) \times \mathbf{Q}, \\
\theta^+[\alpha, r](t) = (-\infty, t + \alpha(r)], \qquad L = \Phi(\mathbf{Q}^3) \times \mathbf{Q}, \\
\theta^-[\beta, r](t) = [t - \beta(r), \infty), \qquad L = \Phi(\mathbf{Q}^3) \times \mathbf{Q}, \\
\alpha, \beta\colon \mathbf{Q}^3 \to \mathbf{Q}, \quad r \in \mathbf{Q}^3, \quad t \in \mathbf{Q}.
\end{gathered}$$

The vicinity mappings $\tau \in F(\mathbf{Q})$ are used to determine the following characteristics of subsets of rational numbers.

Definition 18.

(i) The *neighborhood* of the set $U \subseteq \mathbf{Q}$ is $\mathrm{Close}[\tau](U) = \bigcup_{u\in U} (\{u\} \cup \tau(U))$, $\mathrm{Close}[\tau](\emptyset) = \emptyset$.

(ii) The *touching points* of the set $U \subseteq \mathbf{Q}$ is $\mathrm{Touch}[\tau](U) = \{u \in \mathbf{Q} \mid \mathrm{Close}[\tau](\{u\}) \cap U \neq \emptyset\}$.

(iii) The *soft maximal elements* of the set $U \subseteq \mathbf{Q}$ is $\mathrm{Sup}[\tau](U) = U \cap \mathrm{Touch}[\tau](\mathrm{Up}(U))$.

(iv) The *soft minimal elements* of the set $U \subseteq \mathbf{Q}$ is $\mathrm{Inf}[\tau](U) = U \cap \mathrm{Touch}[\tau](\mathrm{Down}(U))$.

Along with the introduced notation, sometimes when the set U is given by the values of some function f on the set V, i.e., $U = f(V)$, equivalent notations of this type will be used $\mathrm{Sup}[\tau](U) = \mathop{\mathrm{Sup}}_{v\in V} [\tau] f(v)$ and $\mathrm{Up}(U) = \mathop{\mathrm{Up}}_{v\in V} f(v)$.

In determining soft maximal and soft minimal elements, one may also use the neighborhood of the set instead of the touching points. However, such a technique does not give an essentially new concept, which follows from the following proposition.

Proposition 1. $\mathrm{Close}[\tau^{\leftarrow}](U) = \mathrm{Touch}[\tau](U)$.

In the future, constructions in the form of the intersection of a family of rational segments will be often encountered, therefore, we consider this question in more detail. So, let us consider a nonempty set W and two given mappings $F, G\colon W \to \mathbf{Q}$. For any $w \in W$, the inequality $F(w) \leq G(w)$ holds. We denote $S = \bigcap\limits_{w\in W} [F(w), G(w)]$. Now, we are interested in two questions. How can we write the set S without using the intersection? Under what conditions will the set S be nonempty?

Proposition 2. $\bigcap\limits_{w\in W} [F(w), G(w)] = [F(W), G(W)]$.

Now, we state the following proposition without discussing the proof.

Proposition 3. *Let $\alpha, \beta \in \mathbf{R}$; $\alpha, \beta > 0$ and the vicinity mapping is given by the formula $\omega[\alpha](u) = [u - \alpha, u + \alpha]$. If $F(W)$ is bounded from above, and $G(W)$ is bounded from below, then:*

1) $\mathrm{Inf}[\omega[\beta]](G(W)) \neq \emptyset$;

2) $\mathrm{Sup}[\omega[\alpha]](F(W)) \neq \emptyset$;

3) *if $a \in \mathrm{Inf}[\omega[\beta]](G(W))$, $b \in G(W)$ and $b \leq a$, then $b \in \mathrm{Inf}[\omega[\beta]](G(W))$;*

4) *if $a \in \mathrm{Sup}[\omega[\alpha]](F(W))$, $b \in F(W)$ and $b \geq a$, then $b \in \mathrm{Sup}[\omega[\alpha]](G(W))$;*

5) $\mathrm{Inf}[\omega[\beta]](G(W))\boldsymbol{All}(<)G(W) \setminus \mathrm{Inf}[\omega[\beta]](G(W))$;

6) $\mathrm{Sup}[\omega[\alpha]](F(W))\boldsymbol{All}(>)F(W) \setminus \mathrm{Sup}[\omega[\alpha]](F(W))$;

7) $[F(W), G(W)] = [\mathrm{Sup}[\omega[\alpha]](F(W)), \mathrm{Inf}[\omega[\beta]](G(W))]$;

8) *if there exists a number $\varepsilon > 0$ such that for any number $u \in \mathrm{Sup}[\omega[\alpha]](F(W))$ and for any number $v \in \mathrm{Inf}[\omega[\beta]](G(W))$, the inequality $v - u \geq \varepsilon$ holds, then $[F(W), G(W)] \neq \emptyset$.*

Here are some simple properties of the mappings $\mathrm{Sup}[\omega[\alpha]]$ and $\mathrm{Inf}[\omega[\beta]]$.

Proposition 4. *Let two bounded mappings $f, g\colon X \to \mathbf{Q}$ and $k, \alpha, \beta, \gamma > 0$ be given. Then:*

1) $(\mathop{\mathrm{Sup}}\limits_{x\in X}[\omega[\alpha]]f(x) + \mathop{\mathrm{Sup}}\limits_{x\in X}[\omega[\beta]]g(x))\boldsymbol{Exist}(\geq)\mathop{\mathrm{Sup}}\limits_{x\in X}[\omega[\gamma]](f(x) + g(x))$;

2) $(\mathop{\mathrm{Inf}}\limits_{x\in X}[\omega[\alpha]]f(x) + \mathop{\mathrm{Inf}}\limits_{x\in X}[\omega[\beta]]g(x))\boldsymbol{Exist}(\leq)\mathop{\mathrm{Inf}}\limits_{x\in X}[\omega[\gamma]](f(x) + g(x))$;

3) $\mathop{\mathrm{Sup}}\limits_{x\in X}[\omega[\gamma]](f(x) + g(x))\boldsymbol{All}(\leq^{\alpha+\beta})(\mathop{\mathrm{Sup}}\limits_{x\in X}[\omega[\alpha]]f(x) + \mathop{\mathrm{Sup}}\limits_{x\in X}[\omega[\beta]]g(x))$;

4) $\mathop{\mathrm{Inf}}\limits_{x\in X}[\omega[\gamma]](f(x) + g(x))\boldsymbol{All}(\geq^{\alpha+\beta})(\mathop{\mathrm{Inf}}\limits_{x\in X}[\omega[\alpha]]f(x) + \mathop{\mathrm{Inf}}\limits_{x\in X}[\omega[\beta]]g(x))$;

5) $k(\mathop{\mathrm{Sup}}\limits_{x\in X}[\omega[\alpha]]f(x)) = \mathop{\mathrm{Sup}}\limits_{x\in X}[\omega[k\alpha]](kf(x))$;

6) $k(\mathop{\mathrm{Inf}}\limits_{x\in X}[\omega[\beta]]g(x)) = \mathop{\mathrm{Inf}}\limits_{x\in X}[\omega[k\beta]](kg(x))$;

7) $-\mathop{\mathrm{Sup}}\limits_{x\in X}[\omega[\alpha]](-f(x)) = \mathop{\mathrm{Inf}}\limits_{x\in X}[\omega[\alpha]]f(x)$.

P r o o f.

1. Let us consider an arbitrary point $u \in \operatorname*{Sup}_{x\in X}\,[\omega[\gamma]](f(x)+g(x))$. Then, there exists $x \in X$ such that $u = f(x) + g(x)$ and the inequality $u + \gamma \geq f(y) + g(y)$ holds for any $y \in X$. Now, consider an arbitrary point $v \in \operatorname*{Sup}_{x\in X}\,[\omega[\alpha]]f(x)$.

If $v < f(x)$, then $f(x) \in \operatorname*{Sup}_{x\in X}\,[\omega[\alpha]]f(x)$, otherwise $v \geq f(x)$. Therefore, there always exists a point $v^* \in \operatorname*{Sup}_{x\in X}\,[\omega[\alpha]]f(x)$ such that $v^* \geq f(x)$. Similarly, there always exists a point $w^* \in \operatorname*{Sup}_{x\in X}[\omega[\beta]]g(x)$ such that $w^* \geq g(x)$. Hence, $v^* + w^* \geq f(x) + g(x)$.

2. The proof is similar to the above one.

3. Let us take two arbitrary points $u \in \operatorname*{Sup}_{x\in X}[\omega[\alpha]]f(x)$ and $v \in \operatorname*{Sup}_{x\in X}[\omega[\beta]]g(x)$. Since, $\operatorname*{Sup}_{x\in X}[\omega[\alpha]]f(x) \subseteq f(X)$ and $\operatorname*{Sup}_{x\in X}[\omega[\beta]]g(x) \subseteq g(X)$, then there exist two points $y, z \in X$ such that $u = f(y)$ and $v = g(z)$. For any $c \in f(X)$ and any $b \in g(X)$, the inequalities $f(y) \geq c - \alpha$ and $g(z) \geq b - \beta$ hold. We choose points c and b agreed, namely, $c = f(x)$ and $b = g(x)$. Hence, we finally get the inequality $f(y) + g(z) \geq f(x) + g(x) - \alpha - \beta$.

4. This can be proven in a similar way. The remaining proofs are obvious. □

Now, we discuss below some properties of rational segments and intervals.

Proposition 5. *Let $A, B, C, D \subseteq \mathbf{Q}$ and $k, \delta > 0$. Then, the following properties are true:*

1) *if $(A, B) \neq \emptyset$, $(C, D) \neq \emptyset$, then $(A, B) + (C, D) \subseteq (A + C, B + D)$;*

2) *if $[A, B] \neq \emptyset$, $[C, D] \neq \emptyset$, then $[A, B] + [C, D] \subseteq [A + C, B + D]$;*

3) *if $[A, B] \neq \emptyset$, $[C, D] \neq \emptyset$, then $[A + C, B + D] \subseteq [A, B] + [C, D] + [-\delta, \delta]$;*

4) $k[A, B] = [kA, kB]$, $k(A, B) = (kA, kB)$*;*

5) $-[A, B] = [-B, -A]$, $-(A, B) = (-B, -A)$*;*

6) *let $[A, B] \neq \emptyset$ and $x, y \in [A, B]$, then, $x - y \in [A - B, B - A]$;*

7) *let $A \subseteq C$, $B \subseteq D$, and $[C, D] \neq \emptyset$, then $[C, D] \subseteq [A, B]$.*

P r o o f.

1. Let $x \in (A, B)$ and $y \in (C, D)$. This means that for arbitrary points $a \in A$, $b \in B$, $c \in C$, $d \in D$; the inequalities $a < x < b$, $c < y < d$ are valid. Adding these inequalities, we obtain inequality $a + c < x + y < b + d$. Therefore, (A, B)+ $(C, D) \subseteq (A + C, B + D)$.

2. The proof can be obtained following similar steps as the proof of 1.

3. Let $z \in [A + C, B + D]$. This means that for arbitrary points $a \in A$, $b \in B$, $c \in C$, $d \in D$, the inequality $a + c < z < b + d$ is valid. Since $[A, B] \neq \emptyset$, $[C, D] \neq \emptyset$, there are two rational numbers $x^0 \in [A, B]$ and $y^0 \in [C, D]$.

Now, we take four points $a \in \operatorname{Sup}[\omega[\frac{\delta}{2}]](A)$, $c \in \operatorname{Sup}[\omega[\frac{\delta}{2}]](C)$, $b \in \operatorname{Inf}[\omega[\frac{\delta}{2}]](B)$ and $d \in$ $\in \operatorname{Inf}[\omega[\frac{\delta}{2}]](D)$.

Then, $a + \frac{\delta}{2} \in \operatorname{Up}(A)$, $c + \frac{\delta}{2} \in \operatorname{Up}(C)$, $b - \frac{\delta}{2} \in \operatorname{Down}(B)$, $d - \frac{\delta}{2} \in \operatorname{Down}(D)$ are valid. Now, we consider the following four points:

$$a^0 = \min\{a + \frac{\delta}{2}, x^0\}, \quad c^0 = \min\{c + \frac{\delta}{2}, y^0\}, \quad b^0 = \max\{b - \frac{\delta}{2}, x^0\}, \quad d^0 = \max\{d - \frac{\delta}{2}, y^0\}.$$

Then, the following inequalities are valid.

$$A\mathbf{All}(\leq)a^0 \leq b^0\mathbf{All}(\leq)B, \qquad C\mathbf{All}(\leq)c^0 \leq d^0\mathbf{All}(\leq)D,$$
$$\text{and } a^0 + c^0 - \delta \leq z \leq b^0 + d^0 + \delta.$$

There is a rational number $t \in [0, 1]$ such that

$$z = a^0 + c^0 - \delta + t(b^0 + d^0 - a^0 - c^0 + 2\delta).$$

We set two points $x^* = a^0 + t(b^0 - a^0)$ and $y^* = c^0 + t(d^0 - c^0)$.
Then, we finally have $z = x^* + y^* + \delta(2t - 1)$, $-\delta \leq \delta(2t - 1) \leq \delta$.
Hence, $A\mathbf{All}(\leq)x^* \wedge x^*\mathbf{All}(\leq)B$ and $C\mathbf{All}(\leq)y^* \wedge y^*\mathbf{All}(\leq)D$.
The other proofs are easy to be obtained. □

§ 4. The gradient of a function in the multidimensional case

Let a function $f \in \Phi(\mathbf{Q}^3)$, a vicinity mapping $\tau \in F(\mathbf{Q}^3)$, and a soft vicinity mapping of the form $\mu\colon \mathbf{Q}^3 \to F(\mathbf{Q})$ be given.

Definition 19. A vector $u \in \mathbf{Q}^3$ is called a (τ, μ)-*gradient* of the function f at the rational point $r \in \mathrm{Dom}(\mathbf{Q}^3, f) \cap \mathrm{Dom}(\mathbf{Q}^3, \tau)$ if, for any $w \in \tau(r) \cap \mathrm{Dom}(\mathbf{Q}^3, f)$ the inclusion $\langle u, w - r\rangle \in \in \mu[r](f(w) - f(r))$ is valid.

The set of (τ, μ)-gradients of the function f at the point r is denoted by $\mathrm{Grad}(f, r, \tau, \mu)$. We denote $\mathrm{Dom}(\mathbf{Q}^3, \mathrm{Grad}(f, ., \tau, \mu)) = \{r \in \mathbf{Q}^3 \mid \mathrm{Grad}(f, r, \tau, \mu) \neq \emptyset\}$.

The definition of a soft gradient is quite similar in its ideas to various generalizations of a gradient and differential in the non-smooth analysis [4, 8, 11].

The meaning of the concept of a soft gradient is very simple. If a vector r is fixed, then $f(w) - f(r)$ is a function of an argument w. The formula $\langle u, w - r\rangle$ defines a linear function of an argument w. The terms of Definition 19 mean the approximation of the function $f(w) - f(r)$ by the function $\langle u, w - r\rangle$ on the set $\tau(r) \cap \mathrm{Dom}(\mathbf{Q}^3, f)$ with accuracy $\mu[r]$.

Definition 20. The mapping $g\colon \mathbf{Q}^3 \to 2^{\mathbf{Q}^3}$ is said to be a *selector* for soft gradient $\mathrm{Grad}(f, ., \tau, \mu)$ if:

(i) each image $g(r)$ consists of one point or is empty;

(ii) $\mathrm{Dom}(\mathbf{Q}^3, g) = \mathrm{Dom}(\mathbf{Q}^3, \mathrm{Grad}(f, ., \tau, \mu))$;

(iii) for any point $r \in \mathrm{Dom}(\mathbf{Q}^3, \mathrm{Grad}(f, ., \tau, \mu))$; the inclusion $g(r) \in \mathrm{Grad}(f, r, \tau, \mu)$ is valid.

The set of selectors for soft gradient $\mathrm{Grad}(f, ., \tau, \mu)$ is denoted by $\mathrm{grad}(f, \tau, \mu)$. It is to be noted that the property of the monotonicity of a soft gradient is true under very general conditions.

Proposition 6. *Let us consider two vicinity mappings $\tau, \gamma \in F(\mathbf{Q}^3)$, two soft vicinity mappings μ, λ of the form $\mu, \lambda\colon \mathbf{Q}^3 \to F(\mathbf{Q})$, $\tau \supseteq \gamma$, and for any $r \in \mathbf{Q}^3$ we have $\mu[r] \supseteq \lambda[r]$. Then for any $r \in \mathbf{Q}^3$, the inclusion $\mathrm{Grad}(f, r, \tau, \lambda) \subseteq \mathrm{Grad}(f, r, \gamma, \mu)$ is true.*

Let $\mu = \theta[\alpha, \beta, r]$ and we can remember that $\theta[\alpha, \beta, r](t) = [t - \beta(r), t + \alpha(r)]$. Then, we have the following:

$$\begin{aligned}
&\mathrm{Grad}(f, r, \tau, \theta[\alpha, \beta, .]) = \\
&= \{u \in \mathbf{Q}^3 \mid \langle u, w - r\rangle \in [f(w) - f(r) - \beta(r), f(w) - f(r) + \alpha(r)], \forall w \in \tau(r) \cap \mathrm{Dom}(\mathbf{Q}^3, f)\} = \\
&\qquad = \{u \in \mathbf{Q}^3 \mid \langle u, w - r\rangle - f(w) + f(r) \in [-\beta(r), \alpha(r)], \quad \forall w \in \tau(r) \cap \mathrm{Dom}(\mathbf{Q}^3, f)\}.
\end{aligned}$$

It is easy to find that the set $\mathrm{Grad}(f, r, \tau, \theta[\alpha, \beta, .])$ is described by a system of linear inequalities and a number of these inequalities is equal to the power set $P(\tau(r) \cap \mathrm{Dom}(\mathbf{Q}^3, f)\})$.

Now, we present some properties of the mapping $\mathrm{Grad}(f, r, \tau, \theta[\alpha, \beta, .])$.

Proposition 7. *Suppose* $\tau(r) \cap \mathrm{Dom}(\mathbf{Q}^3, f)\} \neq \emptyset$.

1. *Let* $\mathrm{Grad}(f, r, \tau, \theta[\alpha, \beta, .]) \neq \emptyset$ *and* $k \in \mathbf{Q}, k > 0$. *Then,*
$$\mathrm{Grad}(kf, r, \tau, \theta[k\alpha, k\beta, .]) = k\,\mathrm{Grad}(f, r, \tau, \theta[\alpha, \beta, .]).$$
2. *Let* $\mathrm{Grad}(f, r, \tau, \theta[\alpha, \beta, .]) \neq \emptyset$ *and* $\mathrm{Grad}(g, r, \tau, \theta[\delta, \gamma, .]) \neq \emptyset$. *Then,*
$$\mathrm{Grad}(f, r, \tau, \theta[\alpha, \beta, .]) + \mathrm{Grad}(g, r, \tau, \theta[\delta, \gamma, .]) \subseteq \mathrm{Grad}(f + g, r, \tau, \theta[\alpha + \delta, \beta + \gamma, .]).$$
3. $\mathrm{Grad}(-f, r, \tau, \theta[\alpha, \beta, .]) = -\mathrm{Grad}(f, r, \tau, \theta[\alpha, \beta, .])$.
4. *If f is a constant function, then* $0 \in \mathrm{Grad}(f, r, \tau, \theta[\alpha, \beta, .])$.

A soft gradient can also be used to formulate the conditions for an approximate local extremum of a function.

Definition 21. For a function $f \in \Phi(\mathbf{Q}^3)$, we call the point $r \in \mathrm{Dom}(\mathbf{Q}^3, f)$ as

1) (α, β, τ)-*stationary* if for any point $w \in \tau(r) \cap \mathrm{Dom}(\mathbf{Q}^3, f)$, the inequality $f(r) - \alpha(r) \leq$ $\leq f(w) \leq f(r) + \beta(r)$ holds;
2) (α, τ)-*maximal* if for any point $w \in \tau(r) \cap \mathrm{Dom}(\mathbf{Q}^3, f)$, the inequality $f(w) \leq f(r) + \alpha(r)$ holds;
3) (α, τ)-*minimal* if for any point $w \in \tau(r) \cap \mathrm{Dom}(\mathbf{Q}^3, f)$, the inequality $f(r) - \alpha(r) \leq f(w)$ holds.

We state the following proposition without discussing the proof.

Proposition 8. *For any function* $f \in \Phi(\mathbf{Q}^3)$, *the following propositions are valid.*

1. *The point* $r \in \mathrm{Dom}(\mathbf{Q}^3, f)$ *is* (α, β, τ)-*stationary for the function* $f \in \Phi(\mathbf{Q}^3)$ *if and only if* $0 \in \mathrm{Grad}(f, r, \tau, \theta[\alpha, \beta, .])$.
2. *The point* $r \in \mathrm{Dom}(\mathbf{Q}^3, f)$ *is* (α, τ)-*maximal for the function* $f \in \Phi(\mathbf{Q}^3)$ *if and only if* $0 \in \mathrm{Grad}(f, r, \tau, \theta^-[\alpha, .])$.
3. *The point* $r \in \mathrm{Dom}(\mathbf{Q}^3, f)$ *is* (α, τ)-*minimal for the function* $f \in \Phi(\mathbf{Q}^3)$ *if and only if* $0 \in \mathrm{Grad}(f, r, \tau, \theta^+[\alpha, .])$.

A soft gradient is a universal concept with which one can build various analogues results of derivatives from classical analysis. For example, we can consider an analogue of directional derivatives. Let $l \in \mathbf{Q}^3$. For the vector $r \in \mathbf{Q}^3$, we define the set $L(r, l) = \{u \in \mathbf{Q}^3 \mid \exists t \in \mathbf{Q}$ such that $u = r + tl\}$. As an analogue of directional derivatives, we consider the set of (τ, μ)-derivatives in the direction l of the function f at the point r, which is defined as $\frac{\partial}{\partial t}(f, r, \tau, \mu, l) = \mathrm{Grad}(f, r, \tau(.) \cap L(., l), \mu)$. Naturally, this formula only makes sense if $\tau(r) \cap L(r, l) \cap \mathrm{Dom}(\mathbf{Q}^3, f) \neq \emptyset$.

From Proposition 6, we can establish the following proposition.

Proposition 9. *Let* $\mathrm{Grad}(f, r, \tau, \mu) \neq \emptyset$. *Then, the following equality holds.*
$$\mathrm{Grad}(f, r, \tau, \mu) = \bigcap_{\tau(r) \cap L(r,l) \cap \mathrm{Dom}(\mathbf{Q}^3, f) \neq \emptyset} \frac{\partial}{\partial l}(f, r, \tau, \mu, l).$$

If we take vectors coinciding with the direction vectors of the coordinate axes as the vector $l \in \mathbf{Q}^3$, then we obtain the analogues of partial derivatives

$$\frac{\partial}{\partial x}(f, r, \tau, \mu) = \frac{\partial}{\partial l}(f, r, \tau, \mu, i),$$
$$\frac{\partial}{\partial y}(f, r, \tau, \mu) = \frac{\partial}{\partial l}(f, r, \tau, \mu, j),$$
$$\frac{\partial}{\partial z}(f, r, \tau, \mu) = \frac{\partial}{\partial l}(f, r, \tau, \mu, k),$$

where, $i = (1, 0, 0)$, $j = (0, 1, 0)$, $k = (0, 0, 1)$.

From Proposition 6 the following proposition immediately follows.

Proposition 10. *Let*

$$\frac{\partial}{\partial x}(f, r, \tau, \mu) \neq \emptyset, \quad \frac{\partial}{\partial y}(f, r, \tau, \mu) \neq \emptyset, \quad \frac{\partial}{\partial z}(f, r, \tau, \mu) \neq \emptyset, \quad \mathrm{Grad}(f, r, \tau, \mu) \neq \emptyset.$$

Then, the inclusion

$$\mathrm{Grad}(f, r, \tau, \mu) \subseteq \frac{\partial}{\partial x}(f, r, \tau, \mu) \cap \frac{\partial}{\partial y}(f, r, \tau, \mu) \cap \frac{\partial}{\partial z}(f, r, \tau, \mu)$$

holds.

We now consider the question of what values the subset of $\mathrm{Grad}(f, r, \tau, \mu)$ can take. Directly from Definition 19, we obtain the following result.

Proposition 11. *Let* $\mathrm{Grad}(f, r, \tau, \mu) \neq \emptyset$. *Then, for any point* $w \in \tau(r) \cap \mathrm{Dom}(\mathbf{Q}^3, f)$ *the inclusion* $f(w) - f(r) \in \bigcap\limits_{u \in \mathrm{Grad}(f,r,\tau,\mu)} \mu^{\leftarrow}[r](\langle u, w - r\rangle)$ *holds.*

Now suppose that for any $r \in \mathbf{Q}^3$, the condition $0 \in \mu^{\leftarrow}[r](0)$ is satisfied and a set $U \subseteq \mathbf{Q}^3$ is taken such that for any point $w \in \tau(r)$, we have $\bigcap\limits_{u \in U} \mu^{\leftarrow}[r](\langle u, w - r\rangle) \neq \emptyset$. We define a function g as follows: $g(r) = 0$, $g(w) \in \bigcap\limits_{u \in U} \mu^{\leftarrow}[r](\langle u, w - r\rangle)$, for any point $w \in \tau(r) \setminus \{r\}$. Then, for any $u \in U$ and any $w \in \tau(r)$, we have $g(w) \in g(r) + \mu^{\leftarrow}[r](\langle u, w - r\rangle)$, or in equivalent form $\langle u, w - r\rangle \in \mu[r](g(w) - g(r))$.

It follows that $U \subseteq \mathrm{Grad}(g, r, \tau, \mu)$. So, the following proposition can be proven easily.

Proposition 12. *Suppose that for any* $r \in \mathbf{Q}^3$, *the condition* $0 \in \mu^{\leftarrow}[r](0)$ *is satisfied and the set* $U \subseteq \mathbf{Q}^3$ *satisfies the condition: for any* $w \in \tau(r)$, *the set* $\bigcap\limits_{u \in U} \mu^{\leftarrow}[r](\langle u, w - r\rangle)$ *is not empty. Then, there exists a function* g *such that* $U \subseteq \mathrm{Grad}(g, r, \tau, \mu)$.

Thus, the condition of Proposition 12 is necessary and sufficient for the set U to be a subset of the soft gradient of some function. One of the arguments justifying the need to build a new analysis was the remark about the impossibility of numerically finding the limit. Therefore, the question naturally arises: is it possible to numerically find a soft gradient? The answer to this question substantially depends on how the function is set and how the soft gradient parameters are set. If the function is given only on a finite set of points, which is characteristic of many practical problems, then the problem of finding the set $\mathrm{Grad}(f, r, \tau, \theta[\alpha, \beta, .])$ reduces to solving a finite system of linear inequalities of the form given below.

$$f(w) - f(r) - \beta(r) \le \langle u, w - r\rangle \le f(w) - f(r) + \alpha(r) \quad \forall w \in \tau(r) \cap \mathrm{Dom}(\mathbf{Q}^3, f).$$

If the function is given on an infinite set, for example, in the form of an algorithm, then we can choose the vicinity mapping τ so that the set $\tau(r)$ is always finite. In this case, we also arrive at a finite system of linear inequalities of the indicated type. Finally, a situation may arise when it is not necessary to find the whole set of gradients $\mathrm{Grad}(f, r, \tau, \theta[\alpha, \beta, .])$ but it is required for a given vector u to find the minimum values of the parameters $\alpha(r)$ and $\beta(r)$ which guarantee the inclusion $u \in \mathrm{Grad}(f, r, \tau, \theta[\alpha, \beta, .])$. For a finite system of inequalities, this problem is also easily solved.

§ 5. Soft integral and path

In this section, we study some properties of soft line integrals using paths. We procure the following two paragraphs from [4].

First, we find out on which set it makes sense to consider the integration problem. Knowing the value of the soft (τ, μ)-gradient of the function f at the point r imposes restrictions on the values of the function f on the set $\tau(r)$. If the values of the soft gradient on the set $\tau(r)$ are known, then this imposes restrictions on the values of the function f on the set $\tau(\tau(r)) = \tau^2(r)$, and so on. As a result, we come to a set $\mathrm{Pro}(r, \tau)= \bigcup\limits_{n\ge 1} \tau^n(r)$.

This set can be described in the language of graph theory. Each vector is considered a vertex. An edge is a pair of vectors u, v if $v \in \tau(u)$. The set $\mathbf{Q}^3$ with vicinity mapping τ defines an ordered graph. The set $\mathrm{Pro}(r, \tau)$ is simply the set of vertices v for which there exists a path which starts at r and ends at v. In other words, $\mathrm{Pro}(r, \tau)$ is the connected component generated by the vector r. Therefore, it is natural to pose the integration problem on the set $\mathrm{Pro}(r, \tau)$. Since the term “path” was used, we give its definition below.

Definition 22 (see [4]). A sequence of vectors $R = (r_1, r_2, \dots, r_n), n > 1$, is called a *$\tau$-path* if $r_{i+1} \in \tau(r_i)$ holds for any $i = 1, 2, 3, \dots, n - 1$. The set of τ-paths with a starting point r and an ending point v is denoted by $\mathrm{Path}(r, v, \tau)$.

It may be necessary to consider not all paths from a vertex to a vertex, but only some that belong to a certain set, for example, a curve. If we denote this set by $\varphi \subseteq \mathbf{Q}^3$, then instead of the vicinity mapping τ, we can take the mapping $\tau(r) \cap \varphi$ and then, under the condition $r \in \varphi$, all points of any path from the set $\mathrm{Path}(r, v, \tau \cap \varphi)$ will lie in the set φ. Therefore, the use of arbitrary vicinity mapping τ is a universal tool of describing any path.

Let $g\colon \mathbf{Q}^3 \to \mathbf{Q}^3$, τ be a vicinity mapping, i. e., $\tau \in F(\mathbf{Q}^3)$, μ be a soft vicinity mapping of the form $\mu\colon \mathbf{Q}^3 \to F(\mathbf{Q})$ and $R = (r_1, r_2, \dots, r_n) \in \mathrm{Path}(u, v, \tau)$. Generally speaking, a soft mapping μ can also have a wider set of parameters. In this case, these parameters will be indicated additionally in the arguments of this mapping μ.

Definition 23 (see [4]). The *soft* (τ, μ)*-integral* of the mapping g from u to v is the set

$$\Im_u^v[g, \tau, u] = \bigcap_{R \in \mathrm{Path}(u,v,\tau)} \sum_{i=1}^{n-1} \mu^{\leftarrow}[r_i](\langle g(r_i), r_{i+1} - r_i\rangle).$$

Throughout this section, we will always assume that a mapping g is defined on the set $\mathrm{Pro}(r, \tau)$. The physical meaning of the integral sum $\sum\limits_{i=1}^{n-1} \mu^{\leftarrow}[r_i](\langle g(r_i), r_{i+1} - r_i\rangle)$ is completely analogous to the corresponding sum of a line integral with the only difference that each element of the sum $\langle g(r_i), r_{i+1} - r_i\rangle$ is calculated not exactly, but approximately and instead of the exact number, some set $\mu^{\leftarrow}[r_i](\langle g(r_i), r_{i+1} - r_i\rangle)$ of approximate values of this element. The method of forming the value of a line integral and a soft integral is essentially different. A line integral is the limit of integral sums with infinite fragmentation of the step of the path. This method requires an infinite number of paths.

A soft integral is simply the values of approximate integral sums common to all the paths considered. This method works for any power on paths set.

Theorem 2. *Suppose that* $\mathrm{Path}(u, v, \tau) \neq \emptyset$ *and there exists a function* f *such that for any path* $R = (r_1, r_2, \dots, r_n) \in \mathrm{Path}(u, v, \tau)$*, points* r_i *belong to the domain of definition of the gradient of the function* f*, i. e.,* $r_i \in \mathrm{Dom}(\mathbf{Q}^3, \mathrm{Grad}(f, ., \tau, \mu)$ *for* $i = 1, 2, \dots, n$*. Then, the inclusion* $f(v) - f(u) \in \bigcap\limits_{g\in\mathrm{Grad}(f,\tau,\mu)} \Im_u^v[g, \tau, u]$ *holds.*

P r o o f. Let us consider an arbitrary path $R = (r_1, r_2, \dots, r_n) \in \mathrm{Path}(u, v, \tau)$. We also consider an arbitrary selector $g \in \mathrm{Grad}(f, \tau, \mu)$. By the definition of a soft gradient, for any $i = 1, 2, \dots, n-1$, the inclusion $\langle g(r_i), r_{i+1} - r_i\rangle \in \mu[r_i](f(r_{i+1}) - f(r_i))$ holds, or in equivalent form we have $f(r_{i+1}) - f(r_i) \in \mu^{\leftarrow}[r_i](\langle g(r_i), r_{i+1} - r_i\rangle)$.

Since the path was chosen arbitrarily, on the right side of the inclusion we can take the intersection along all admissible paths. Then, since the selector is also chosen arbitrarily, we can take the intersection for all selectors as well. Hence, the proof is done. □

The above theorem can also be treated as a theorem on the existence of a soft integral. If a mapping $g\colon \mathbf{Q}^3 \to \mathbf{Q}^3$ is a selector of some soft gradient $\mathrm{Grad}(f, ., \tau, \mu)$ and $\mathrm{Path}(u, v, \tau) \neq \emptyset$, where $u, v, \in \mathrm{Dom}(\mathbf{Q}^3, g)$, then the soft (τ, μ)-integral of the mapping g from u to v exists, i. e., $\Im_u^v[g, \tau, u] \neq \emptyset$.

Another result of theorem 2 turns out if we consider a closed path, i. e., the case when $u = v$. Then, under the conditions of theorem 2, we obtain $0 \in \bigcap\limits_{g\in\mathrm{Grad}(f,\tau,\mu)} \Im_u^v[g, \tau, u]$.

The construction of a soft integral was carried out in a very general way and this certainly constrains the ability to establish the properties of a soft integral. For further research, we specify some parameters of a soft integral. As a soft vicinity mapping $\mu\colon \mathbf{Q}^{\oplus} \to F(\mathbf{Q})$, we consider the soft vicinity mapping $\omega[\varepsilon](r) = \{v \in \mathbf{Q} \mid |v - r| \le \epsilon\}$, $\epsilon \in \mathbf{Q}^{\oplus}, r \in \mathbf{Q}$. Inverse of it is easy to compute as $\omega^{\leftarrow}[\varepsilon](r) = \omega[\varepsilon](r)$.

Now, the soft integral takes the following form:

$$\begin{aligned} \Im_u^v[g, \tau, \omega[\varepsilon]] &= \bigcap_{R\in\mathrm{Path}(u,v,\tau)} \sum_{i=1}^{n-1} [\langle g(r_i), r_{i+1} - r_i\rangle - \varepsilon, \langle g(r_i), r_{i+1} - r_i\rangle + \varepsilon] \\ &= \bigcap_{R\in\mathrm{Path}(u,v,\tau)} \sum_{i=1}^{n-1} (\langle g(r_i), r_{i+1} - r_i\rangle + n\varepsilon[-1, 1]) \end{aligned}$$

Following proposition can be obtained from Proposition 2.

Proposition 13. *Let* $\alpha, \beta, \varepsilon \in \mathbf{Q}$ *and* $\alpha, \beta, \varepsilon > 0$*, then*

$$\Im_u^v[g, \tau, \omega[\varepsilon]] = \Bigg[\operatorname*{Sup}_{R\in\mathrm{Path}(u,v,\tau)} [\omega[\alpha]](\sum_{i=1}^{n-1}(\langle g(r_i), r_{i+1} - r_i\rangle - n\varepsilon), \\ \operatorname*{Inf}_{R\in\mathrm{Path}(u,v,\tau)} [\omega[\beta]](\sum_{i=1}^{n-1}(\langle g(r_i), r_{i+1} - r_i\rangle + n\varepsilon) \Bigg]$$

The soft integral can also be written in another form. Let $R = (r_1, r_2, \ldots, r_n) \in \mathrm{Path}(u, v, \tau)$. We first define the following notions.

$$\begin{aligned} N(R) &= n, \\ \sigma(R, G) &= \sum_{i=1}^{N(R)-1} \langle g(r_i), r_{i+1} - r_i \rangle, \underline{\sigma}(R, g, \varepsilon) = \sigma(R, g) - N(R)\varepsilon, \\ \overline{\sigma}(R, g, \varepsilon) &= \sigma(R, g) + N(R)\varepsilon. \end{aligned}$$

Proposition 14. *Let $\varepsilon \in \mathbf{Q}$ and $\varepsilon > 0$, then*

$$\Im_u^v[g, \tau, \omega[\varepsilon]] = [\underline{\sigma}(\mathrm{Path}(u, v, \tau), g, \varepsilon), \overline{\sigma}(\mathrm{Path}(u, v, \tau), g, \varepsilon)].$$

Now, we discuss some properties of the functions $\underline{\sigma}$ and $\overline{\sigma}$.

Proposition 15. *Let $\alpha, \beta, k \in \mathbf{Q}$, $R \in \mathrm{Path}(u, v, \tau)$ and $\alpha, \beta, k > 0$. Then,*

(1) $\underline{\sigma}(R, g + f, \alpha + \beta) = \underline{\sigma}(R, g, \alpha) + \underline{\sigma}(R, f, \beta)$;

(2) $\overline{\sigma}(R, g + f, \alpha + \beta) = \overline{\sigma}(R, g, \alpha) + \overline{\sigma}(R, f, \beta)$;

(3) $\underline{\sigma}(R, kg, k\alpha) = k\underline{\sigma}(R, g, \alpha)$;

(4) $\overline{\sigma}(R, kg, k\alpha) = k\overline{\sigma}(R, g, \alpha)$;

(5) $\underline{\sigma}(R, -g, \alpha) = -\overline{\sigma}(R, g, \alpha)$;

(6) $\overline{\sigma}(R, -g, \alpha) = -\underline{\sigma}(R, g, \alpha)$;

(7) $\underline{\sigma}(\mathrm{Path}(u, v, \tau), g + f, \alpha + \beta) \subseteq \underline{\sigma}(\mathrm{Path}(u, v, \tau), g, \alpha) + \underline{\sigma}(\mathrm{Path}(u, v, \tau), f, \beta)$;

(8) $\overline{\sigma}(\mathrm{Path}(u, v, \tau), g + f, \alpha + \beta) \subseteq \overline{\sigma}(\mathrm{Path}(u, v, \tau), g, \alpha) + \overline{\sigma}(\mathrm{Path}(u, v, \tau), f, \beta)$;

(9) $\underline{\sigma}(\mathrm{Path}(u, v, \tau), kg, k\alpha) = k\underline{\sigma}(\mathrm{Path}(u, v, \tau), g, \alpha)$;

(10) $\overline{\sigma}(\mathrm{Path}(u, v, \tau), kg, k\alpha) = k\overline{\sigma}(\mathrm{Path}(u, v, \tau), g, \alpha)$;

(11) $\underline{\sigma}(\mathrm{Path}(u, v, \tau), -g, \alpha) = -\overline{\sigma}(\mathrm{Path}(u, v, \tau), g, \alpha)$;

(12) $\overline{\sigma}(\mathrm{Path}(u, v, \tau), -g, \alpha) = -\underline{\sigma}(\mathrm{Path}(u, v, \tau), g, \alpha)$.

Using the above results, it is easy to obtain the following properties of the soft integral.

Proposition 16. *Assume that $\Im_u^v[g, \tau, \omega[\varepsilon]] \neq \emptyset$, $\Im_u^v[f, \tau, \omega[\lambda]] \neq \emptyset$ and $k, \varepsilon, \lambda \in \mathbf{Q}$, $k, \varepsilon, \lambda > 0$. Then,*

(1) $\Im_u^v[g, \tau, \omega[\varepsilon]] + \Im_u^v[f, \tau, \omega[\lambda]] \subseteq \Im_u^v[g + f, \tau, \omega[\varepsilon + \lambda]]$;

(2) $\Im_u^v[kg, \tau, \omega[k\varepsilon]] = k\Im_u^v[g, \tau, \omega[\varepsilon]]$;

(3) $\Im_u^v[-g, \tau, \omega[\varepsilon]] = -\Im_u^v[g, \tau, \omega[\varepsilon]]$;

(4) *for any two paths $R, P \in \mathrm{Path}(u, v, \tau)$; the next inclusion holds:*

$$\Im_u^v[g, \tau, \omega[\varepsilon]] \subseteq [\underline{\sigma}(R, g, \varepsilon), \overline{\sigma}(P, g, \varepsilon)].$$

P r o o f.
1. From Propositions 4 and 14, the following properties hold.

$$\begin{aligned}
&\Im_u^v[g,\tau,\omega[\varepsilon]] + \Im_u^v[f,\tau,\omega[\lambda]] \\
&= [\underline{\sigma}(\mathrm{Path},g,\varepsilon), \overline{\sigma}(\mathrm{Path},g,\varepsilon)] + [\underline{\sigma}(\mathrm{Path},f,\lambda), \overline{\sigma}(\mathrm{Path},f,\lambda)] \\
&\subseteq [\underline{\sigma}(\mathrm{Path},g,\varepsilon) + \underline{\sigma}(\mathrm{Path},f,\lambda), \overline{\sigma}(\mathrm{Path},g,\varepsilon) + \overline{\sigma}(\mathrm{Path},f,\lambda)] \\
&\subseteq [\underline{\sigma}(\mathrm{Path},g+f,\varepsilon+\lambda), \overline{\sigma}(\mathrm{Path},g+f,\varepsilon+\lambda)] = \Im_u^v[g+f,\tau,\omega[\varepsilon+\lambda]].
\end{aligned}$$

In these results, the argument $\mathrm{Path}(u,v,\tau)$ in all the functions $\underline{\sigma}$, $\overline{\sigma}$ is the same and is replaced by the symbol Path for brevity. We skip the other proofs. □

The soft integral $\Im_u^v[g,\tau,\omega[\varepsilon]]$ is a rational segment. Let us estimate the size of this segment and the segment itself. We introduce a set of paths that give the minimum number of steps in a path belonging to the set $\mathrm{Path}(u,v,\tau)$.

We define $\mathrm{MinPath}(u,v,\tau) = \{P \in \mathrm{Path}(u,v,\tau) \mid N(P) = \min\limits_{R \in \mathrm{Path}(u,v,\tau)} N(R)\}$.

Proposition 17. *Assume that* $\Im_u^v[g,\tau,\omega[\varepsilon]] \neq \emptyset$. *Then,*

(1) $\Im_u^v[g,\tau,\omega[\varepsilon]] \subseteq \sigma(R,g) + \varepsilon N(R)[-1,1] \quad \forall R \in \mathrm{MinPath}(u,v,\tau)$;

(2) *for any* $x, y \in \Im_u^v[g,\tau,\omega[\varepsilon]]$ *the inequality* $\mid x - y \mid \leq 2\varepsilon \min\limits_{R \in \mathrm{Path}(u,v,\tau)} N(R)$ *holds.*

We now consider the question of additivity with respect to the upper limit of a soft integral. We need a concatenation operation for paths.

Definition 24 (see [4]). Let $R = (r_1, r_2, \ldots, r_n) \in \mathrm{Path}(u,v,\tau)$, $P = (p_1, p_2, \ldots, p_m) \in$ $\in \mathrm{Path}(v,w,\tau)$. Then, the *concatenation* of τ-paths R and P is called a τ-path Q, where $Q = R \oplus P = (r_1, r_2, \ldots, r_n, p_2, \ldots, p_m) = (r_1, r_2, \ldots, r_{n-1}, p_1, \ldots, p_m)$.

An obvious proposition directly follows from the above definition.

Proposition 18. *Let* $\mathrm{Path}(u,v,\tau) \neq \emptyset$ *and* $\mathrm{Path}(v,w,\tau) \neq \emptyset$. *Then,*

$$\mathrm{Path}(u,v,\tau) \oplus \mathrm{Path}(v,w,\tau) \subseteq \mathrm{Path}(u,w,\tau).$$

Proposition 19. *Let* $u \neq v \neq w$, $\Im_u^v[g,\tau,\omega[\varepsilon]] \neq \emptyset$, $\Im_v^w[g,\tau,\omega[\varepsilon]] \neq \emptyset$, $\Im_u^w[g,\tau,\omega[\varepsilon]] \neq \emptyset$ *and* $\varepsilon, \delta > 0$. *Then,*

$$\Im_u^v[g,\tau,\omega[\varepsilon]] + \Im_v^w[g,\tau,\omega[\varepsilon]] + [-\delta,\delta] \supseteq \Im_u^w[g,\tau,\omega[\varepsilon]].$$

P r o o f. From Proposition 18 it follows that $\mathrm{Path}(u,v,\tau) \oplus \mathrm{Path}(v,w,\tau) \subseteq \mathrm{Path}(u,w,\tau)$.

From here we get $\underline{\sigma}(\mathrm{Path}(u,v,\tau),g,\varepsilon) + \underline{\sigma}(\mathrm{Path}(v,w,\tau),g,\varepsilon) \subseteq \underline{\sigma}(\mathrm{Path}(u,w,\tau),g,\varepsilon)$, and $\overline{\sigma}(\mathrm{Path}(u,v,\tau),g,\varepsilon) + \overline{\sigma}(\mathrm{Path}(v,w,\tau),g,\varepsilon) \subseteq \overline{\sigma}(\mathrm{Path}(u,w,\tau),g,\varepsilon)$.

For the sum of the integrals from Proposition 5 we have

$$\begin{aligned}
&\Im_u^v[g,\tau,\omega[\varepsilon]] + \Im_v^w[g,\tau,\omega[\varepsilon]] + [-\delta,\delta] \supseteq \\
&\supseteq [\underline{\sigma}(\mathrm{Path}(u,v,\tau),g,\varepsilon) + \underline{\sigma}(\mathrm{Path}(v,w,\tau), \overline{\sigma}(\mathrm{Path}(u,v,\tau),g,\varepsilon) + \overline{\sigma}(\mathrm{Path}(v,w,\tau),g,\varepsilon)] \supseteq \\
&\qquad \supseteq [\underline{\sigma}(\mathrm{Path}(u,w,\tau),g,\varepsilon), \overline{\sigma}(\mathrm{Path}(u,w,\tau),g,\varepsilon)] = \Im_u^w[g,\tau,\omega[\varepsilon]].
\end{aligned}$$

□

§ 6. Comparison

The area of soft rational analysis was established by the second author of this paper in [8]. Here, several properties of continuous function were introduced in the sense of soft rational analysis. These properties are fundamental building blocks of soft rational analysis. But, it is known to us that derivatives and integrals are two of the main building blocks of real analysis and allied areas of mathematics, thus it is important to introduce concepts of derivatives, integrals and then to study their properties from the viewpoint of soft rational analysis.

The concepts of derivatives and integrals were introduced by Molodtsov [4] and their many fundamental properties were studied. In [4], the notion of τ-path, soft (τ, μ)-integral of mapping g from point u to point v, etc. were introduced. The idea of proximity mapping was considered in [4]. Several properties of paths were studied from the perspective of soft rational analysis and proximity mappings.

In this paper, we investigate some new properties of soft rational line integrals by introducing a new concept of a soft gradient. Using a soft gradient, properties of paths are studied under concatenation. Moreover, properties of paths are investigated for two mappings g and f under various structures viz. $g + f$, kg, etc. where $k \in \mathbf{Q}$ and $k > 0$. We also show some properties of (τ, μ)-integrals for two mappings g and f with some of the above structures. In this manner, it is important to ensure that the results of this paper may be found similar in nature to some results of [4], but in reality, they are completely different and newly established results. The differences between the second author's work in [4] and this paper are based on two ideas: proximity mappings and gradients in soft set setting from the perspective of rational analysis. Our present paper studies several new fundamental results in the paths of soft rational line integrals with the help of the concepts viz. soft gradients, (α, β, τ)-stationary points, etc. Some of the inclusion properties are studied in the settings of soft gradients, and these properties are used to investigate several properties of paths under the various structures viz. concatenation, $g + f$, kg, etc. where $k \in \mathbf{Q}$ and $k > 0$. This paper extends the concept of a soft rational line integral as a continuation [4] but in an independent manner with several new notions as stated above.

§ 7. Conclusion

In this paper, some concepts of a soft rational gradient and a soft rational integral are proposed. Some properties of a soft rational gradient are established. A necessary and sufficient condition is found so that a certain set can be a subset of the gradient of a certain function. Moreover, the important properties of the soft rational integral are established. The inclusion for the integral of a gradient is proved. Semi-additivity and positive uniformity of a soft rational integral are established. Estimates are obtained for a soft rational integral and the size of its segment. Semi-additivity with respect to the upper limit of integration is proved.

However, only the first step has been taken towards the construction of soft rational analysis, and the theory of soft rational integration. Many interesting questions are still unexplored. Some of them are as discussed below.

Q.1. What are the necessary and sufficient conditions for the existence of a soft integral?

Q.2. What are the stability issues of a soft gradient and a soft integral?

Q.3. What is the connection between a one-dimensional soft integral and a multidimensional soft integral?

Q.4. How to build soft analogues for integration by the area and volume? How to find out the possibility of constructing analogues of the Green's formula and Divergence theorem?

Q.5. What are the connections (in the limit) between a soft integral and classical Riemann, Lebesgue, Perron and Kurzweil integrals?

Q.6. What will be the consequences of applying the soft rational analysis in various applied sciences: in theoretical mechanics, in quantum mechanics, etc.?

Thus, we hope that this paper will find suitable scopes for further research for the benefit of theoretical computer science.

§ 8. Conflict of interest

The authors declare that there is no conflict of interest. Moreover, both authors contributed equally to this paper.

§ 9. Acknowledgement

The first author of this paper is thankful to the Almighty and his father Late S. Acharjee. He is also thankful to his mother Mrs. M. Acharjee for her constant support. Moreover, he is also thankful to his student Mr. Kabindra Goswami for his kind help in formatting the paper. The second author is thankful to Russian Foundation for Basic Research as this research was partially supported by the Russian Foundation for Basic Research (project 19-01-00625). Also, both authors are thankful to the reviewers of the paper.

Funding. The research of D. A. Molodtsov was carried out with partial support from the Russian Foundation for Basic Research (project 19-01-00625).

REFERENCES

1. Molodtsov D. A. *Ustoichivost' printsipov optimal'nosti* (Stability of the principles of optimality), Moscow: Nauka, 1987. https://zbmath.org/?q=an:0632.49001
2. Molodtsov D. A. Soft set theory — First results, *Computers and Mathematics with Applications*, 1999, vol. 37, no. 4-5, pp. 19–31. https://doi.org/10.1016/S0898-1221(99)00056-5
3. Molodtsov D. A. Soft differential equations, *Computational Mathematics and Mathematical Physics*, 2000, vol. 40, no. 8, pp. 1116–1128. https://zbmath.org/?q=an:1002.34052
4. Molodtsov D. A. Principles of rational analysis — derivatives and integrals, *Nechetkie Sistemy i Myagkie Vychisleniya*, 2020, vol. 15, no. 1, pp. 5–25 (in Russian). https://doi.org/10.26456/fssc70
5. Molodtsov D. A. *Priblizhennyi integral v mnogomernom sluchae* (Approximate integral in multidimensional case), Moscow: Dorodnitsyn Computing Centre of the Russian Academy of Sciences, 1989.
6. Molodtsov D. A. *K obshchei teorii abstraktnogo interval'nogo integrala* (To the general theory of abstract interval integral), Moscow: Dorodnitsyn Computing Centre of the Russian Academy of Sciences, 1991.
7. Molodtsov D. A., Sokolov A. A., Kovkov D. V. Introduction to the soft analysis, *Nechetkie Sistemy i Myagkie Vychisleniya*, 2007, vol. 2, no. 1, pp. 5–28 (in Russian). https://elibrary.ru/item.asp?id=16216165
8. Molodtsov D. A. Principles of rational analysis — continuity of functions, *Nechetkie Sistemy i Myagkie Vychisleniya*, 2019, vol. 14, no. 2, pp. 126–141 (in Russian). https://doi.org/10.26456/fssc57
9. Molodtsov D. A. Higher order derivatives in soft analysis, *Nechetkie Sistemy i Myagkie Vychisleniya*, 2019, vol. 14, no. 1, pp. 34–55 (in Russian). https://doi.org/10.26456/fssc51
10. Molodtsov D. A. Soft dynamical extrapolation of the multi-valued dependencies, *Nechetkie Sistemy i Myagkie Vychisleniya*, 2019, vol. 14, no. 1, pp. 5–18 (in Russian). https://doi.org/10.26456/fssc50
11. Dem'yanov V. F., Shomesova V. K. Conditional subdifferentials of convex functions, *Soviet Mathematics. Doklady*, 1978, vol. 19, pp. 1181–1185. https://zbmath.org/?q=an:0461.49005

Received 30.10.2020

Santanu Acharjee, Assistant Professor, Department of Mathematics, Gauhati University, Guwahati-781014, Assam, India.
ORCID: https://orcid.org/0000-0003-4932-3305
E-mail: sacharjee326@gmail.com

Dmitrii Anatol'evich Molodtsov, Doctor of Physics and Mathematics, Leading Researcher, Dorodnitsyn Computing Centre of the Russian Academy of Sciences, ul. Vavilova, 40, Moscow, 119333, Russia.
E-mail: dmitri_molodtsov@mail.ru

Citation: S. Acharjee, D. A. Molodtsov. Soft rational line integral, *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki*, 2021, vol. 31, issue 4, pp. 578–596.

С. Ачарджи, Д. А. Молодцов

Мягкий рациональный криволинейный интеграл

Ключевые слова: мягкий рациональный анализ, мягкий градиент, мягкий интеграл, мягкое множество.

УДК 517.977

DOI: 10.35634/vm210404

Теория мягких множеств — это новая область математики, которая имеет дело с неопределенностями. Приложения теории мягких множеств широко распространены в различных областях науки и социальных наук, таких как принятие решений, информатика, распознавание образов, искусственный интеллект и т. д. Важность мягких теоретико-множественных версий математического анализа ощущается в нескольких областях информатики. В этой статье предлагаются некоторые концепции мягкого градиента функции и мягкого интеграла, аналога криволинейного интеграла в классическом анализе. Установлены основные свойства мягких градиентов. Найдено необходимое и достаточное условие, при котором множество может быть подмножеством мягкого градиента некоторой функции. Доказано включение мягкого градиента в мягкий интеграл. Установлены полуаддитивность и положительная однородность мягкого интеграла. Получены оценки мягкого интеграла и размера его отрезка. Полуаддитивность относительно верхнего предела интегрирования доказана. Кроме того, эта статья расширяет теоретические развитие мягкого рационального криволинейного интеграла и связанных областей для повышения функциональности с точки зрения вычислительных систем.

Финансирование. Исследование Д. А. Молодцова было проведено при частичной поддержке Российского фонда фундаментальных исследований (проект 19-01-00625).

СПИСОК ЛИТЕРАТУРЫ

1. Молодцов Д. А. Устойчивость принципов оптимальности. М.: Наука, 1987. https://zbmath.org/?q=an:0632.49001
2. Molodtsov D. A. Soft Set Theory — First results // Computers and Mathematics with Applications. 1999. Vol. 37. No. 4-5. P. 19–31. https://doi.org/10.1016/S0898-1221(99)00056-5
3. Молодцов Д. А. Мягкое дифференциальное уравнение // Журнал вычислительной математики и математической физики. 2000. Т. 40. № 8. С. 1163–1175. http://mi.mathnet.ru/zvmmf1458
4. Молодцов Д. А. Начала рационального анализа - производные и интегралы // Нечеткие системы и мягкие вычисления. 2020. Т. 15. № 1. С. 5–25. https://doi.org/10.26456/fssc70
5. Молодцов Д. А. Приближенный интеграл в многомерном случае. М.: Вычислительный центр им. А. А. Дородницына РАН, 1989.
6. Молодцов Д. А. К общей теории абстрактного интервального интеграла. М.: Вычислительный центр им. А. А. Дородницына РАН, 1991.
7. Молодцов Д. А., Соколов А. А., Ковков Д. В. Введение в мягкий анализ // Нечеткие системы и мягкие вычисления. 2007. Т. 2. № 1. С. 5–28. https://elibrary.ru/item.asp?id=16216165
8. Молодцов Д. А. Начала рационального анализа - непрерывность функций // Нечеткие системы и мягкие вычисления. 2019. Т. 14. № 2. С. 126–141. https://doi.org/10.26456/fssc57
9. Молодцов Д. А. Производные высших порядков в мягком анализе // Нечеткие системы и мягкие вычисления. 2019. Т. 14. № 1. С. 34–55. https://doi.org/10.26456/fssc51
10. Молодцов Д. А. Мягкая динамическая экстраполяция многозначных зависимостей // Нечеткие системы и мягкие вычисления. 2019. Т. 14. № 1. С. 5–18. https://doi.org/10.26456/fssc50
11. Демьянов В. Ф., Шомесова В. К. Условные субдифференциалы выпуклых функций // Докл. АН СССР. 1978. Т. 242. № 4. С. 753–756. http://mi.mathnet.ru/dan42022

Поступила в редакцию 30.10.2020

Ачарджи Сантану, математический факультет, Университет Гаухати, Гувахати-781014, Ассам, Индия.
ORCID: https://orcid.org/0000-0003-4932-3305
E-mail: sacharjee326@gmail.com

Молодцов Дмитрий Анатольевич, д. ф.-м. н., ведущий научный сотрудник, Вычислительный центр им. А. А. Дородницына Российской академии наук Федерального исследовательского центра «Информатика и управление» Российской академии наук, 119333, Москва, ул. Вавилова, 40.
E-mail: dmitri_molodtsov@mail.ru

Цитирование: С. Ачарджи, Д. А. Молодцов. Мягкий рациональный криволинейный интеграл // Вестник Удмуртского университета. Математика. Механика. Компьютерные науки. 2021. Т. 31. Вып. 4. С. 578–596.