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APPROXIMATION OF VALUE FUNCTION OF DIFFERENTIAL GAME WITH MINIMAL COST

The paper is concerned with the approximation of the value function of the zero-sum differential game with the minimal cost, i. e., the differential game with the payoff functional determined by the minimization of some quantity along the trajectory by the solutions of continuous-time stochastic games with the stopping governed by one player. Notice that the value function of the auxiliary continuous-time stochastic game is described by the Isaacs–Bellman equation with additional inequality constraints. The Isaacs–Bellman equation is a parabolic PDE for the case of stochastic differential game and it takes a form of system of ODEs for the case of continuous-time Markov game. The approximation developed in the paper is based on the concept of the stochastic guide first proposed by Krasovskii and Kotelnikova.

Keywords: differential games with minimal cost, stochastic guide, approximation of the value function, Isaacs–Bellman equation.

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Introduction

The paper is concerned with the zero-sum differential game where the payoff functional is given by the minimization of some function along the trajectory. Following [4], we will call such games differential games with minimal cost (Barron in [4] considered the maximization condition; however it seems that the minimization is more natural in the light of possible application). The differential games with minimal cost include the pursuit-evasion games, where the pursuit should be performed not at the given time but up to the final time instant [29]. Moreover, one can reduce to the games with minimal time the general setting of differential games with final time determined by reaching a target set (see [11] for more information on such type of differential games).

Krasovskii and Subbotin in their seminal book [24] proved the existence of the value function of the zero-sum differential game with minimal cost (see also [3, 4] for the case when the payoff depends also on the players' controls) under certain assumptions which include so called Isaacs' condition. Moreover, they proposed the construction of players' strategies which are optimal at the given point. Notice the Krasovskii and Subbotin used the feedback formalization. There are several other approaches to the differential game theory. Among them are control with guide strategies [24] and the nonanticipative strategies [9, 31]. It is proved that these formalizations are equivalent to the feedback approach in the sense that they produce the same value function [30]. Notice that the value function of the differential game with minimal cost solves the Isaacs–Bellman equation with additional inequality constraints [29]. In the case of time-homogeneous dynamics and payoff functions, the differential game with minimal cost can be reduced to the games with terminal payoff, where the first player can freeze the system [27].

In the paper, we study approximations of the value function based on solutions of an auxiliary continuous-time stochastic game. We follow the approach proposed by Krasovskii and Kotelnikova [21–23]. In those papers, the approximation of the value function of the differential game with terminal payoff was constructed based on solutions of the stochastic differential game. The key tool in the aforementioned papers is the concept of stochastic guide that allows to design suboptimal players' strategy using optimal strategies in the auxiliary stochastic game. Paper [2] extends the approach of [21–23] to the case when the dynamics of the auxiliary game is given by a Lévy–Khintchine generator (see [18] for the general theory of systems governed by Lévy–Khintchine generators). This class of auxiliary games includes stochastic differential games and continuous-time Markov games (i.e., continuous-time stochastic games with dynamics governed by a Markov chain). Notice that the optimal strategies in the auxiliary stochastic games often can be constructed using the solution of Isaacs–Bellman equation. Using the fact that the Isaacs–Bellman equation for the stochastic differential game with terminal payoff is second-order parabolic PDE, whereas the value function for the zero-sum continuous-time Markov game with terminal payoff is determined by the system of ODEs, one can approximate the value function of the zero-sum differential game with the terminal payoff by solutions of parabolic PDEs and solutions of system of ODEs [2, 21–23].

We extend the methodology of the stochastic guide based on players' strategies optimal for the auxiliary continuous-time stochastic game to the case of differential game with minimal cost. To this end, we use the auxiliary game where the stopping time is controlled by the player who tries to minimize the outcome. The value function of this auxiliary continuous-time game is described by the Isaacs–Bellman equation with additional inequality constraints. Recall that the stochastic differential game implies that the Isaacs–Bellman equation is a parabolic PDE, whilst for the case of Markov game the Isaacs–Bellman equation takes the form of system of ODEs. The concept of continuous-time stochastic game with stopping controlled by one player as well as the setting of Isaacs–Bellman equations with inequality constraints are apparently entirely new. However, this problem is strongly related to the setting of stochastic game with stopping examined in the earlier works. First, let us mention so called Dynkin games (see [10, 12, 13, 17, 25, 26, 28] and reference therein for the recent advances in the field of continuous-time Dynkin game). Notice that the Dynkin game implies that both players choose their stopping time whilst the dynamics is uncontrolled. The further development of such setting leads to the game theoretical problems where both players can control the dynamics and stopping time [5–8], as well as to the setting when one player controls the drift whereas the second one chooses the stopping time [16]. Moreover, let us mention in this direction the setting where the stopping time is not controlled but distributed on a finite time interval [19, 20].

The paper is organized as follows. First, in Section 1 we introduce the differential game with minimal cost and the differential game with the stopping governed by one player. Section 2 provides the description of the auxiliary continuous-time stochastic games. Furthermore, in that section we introduce the concept of stability in the auxiliary games and state the main results. They provide the evaluations of the value function for the differential games with stopping time governed by one player by functions that are stable for the auxiliary game and the equivalence of the differential game with minimal cost and the differential game with stopping time controlled by one player. Section 3 is concerned with the construction of the first player's strategy in the differential game with stopping time governed by the first player to evaluate the upper value function. The second player's strategy providing the evaluation of the lower value function is presented in Section 4. The equivalence between the differential game with minimal cost and the differential game with stopping controlled by one player is proved in Section 5. Finally, we write out the Isaacs–Bellman equation with additional inequality constraints for the auxiliary continuous-time stochastic game. It is a sufficient condition on a given function to be a value function of this auxiliary game (see Section 6). We complete this section with concrete examples of the Isaacs–Bellman equation with additional inequality constraints for stochastic differential game and Markov game.

§ 1. Problem statement

We examine the differential game with the dynamics

$$\frac{d}{dt}x(t) = f(t, x(t), u(t), v(t)), \quad t \in [0, T], \quad x(t) \in \mathbb{R}^d, \quad u \in U, \quad v \in V \tag{1.1}$$

and the payoff functional

$$\gamma(x(\cdot)) = \min_{t\in[t_0,T]} g(t, x(t)). \tag{1.2}$$

Such types of games were previously studied in [4, 24, 29, 30].

For the purpose of approximations, it is convenient to replace this game with the differential game with the dynamics given by (1.1) and payoff

$$g(\tau, x(\tau)), \tag{1.3}$$

where the time τ is controlled by the first player. Theorem 3 below shows that if the first player observes the whole history, games (1.1), (1.2) and (1.1), (1.3) are equivalent.

We impose the following conditions on control spaces and function f and g:

1) U and V are metric compacts;

2) the functions f and g are continuous and bounded;

3) the function g is uniformly continuous w. r. t. x;

4) the function f is Lipschitz continuous w. r. t. the phase variable x for some constant K;

5) (Isaacs' condition) for every $t \in [0, T]$, $x, w \in \mathbb{R}^d$,

$$\min_{u\in U} \max_{v\in V} \langle w, f(t, x, u, v)\rangle = \max_{v\in V} \min_{u\in U} \langle w, f(t, x, u, v)\rangle.$$

We denote the upper bound of norm of f by R i.e., for every $t \in [0, T]$, $x \in \mathbb{R}^d$, $u \in U$, $v \in V$,

$$\|f(t, x, u, v)\| \le R. \tag{1.4}$$

We consider differential game (1.1), (1.3) in the class of stochastic strategies with memory. In this case the parameter τ becomes a stopping time controlled by the first player (see [15, p. 120] for definition of a stopping time).

Definition 1. We say that $\mathfrak{u} = (\Omega^U, \mathcal{F}^U, \{\mathcal{F}_t^U\}_{t\in[t_0,T]}, P^U_{x(\cdot)}, u_{x(\cdot)}, \tau_{x(\cdot)})$ is a strategy of the first player on $[t_0, T]$ provided that the following conditions hold:

(i) $(\Omega^U, \mathcal{F}^U, \{\mathcal{F}_t^U\}_{t\in[t_0,T]})$ is a filtered measurable space;

(ii) for each $x(\cdot) \in C([t_0, T])$, $P^U_{x(\cdot)}$ is a probability on $\mathcal{F}^U$, $u_{x(\cdot)}$ is a $\{\mathcal{F}_t^U\}_{t\in[t_0,T]}$-progressively measurable process with values in U, when $\tau_{x(\cdot)}$ is a stopping time w.r.t. the filtration $\{\mathcal{F}_t\}_{t\in[t_0,T]}$ with values in $[t_0, T]$;

(iii) if, for some $r \in [t_0, T]$, $x(t) = y(t)$ when $t \in [t_0, r]$, then

$$P^U_{x(\cdot)}(A) = P^U_{y(\cdot)}(A) \text{ for every } A \in \mathcal{F}^U_r,$$

$$\tau_{x(\cdot)} \wedge r = \tau_{y(\cdot)} \wedge r \; P^U_{x(\cdot)}\text{-a.s.}$$

and, for any $t \in [t_0, r]$,

$$u_{x(\cdot)}(t) = u_{y(\cdot)}(t) \quad P^U_{x(\cdot)}\text{-a.s.}$$

Definition 2. Given a strategy of the first player $\mathfrak{u}$ and an initial position (t_0, x_0), we say that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[s,r]}, P, X, u, v, \tau)$ is a motion corresponding to the strategy $\mathfrak{u}$ and the initial position (t_0, x_0) if the following conditions are fulfilled:

(i) $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[t_0,T]}, P)$ is a filtered probability space;

(ii) $\Omega = \Omega^U \times \Omega'$, $\mathcal{F} = \mathcal{F}^U \otimes \mathcal{F}'$, $\mathcal{F}_t = \mathcal{F}_t^U \otimes \mathcal{F}_t'$, where $(\Omega', \mathcal{F}', \{\mathcal{F}_t'\}_{t\in[t_0,T]})$ is a filtered measurable space;

(iii) u and v are $\{\mathcal{F}_t\}_{t\in[t_0,T]}$-progressively measurable processes taking values in U and V respectively;

(iv) X is $\{\mathcal{F}_t\}_{t\in[t_0,T]}$-adapted process with values in $\mathbb{R}^d$;

(v) for P-a.e. $\omega = (\omega^U, \omega') \in \Omega$,

$$\frac{d}{dt}X(t,\omega) = f(t, X(t,\omega), u(t,\omega), v(t,\omega)),\ X(t,\omega) = x_0,$$

whereas

$$u(t, \omega^U, \omega') = u_{X(t,\omega^U,\omega')}(t, \omega^U);$$

(vi) τ is the stopping time w.r.t. the filtration $\{\mathcal{F}_t\}_{t\in[t_0,T]}$ taking values in $[t_0, T]$ such that, for P-a.e. $\omega = (\omega^U, \omega') \in \Omega$,

$$\tau(\omega^U, \omega') = \tau_{X(\omega^U,\omega')}(\omega^U);$$

(vii) P is such that, for every $A \in \mathcal{F}^U$ and $x(\cdot) \in C([t_0, T], \mathbb{R}^d)$,

$$P(A \times \Omega' | X(\cdot) = x(\cdot)) = P^U_{x(\cdot)}(A).$$

If the first player uses the strategy $\mathfrak{u}$, then his/her guarantee at the position (t_0, x_0) is evaluated by the quantity

$$J^1(t_0, x_0, \mathfrak{u}) = \sup \mathbb{E} g(\tau, X(\tau)), \tag{1.5}$$

where $\sup$ is taken w.r.t. $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[t_0,T]}, P, X, u, v, \tau)$ providing a motion corresponding to the strategy $\mathfrak{u}$ and the initial position (t_0, x_0), whilst $\mathbb{E}$ stands for the expectation according to the probability P.

The definition of the second player's strategies is slightly non-symmetric.

Definition 3. We say that $\mathfrak{v} = (\Omega^V, \mathcal{F}^V, \{\mathcal{F}_t^V\}_{t\in[t_0,T]}, P^V_{x(\cdot)}, v_{x(\cdot)})$ is a strategy of the second player on $[t_0, T]$ if the following conditions are fulfilled:

(i) $(\Omega^V, \mathcal{F}^V, \{\mathcal{F}_t^V\}_{t\in[t_0,T]})$ is a filtered measurable space;

(ii) for each $x(\cdot) \in C([t_0, T])$, $P^V_{x(\cdot)}$ is a probability on $\mathcal{F}^V$, $v_{x(\cdot)}$ is a $\{\mathcal{F}_t^V\}_{t\in[t_0,T]}$-progressively measurable process with values in V;

(iii) if, for some $r \in [t_0, T]$, $x(t) = y(t)$ when $t \in [t_0, r]$, then

$$P^V_{x(\cdot)}(A) = P^V_{y(\cdot)}(A) \text{ for every } A \in \mathcal{F}^V_r$$

and, for any $t \in [t_0, r]$,

$$v_{x(\cdot)}(t) = v_{y(\cdot)}(t) \quad P^V_{x(\cdot)} - \text{a.s.}$$

Motion of the system corresponding to the second player's strategy $\mathfrak{v}$ and the initial position (t_0, x_0) is introduced as follows.

Definition 4. We say that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[s,r]}, P, X, u, v, \tau)$ is a motion realizing the strategy $\mathfrak{v}$ and the initial position (t_0, x_0) provided that

(i) $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[t_0,T]}, P)$ is a filtered probability space;

(ii) $\Omega = \Omega^V \times \Omega'$, $\mathcal{F} = \mathcal{F}^V \otimes \mathcal{F}'$, $\mathcal{F}_t = \mathcal{F}_t^V \otimes \mathcal{F}_t'$, where $(\Omega', \mathcal{F}', \{\mathcal{F}_t'\}_{t\in[t_0,T]})$ is a filtered measurable space;

(iii) u and v are $\{\mathcal{F}_t\}_{t\in[t_0,T]}$-progressively measurable processes taking values in U and V respectively;

(iv) X is a $\{\mathcal{F}_t\}_{t\in[t_0,T]}$-adapted process with values in $\mathbb{R}^d$;

(v) for P-a.e. $\omega = (\omega^V, \omega') \in \Omega$,

$$\frac{d}{dt}X(t) = f(t, X(t,\omega), u(t,\omega), v(t,\omega)),\ X(t,\omega) = x_0,$$

whereas

$$v(t, \omega^V, \omega') = v_{X(t,\omega^V,\omega')}(t, \omega^V);$$

(vi) τ is the stopping time w. r. t. the filtration $\{\mathcal{F}_t\}_{t\in[t_0,T]}$ taking values in $[t_0, T]$;

(vii) P is such that, for every $A \in \mathcal{F}^V$ and $x(\cdot) \in C([t_0, T], \mathbb{R}^d)$,

$$P(A \times \Omega' | X(\cdot) = x(\cdot)) = P^V_{x(\cdot)}(A).$$

The guarantee of the second player in the case when he/she plays with the strategy $\mathfrak{v}$ from the initial position (t_0, x_0) is given by

$$J^2(t_0, x_0, \mathfrak{v}) = \inf \mathbb{E} g(\tau, X(\tau)), \tag{1.6}$$

where, as above, inf is taken w. r. t. $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[t_0,T]}, P, X, u, v, \tau)$ providing a motion corresponding to the strategy $\mathfrak{v}$ and the initial position (t_0, x_0), when $\mathbb{E}$ stands for the expectation according to the probability P.

One can also introduce the upper and lower value functions:

$$\mathrm{Val}^+(t_0, x_0) \triangleq \inf\{J^1(t_0, x_0, \mathfrak{u})\colon \mathfrak{u} \text{ is a strategy of the first player}\},$$

$$\mathrm{Val}^-(t_0, x_0) \triangleq \sup\{J^2(t_0, x_0, \mathfrak{v})\colon \mathfrak{v} \text{ is a strategy of the second player}\}.$$

Obviously,

$$\mathrm{Val}^+(t_0, x_0) \geq \mathrm{Val}^-(t_0, x_0).$$

§ 2. Main results

In this section, we introduce the auxiliary stochastic controlled system and the concept of u- and v-stability for this system (see Definitions 6, 7). Using these notions, we formulate the main results (see Theorem 1, 2).

It is convenient to introduce the dynamics of the auxiliary system using the generator technique. Let $\mathcal{D}^i$ be a linear subspace of $C^2(\mathbb{R}^d)$ containing all bounded twice differentiable functions as well as linear and quadratic functions. Notice that the evolution given of original system (1.1) corresponds to the generator

$$L^1_t[u,v]\phi(x) \triangleq \langle f(t,x,u,v), \nabla\phi(x)\rangle. \tag{2.1}$$

This means that, given open-loop controls $u(\cdot)$ and $v(\cdot)$ and the motion of system (1.1) generated by these controls, we have that, for every $\phi \in \mathcal{D}^1 \triangleq C^1(\mathbb{R}^d)$,

$$\phi(x(r)) - \phi(x(s)) = \int_s^r L^1_t[u(t),v(t)]\phi(x(t))dt = \int_s^r \langle f(t,x(t),u(t),v(t)), \nabla\phi(x(t))\rangle dt.$$

The auxiliary control system is described by the Lévy–Khintchine type generator L^2 acting on $\phi \in \mathcal{D}^2$ by the rule:

$$\begin{aligned} L^2_t[u,v]\phi(x) \triangleq \frac{1}{2}\langle G^a(t,x,u,v)\nabla, \nabla\rangle\phi(x) + \langle f^a(t,x,u,v), \nabla\rangle\phi(x) \\ + \int_{\mathbb{R}^d}[\phi(x+y) - \phi(x) - \langle y, \nabla\phi(x)\rangle \mathbf{1}_{B_1}(y)]\nu^a(t,x,u,v,dy). \end{aligned} \tag{2.2}$$

Here B_1 denotes the unit ball centered at the origin, $\mathbf{1}$ stands for the indicator function, whilst the superindex a serves to underline the fact that L^2 determines the auxiliary system. The motion according to this generator is defined using so called martingale problem (see Definition 5 below).

For the auxiliary control system, we consider the relaxed stochastic controls. In this case, a player's control is regarded as a stochastic process taking values in the set of distributions over the control space. Below, if A is a metric space, then $\mathcal{P}(A)$ stands for the set of probabilities over the set A. Furthermore, if $a \in A$, then δ_a denotes the measure concentrated in a. The mapping $a \mapsto \delta_a$ provides the natural embedding of A into $\mathcal{P}(A)$.

Definition 5. Let $s, r \in [0,T]$, $s < r$. We say that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[s,r]}, P, X, \xi, \zeta)$ is a control process admissible for the generator L^2 on $[s,r]$ provided that

1) $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[s,r]}, P)$ is a probability space;
2) X is a $\{\mathcal{F}_t\}_{t\in[s,r]}$-adapted stochastic process with values in $\mathbb{R}^d$;
3) ξ (respectively, ζ) is $\{\mathcal{F}_t\}_{t\in[s,r]}$-progressively measurable processes taking values in $\mathcal{P}(U)$ (respectively, in $\mathcal{P}(V)$);
4) for any $\phi \in \mathcal{D}^2$,

$$\phi(X(t)) - \int_s^t \int_U \int_V L^2_\tau[u,v]\phi(X(\tau))\xi(du)\zeta(dv)d\tau$$

is a $\{\mathcal{F}_t\}_{t\in[s,r]}$-martingale.

The definition of the u-stability is a stochastic analog of the definition introduced in [29, p. 205].

Definition 6. We say that a lower semicontinuous function $\varphi\colon [0,T]\times\mathbb{R}^d\to\mathbb{R}$ is u-stable with respect to the generator L^2 if $\varphi(T,x)=g(T,x)$ on $[0,T]\times\mathbb{R}^d$ and, given $s,r\in[0,T]$, $s<r$, one can choose a filtered measurable space $(\widehat{\Omega}^{s,r},\widehat{\mathcal{F}}^{s,r},\{\widehat{\mathcal{F}}^{s,r}_t\}_{t\in[s,r]})$ such that, for every $y\in\mathbb{R}^d$ and $v\in V$, there exist a probability $\widehat{P}^{s,r}_{y,v}$, processes $\widehat{Y}^{s,r}_{y,v}$, $\hat{\xi}^{s,r}_{y,v}$ and a stopping time w. r. t. the filtration $\{\widehat{\mathcal{F}}^{s,r}_t\}_{t\in[s,r]}$ $\hat{\tau}^{s,r}_{y,v}$ taking value in $[s,r]\cup\{+\infty\}$ provided that

(U1) $(\widehat{\Omega}^{s,r},\widehat{\mathcal{F}}^{s,r},\{\widehat{\mathcal{F}}^{s,r}_t\}_{t\in[s,r]},\widehat{P}^{s,r}_{y,v},\widehat{Y}^{s,r}_{y,v},\hat{\xi}^{s,r}_{y,v},\delta_v)$ is a control process admissible for the generator L^2 on $[s,r]$;

(U2) $\widehat{Y}^{s,r}_{y,v}(s)=y$, $\widehat{P}^{s,r}_{y,v}$-a.s.;

(U3)
$$\varphi(s,y)\geq\widehat{\mathbb{E}}^{s,r}_{y,v}\hat{\varphi}^{s,r}_{y,v},$$
where $\widehat{\mathbb{E}}^{s,r}_{y,v}$ stands for the expectation according to the probability $\widehat{P}^{s,r}_{y,v}$, and the random variable $\hat{\phi}^{s,r}_{y,v}$ is defined by the rule

$$\hat{\phi}^{s,r}_{y,v}\triangleq\begin{cases} g(\hat{\tau}^{s,r}_{y,v},\widehat{Y}^{s,r}_{y,v}(\hat{\tau}^{s,r}_{y,v})), & \hat{\tau}^{s,r}_{y,v}\in[s,r],\\ \varphi(r,\widehat{Y}^{s,r}_{y,v}(r)) & \hat{\tau}^{s,r}_{y,v}=+\infty.\end{cases}\tag{2.3}$$

The definition of the v-stability property is given as follows.

Definition 7. We say that an upper semicontinuous function $\varphi\colon[0,T]\times\mathbb{R}^d\to\mathbb{R}$ is v-stable with respect to the generator L^2 if $\varphi(t,x)\leq g(t,x)$ and, given $s,r\in[0,T]$, $s<r$, one can choose a filtered measurable space $(\widetilde{\Omega}^{s,r},\widetilde{\mathcal{F}}^{s,r},\{\widetilde{\mathcal{F}}^{s,r}_t\}_{t\in[s,r]})$ such that, for every $y\in\mathbb{R}^d$ and $u\in U$, there exist a probability $\widetilde{P}^{s,r}_{y,u}$, and processes $\widetilde{Y}^{s,r}_{y,u}$, $\tilde{\zeta}^{s,r}_{y,u}$ provided that

(L1) $(\widetilde{\Omega}^{s,r},\widetilde{\mathcal{F}}^{s,r},\{\widetilde{\mathcal{F}}^{s,r}_t\}_{t\in[s,r]},\widetilde{P}^{s,r}_{y,u},\widetilde{Y}^{s,r}_{y,u},\delta_u,\tilde{\zeta}^{s,r}_{y,u})$ is a control process admissible for the generator L^2;

(L2) $\widetilde{Y}^{s,r}_{y,u}(s)=y$, $\widetilde{P}^{s,r}_{y,u}$-a.s.;

(L3) $\varphi(s,y)\leq\widetilde{\mathbb{E}}^{s,r}_{y,u}\varphi(r,\widetilde{Y}^{s,r}_{y,u}(r))$, where $\widetilde{\mathbb{E}}^{s,r}_{y,u}$ stands for the expectation according to the probability $\widetilde{P}^{s,r}_{y,u}$.

To formulate the main results, put

$$\Sigma(t,x,u,v)\triangleq\sum_{j=1}^d G^a_{jj}(t,x,u,v)+\int_{\mathbb{R}^d}\|y\|^2\nu^a(t,x,u,v,dy)\geq 0;$$

$$b(t,x,u,v)\triangleq f^a(t,x,u,v)+\int_{\mathbb{R}^d\setminus B_1}y\nu^a(t,x,u,v,dy).\tag{2.4}$$

The function Σ provides the measure of stochasticity of the generator L^2, when the function b is an effective drift in the auxiliary continuous-time game. Below we assume that the functions G^a, f^a and ν^a are continuous, when the functions Σ and b are also bounded.

Furthermore, let ς_g denote the modulus of continuity of the function g w. r. t. x, i.e.,

$$\varsigma_g(\epsilon)\triangleq\sup\{|g(t,x')-g(t,x'')|\colon t\in[0,T],\quad x',x''\in\mathbb{R}^d,\|x'-x''\|\leq\epsilon\}.$$

The main results of the paper are as follows.

Theorem 1. *There exists a constant C_* dependent only on the dynamics of the original function f and a function $\alpha_*\colon [0,+\infty) \to [0,+\infty)$ vanishing at zero such that, if φ^+ is a u-stable function for the generator L^2, when*

$$\Sigma(t,x,u,v) \leq \varepsilon^2, \tag{2.5}$$

$$\|b(t,x,u,v) - f(t,x,u,v)\| \leq \varepsilon. \tag{2.6}$$

for some positive ε and every $t \in [0,T]$, $x \in \mathbb{R}^d$, $u \in U$, $v \in V$, then, given an initial position $t_0 \in [0,T] \times \mathbb{R}^d$ and a partition Δ of the time interval $[t_0,T]$, one can construct a first player strategy with memory $\mathfrak{u}^\Delta$ satisfying the following inequality: for any $x_0 \in \mathbb{R}^d$,

$$J^1(t_0,x_0,\mathfrak{u}^\Delta) \leq \varphi^+(t_0,x_0) + \varsigma_g(C_*\varepsilon + \alpha_*(d(\Delta))). \tag{2.7}$$

This theorem obviously implies the following.

Corollary 1. *If φ^+ is a u-stable function, and ε is such that (2.5) and (2.6) hold, then*

$$\mathrm{Val}^+(t_0,x_0) \leq \varphi^+(t_0,x_0) + \varsigma_g(C_*\varepsilon).$$

Hereinafter, the constant C_* is the same as in Theorem 1.

The result for the second player' strategy is formulated in a similar way.

Theorem 2. *Assume that φ^- is a v-stable function for the generator L^2 and ε satisfies (2.5), (2.6). Then, given $t_0 \in [0,T]$ and a partition Δ of $[t_0,T]$, there exists a second player's strategy $\mathfrak{v}^\Delta$ such that, for any $x_0 \in \mathbb{R}^d$,*

$$J^2(t_0,x_0,\mathfrak{v}^\Delta) \geq \varphi^-(t_0,x_0) - \varsigma_g(C_*\varepsilon + \alpha_*(d(\Delta))). \tag{2.8}$$

Corollary 2. *If φ^+ is a v-stable function, and ε is such that (2.5) and (2.6) hold, then*

$$\mathrm{Val}^-(t_0,x_0) \geq \varphi^-(t_0,x_0) - \varsigma_g(C_*\varepsilon).$$

The following statement shows that the differential game with minimal cost, i. e., the differential game with the dynamics given by (1.1) and payoff (1.2) is equivalent to the differential game with stopping governed by the first player (1.1), (1.3). For differential game (1.1), (1.2) we use the feedback formalization proposed by Krasovskii and Subbotin. Despite its name, this formalization uses a short history. Namely, the strategy of the first player is a mapping $\mathfrak{u}\colon [0,T] \times \mathbb{R}^d \to U$, whereas the second player's strategy is a function $v\colon [0,T] \times \mathbb{R}^d \to V$. The players' strategies generate the motions as follows. If $(t_0,x_0) \in [0,T] \times \mathbb{R}^d$, $\mathfrak{u}\colon [0,T] \times \mathbb{R}^d \to U$ is a strategy of the first player, $\Delta = \{t_k\}_{k=0}^N$ is a partition of the time interval $[t_0,T]$, then on $[t_k,t_{k+1}]$ the motion satisfies the differential equation

$$\frac{d}{dt}x(t) = f(t,x(t),\mathfrak{u}(t_k,x(t_k)),v(t)),$$

and the initial condition $x(t_0) = x_0$ where $v(\cdot)$ is a second player's control. If we denote the set of the corresponding motions by $\mathcal{X}_1(t_0,x_0,\mathfrak{u},\Delta)$, then the outcome of the first player is evaluated by the quantity:

$$J^1_f[t_0,x_0,\mathfrak{u},\Delta] \triangleq \sup_{x(\cdot)\in\mathcal{X}_1(t_0,x_0,\mathfrak{u},\Delta)} \min_{t\in[t_0,T]} g(t,x(t)).$$

The upper value of the game is defined as follows:

$$\mathrm{Val}^+_0(t_0,x_0) \triangleq \inf_{u,\Delta} J^1_f[t_0,x_0,\mathfrak{u},\Delta].$$

Analogously, one can introduce the set of motions generated by the initial position (t_0, x_0), the second player's strategy $\mathbb{v}$ and the partition Δ $\mathcal{X}_2(t_0, x_0, \mathbb{v}, \Delta)$. The second player's outcome in this case is determined by the formula

$$J_f^2[t_0, x_0, v, \Delta] \triangleq \inf_{x(\cdot)\in\mathcal{X}_2(t_0,x_0,\mathbb{v},\Delta)} \min_{t\in[t_0,T]} g(t, x(t));$$

whilst the lower value function is given by

$$\mathrm{Val}_0^-(t_0, x_0) \triangleq \sup_{v,\Delta} J_f^2[t_0, x_0, v, \Delta].$$

It is shown [24] that differential game (1.1), (1.2) has the value, i. e.,

$$\mathrm{Val}_0^+(t_0, x_0) = \mathrm{Val}_0^-(t_0, x_0) = \mathrm{Val}_0(t_0, x_0).$$

Theorem 3. *Differential game* (1.1)*,* (1.3) *has a value in the class of stochastic strategies with memory. This value is equal to the value of differential game* (1.1)*,* (1.2)*.*

This theorem is proved in Section 5.

§ 3. Control with guide strategies for the first player

In this section, we define the strategy of the first player that is used to prove Theorem 1. Let φ be a u-stable function, t_0 be an initial time, and let $\Delta = \{t_i\}_{i=0}^N$ be a partition of the time interval. Below, we introduce the strategy $\mathfrak{u}^\Delta = (\Omega^{U,\Delta}, \mathcal{F}^{U,\Delta}, \{\mathcal{F}_t^{U,\Delta}\}_{t\in[0,T]}, P_{x(\cdot)}^{U,\Delta}, u^\Delta, \tau^{U,\Delta})$ providing an estimate desired in Theorem 1. This strategy realize the concepts of control with guide and the extremal shift rule first proposed by Krasovskii and Subbotin. To this end, for each $t \in [0, T]$, $x, y \in \mathbb{R}^d$, let $\hat{u}(t, x, y)$, $\hat{v}(t, x, y)$ be such that

$$\max_{v\in V}\langle x - y, f(t, x, \hat{u}(t, x, y), v) = \min_{u\in U}\max_{v\in V}\langle x - y, f(t, x, u, v), \tag{3.1}$$

$$\min_{u\in U}\langle x - y, f(t, x, u, \hat{v}(t, x, y)) = \min_{u\in U}\max_{v\in V}\langle x - y, f(t, x, u, v). \tag{3.2}$$

By the Measurable Maximum Theorem [1, Theorem 18.19], one can assume that the functions $\hat{u}$ and $\hat{v}$ are measurable.

By definition of u-stability (see Definition 6), we choose, for the time interval $[t_k, t_{k+1}]$, a filtered measurable space $(\widehat{\Omega}^{t_k,t_{k+1}}, \widehat{\mathcal{F}}^{t_k,t_{k+1}}, \{\widehat{\mathcal{F}}_t^{t_k,t_{k+1}}\}_{t\in[t_k,t_{k+1}]})$. Without loss of generality, we assume that $\widehat{\mathcal{F}}^{t_k,t_{k+1}} = \widehat{\mathcal{F}}_{t_{k+1}}^{t_k,t_{k+1}}$.

Set,

$$\Omega^{U,\Delta} \triangleq \bigtimes_{k=0}^{N-1} \Omega^{t_k,t_{k+1}}.$$

In the following, we write elements of Ω as N-tuples $(\omega_0, \dots, \omega_{N-1})$.

For $k = 0, \dots, N-1$, $t \in [t_k, t_{k+1}]$, let

$$\mathcal{F}_t^{U,\Delta} \triangleq \bigotimes_{i=0}^{k-1} \widehat{\mathcal{F}}^{t_i,t_{i+1}} \otimes \widehat{\mathcal{F}}_t^{t_k,t_{k+1}} \otimes \bigotimes_{i=k+1}^{N-1} \{\varnothing, \widehat{\Omega}^{t_i,t_{i+1}}\}.$$

Denote

$$\mathcal{F}^{U,\Delta} \triangleq \mathcal{F}_T^{U,\Delta}.$$

Now, let $x(\cdot) \in C([t_0, T]; \mathbb{R}^d)$. We will define $P_{x(\cdot)}^{U,\Delta}(A)$, $u_{x(\cdot)}^\Delta(t)$ and a stopping time $\tau_{x(\cdot)}^{U,\Delta}$ using an auxiliary stochastic process $Y_{x(\cdot)}^1(\cdot)$ and sequence of auxiliary stopping times $\tau_{x(\cdot),k}$, $k = 0, \dots, N-1$ inductively as follows.

1. Put $x_0 = y_0 \triangleq x(t_0)$, $Y(t_0) \triangleq x(t_0)$. For, $t \in [t_0, t_k]$, choose

$$u^{\Delta}_{x(\cdot)}(t) = \hat{u}(t_0, x_0, y_0).$$

Further,

$$v_0 \triangleq \hat{v}(t_0, x_0, y_0).$$

Since φ^+ is a u-stable function, there exist a probability $\widehat{P}^{t_0,t_1}_{x_0,v_0}$, a stochastic control $\hat{\xi}^{t_0,t_1}_{x_0,v_0}$, a motion $\widehat{Y}^{t_0,t_1}_{x_0,v_0}$ and a stopping time $\hat{\tau}^{t_0,t_1}_{x_0,v_0}$ such that conditions (U1)–(U3) are fulfilled for $y = x_0$ and $v = v_0$. Now, we set, for $t \in [t_0, t_1]$,

$$Y^1_{x(\cdot)}(t, (\omega_0, \ldots, \omega_{N-1})) = \widetilde{Y}^{t_0,t_1}(t, \omega_0).$$

For $A_0 \in \widehat{\mathcal{F}}^{t_0,t_1}$, let

$$P^{U,\Delta}_{x(\cdot)}(A_0 \times \widehat{\Omega}^{t_1,t_1} \ldots \times \widehat{\Omega}^{t_{N-1},t_N}) \triangleq \widehat{P}^{t_0,t_1}_{x_0,v_0}(A_0).$$

Notice that this formula defines $P^{U,\Delta}_{x(\cdot)}$ on $\mathcal{F}^{U,\Delta}_{t_1}$. Finally, put

$$\tau_{x(\cdot),0}(\omega_0, \ldots, \omega_{N-1}) \triangleq \hat{\tau}^{s,r}_{y_0,v_0}(\omega_0).$$

2. Now, assume that we have constructed, for some l, $t \in [t_0, t_l]$ and $A \in \mathcal{F}^{U,\Delta}_{t_l}$, the probability $P^{U,\Delta}_{x(\cdot)}(A)$, the control $u_{x(\cdot)}(t)$, the motion $Y^1_{x(\cdot)}(t)$ and the stopping time $\tau_{x(\cdot),l-1}$ taking values in $[t_0, t_l] \cup \{+\infty\}$.

   Denote $x_l \triangleq x(t_l)$. Notice that $\hat{u}(t_l, x_l, Y(t_l))$ and $\hat{v}(t_l, x_l, Y(t_l))$ are random variables with values in U and V respectively. If $t \in [t_l, t_{l+1}]$, then

$$u^{\Delta}_{x(\cdot)}(t) \triangleq \hat{u}(t_l, x_l, Y(t_l)).$$

   Recall that, for each $y \in \mathbb{R}^d$, $v \in V$, there exist a probability $\widehat{P}^{t_l,t_{l+1}}_{x_l,v}$, a stochastic control $\hat{\xi}^{t_l,t_{l+1}}_{y,v}$, a motion $\widehat{Y}^{t_l,t_{l+1}}_{y,v}$ and a stopping time $\hat{\tau}^{t_l,t_{l+1}}_{y,v}$. Choosing

$$y_l = Y^1_{x(\cdot)}(t_l) \text{ and } v_l = \hat{v}(t_l, x_l, y_l),$$

   we define

$$Y^1_{x(\cdot)}(t) \triangleq \widehat{Y}^{t_l,t_{l+1}}_{y_l,v_l}, \quad t \in [t_l, t_{l+1}].$$

   Further, let $\mu_{x(\cdot),l}(dy)$ be a distribution of $Y^1_{x(\cdot)}(t_l)$. For $A_k \in \widehat{\mathcal{F}}^{t_k,t_{k+1}}_{t_{k+1}}$, $k = 0, \ldots, l$, we put

$$P^{U,\Delta}_{x(\cdot)}(A_0 \times \ldots \times A_{l-1} \times A_l \times \widehat{\Omega}^{t_{l+1},t_{l+2}} \times \ldots \times \widehat{\Omega}^{t_{N-1},t_N}) \triangleq \int_{\mathbb{R}^d} P^{U,\Delta}_{x(\cdot)}(A_0 \times \ldots \times A_{l-1} \times \widehat{\Omega}^{t_l,t_{l+1}} \times \widehat{\Omega}^{t_{l+1},t_{l+2}} \times \ldots \times \widehat{\Omega}^{t_{N-1},t_N} | Y^1_{x(\cdot)}(t_l) = y) \widehat{P}^{t_l,t_{l+1}}_{y_l,v_l}(A_l)\mu_{x(\cdot),l}(dy).$$

   Notice that one can extend the probability $P^{U,\Delta}_{x(\cdot)}$ to the whole σ-algebra $\mathcal{F}^{U,\Delta}_{t_{l+1}}$.

   Finally, set

$$\tau_{x(\cdot),l} \triangleq \tau_{x(\cdot),l-1} \wedge \hat{\tau}^{t_l,t_{l+1}}_{y_l,v_l}.$$

To complete the construction, we set

$$\tau^{U,\Delta}_{x(\cdot)} \triangleq \tau_{x(\cdot),N-1} \wedge T.$$

Further, let us examine the properties of a motion corresponding to the strategy $\mathfrak{u}^\Delta$ that is a tuple $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[s,r]}, P, X, u, v, \tau)$. For $t \in [t_l, t_{l+1})$,

$$Y^1(t) \triangleq Y^1_{X(t_l)}(t), \tag{3.3}$$

$$\hat{v}^1(t) \triangleq v_l = \hat{v}(t_l, X(t_l), Y^1(t_l)), \tag{3.4}$$

$$\hat{\xi}^1(t) \triangleq \hat{\xi}^{t_l,t_{l+1}}_{Y^1(t_l),\hat{v}^1(t_l)}. \tag{3.5}$$

The proof of Theorem 1 is based on Lemmas 1–3.

Lemma 1. *The following statements hold.*

1. *For each $\phi \in \mathcal{D}^1$, the process*

$$\phi(X(t)) - \int_{t_0}^{t} L^1_\tau[u(\tau), v(\tau)]\phi(X(\tau))dt$$

   is a $\{\mathcal{F}_t\}_{t\in[t_0,T]}$-martingale.

2. *For each $\phi \in \mathcal{D}^2$, the process*

$$\phi(Y^1(t)) - \int_{t_0}^{t}\int_U L^2_\tau[u, \hat{v}^1(\theta)]\phi(Y^1(\theta))\hat{\xi}^1(\theta, du)d\theta$$

   is a $\{\mathcal{F}_t\}_{t\in[t_0,T]}$-martingale.

The proof of the first statement is a simple consequence of the definition of generator L^1 (see (2.1)), whereas the second statement follows from the very definition of strategy $\mathfrak{u}$ and the process Y^1.

Lemma 2. $\varphi^+(t_0, x_0) \geq \mathbb{E}g(\tau, Y^1(\tau))$.

P r o o f. First, notice that, if τ takes values in $[t_k, t_{k+1})$, then

$$\tau = \hat{\tau}^{t_k,t_{k+1}}_{Y^1(t_k),\hat{v}(t_k,X(t_k),Y(t_k))}, \tag{3.6}$$

$$Y^1(\tau) = \widehat{Y}^{t_k,t_{k+1}}_{Y^1(t_k),\hat{v}(t_k,X(t_k),Y(t_k))}(\tau). \tag{3.7}$$

Moreover, $\varphi^+(T, x) = g(T, x)$.

Now, we use the backward induction by k to prove the following inequality:

$$\varphi^+(t_k, Y^1(t_k)) \geq \mathbb{E}(g(\tau, Y^1(\tau))\mathbb{1}_{\tau\in[t_k,T]}|\mathcal{F}_{t_k}). \tag{3.8}$$

We have that

$$g(\tau, Y^1(\tau))\mathbb{1}_{\tau\in[t_k,T]} = g(T, Y^1(T))\mathbb{1}_{\tau=T} + g(\tau, Y^1(\tau))\mathbb{1}_{\tau\in[t_k,T)}.$$

Using equalities (3.6), (3.7) and condition (U3), we conclude that

$$\begin{aligned}&\mathbb{E}(g(\tau, Y^1(\tau))\mathbb{1}_{\tau\in[t_{N-1},T]}|\mathcal{F}_{t_{N-1}})\\ &\quad= \mathbb{E}(\varphi(T, Y^1(T))\mathbb{1}_{\tau=T} + g(\tau, Y^1(\tau))\mathbb{1}_{\tau\in[t_{N-1},T)}|\mathcal{F}_{t_{N-1}})\\ &\quad\leq \varphi^+(t_{N-1}, Y^1(t_{N-1})).\end{aligned}$$

This proves (3.8) for $k = N - 1$.

Assume now that (3.8) is fulfilled for $k = l + 1$. Notice that

$$\begin{aligned}\mathbb{E}(g(\tau,Y^1(\tau))\mathbb{1}_{\tau\in[t_l,T]}|\mathcal{F}_{t_l})\\ &= \mathbb{E}(g(\tau,Y^1(\tau))\mathbb{1}_{\tau\in[t_l,t_{l+1})}|\mathcal{F}_{t_l}) + \mathbb{E}(g(\tau,Y^1(\tau))\mathbb{1}_{\tau\in[t_{l+1},T]}|\mathcal{F}_{t_l})\\ &= \mathbb{E}(g(\tau,Y^1(\tau))\mathbb{1}_{\tau\in[t_l,t_{l+1})}|\mathcal{F}_{t_l}) + \mathbb{E}(\mathbb{E}(g(\tau,Y^1(\tau))|\mathcal{F}_{t_{l+1}})\mathbb{1}_{\tau\in[t_{l+1},T]}|\mathcal{F}_{t_l})\end{aligned}$$

Applying (3.8) for $k = l + 1$, we arrive at the following inequality:

$$\begin{aligned}\mathbb{E}(g(\tau,Y^1(\tau))\mathbb{1}_{\tau\in[t_l,T]}|\mathcal{F}_{t_l}) \leq \mathbb{E}(g(\tau,Y^1(\tau))\mathbb{1}_{\tau\in[t_l,t_{l+1})}|\mathcal{F}_{t_l}) +\\ + \mathbb{E}(\varphi^+(t_{l+1},Y^1(t_{l+1}))\mathbb{1}_{\tau\in[t_{l+1},T]}|\mathcal{F}_{t_{l+1}}).\end{aligned}$$

As above, using (3.6), (3.7) and condition (U3), we estimate the right-hand side of this inequality from below by $\varphi^+(t_l, Y^1(t_l))$. This proves (3.8) for $k = l$.

Since $Y^1(t_0) = x_0$, estimate (3.8) implies the statement of the lemma. □

Lemma 3. *The following estimate holds true:*

$$\mathbb{E}\|X(\tau) - Y^1(\tau)\|^2 \leq C_*\varepsilon^2 + \alpha_*(d(\Delta)).$$

P r o o f. We shall estimate the following quantity

$$\mathbb{E}\|X(\tau\wedge t_k) - Y^1(\tau\wedge t_k)\|^2.$$

Notice that

$$\mathbb{E}\|X(\tau\wedge t_0) - Y^1(\tau\wedge t_0)\|^2 = 0, \tag{3.9}$$

$$\mathbb{E}\|X(\tau\wedge t_N) - Y^1(\tau\wedge t_N)\|^2 = \mathbb{E}\|X(\tau) - Y^1(\tau)\|^2. \tag{3.10}$$

We have that

$$\begin{aligned}\|X(\tau\wedge t_{k+1}) - Y^1(\tau\wedge t_{k+1})\|^2 &= \|X(\tau\wedge t_k) - Y^1(\tau\wedge t_k)\|^2\\ &+ \|X(\tau\wedge t_{k+1}) - X(\tau\wedge t_k)\|^2 + \|Y^1(\tau\wedge t_{k+1}) - Y^1(\tau\wedge t_k)\|^2\\ &+ 2\langle X(\tau\wedge t_k) - Y^1(\tau\wedge t_k), X(\tau\wedge t_{k+1}) - X(\tau\wedge t_k)\rangle\\ &- 2\langle X(\tau\wedge t_k) - Y^1(\tau\wedge t_k), Y^1(\tau\wedge t_{k+1}) - Y^1(\tau\wedge t_k)\rangle\\ &- 2\langle X(\tau\wedge t_{k+1}) - X(\tau\wedge t_k), Y^1(\tau\wedge t_{k+1}) - Y(\tau\wedge t_k)\rangle\\ \leq \|X(\tau\wedge t_k) - Y^1(\tau\wedge t_k)\|^2\\ &+ 2\|X(\tau\wedge t_{k+1}) - X(\tau\wedge t_k)\|^2 + 2\|Y^1(\tau\wedge t_{k+1}) - Y^1(\tau\wedge t_k)\|^2\\ &+ 2\langle X(\tau\wedge t_k) - Y^1(\tau\wedge t_k), X(\tau\wedge t_{k+1}) - X(\tau\wedge t_k)\rangle\\ &- 2\langle X(\tau\wedge t_k) - Y^1(\tau\wedge t_k), Y^1(\tau\wedge t_{k+1}) - Y^1(\tau\wedge t_k)\rangle.\end{aligned} \tag{3.11}$$

Further, since $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[t_0,T]}, P, X, u, v, \tau)$ is a motion corresponding to the strategy $\mathfrak{u}^\Delta$, we have that

$$\begin{aligned}\mathbb{E}\|X(\tau\wedge t_{k+1}) - X(\tau\wedge t_k)\|^2 &= \mathbb{E}\left\|\int_{\tau\wedge t_k}^{\tau\wedge t_{k+1}} f(t,X(t),u(t),v(t))dt\right\|^2\\ &\leq (t_{k+1} - t_k)^2 \sup_{t\in[0,T]} \|f(t,X(t),u(t),v(t))\| \leq (t_{k+1} - t_k)^2 R^2.\end{aligned} \tag{3.12}$$

Recall that R is an upper bound of $\|f\|$ (see (1.4)).

To estimate $\mathbb{E}\|Y^1(\tau\wedge t_{k+1})-Y^1(\tau\wedge t_k)\|^2$, notice that P-a.s.

$$\|Y^1(\tau\wedge t_{k+1})-Y^1(\tau\wedge t_k)\|^2=\|Y^1((\tau\wedge t_{k+1})\vee t_k)-Y^1(t_k)\|^2.$$

From [2, Lemma 13], it follows that

$$\mathbb{E}\|Y^1(t)-Y^1(t_k)\|^2\leq\varepsilon^2(t-t_k)+\alpha_1(t-t_k)\cdot(t-t_k),$$

where $\alpha_1(\cdot)$ is a nonnegative monotone function vanishing at zero. Using this, we obtain that

$$\begin{aligned}\mathbb{E}\|Y^1((\tau\wedge t_{k+1})\vee t_k)-Y^1(t_k)\|^2\\ \leq\varepsilon^2(t_{k+1}-t_k)+\alpha_1(t_{k+1}-t_k)\cdot(t_{k+1}-t_k).\end{aligned}$$

Hence

$$\mathbb{E}\|Y^1(\tau\wedge t_{k+1})-Y^1(\tau\wedge t_k)\|^2\leq\varepsilon^2(t_{k+1}-t_k)+\alpha_1(t_{k+1}-t_k)\cdot(t_{k+1}-t_k).\tag{3.13}$$

Now let us consider the term

$$\mathbb{E}\langle X(\tau\wedge t_k)-Y^1(\tau\wedge t_k),X(\tau\wedge t_{k+1})-X(\tau\wedge t_k)\rangle.$$

We have that

$$X(\tau\wedge t_{k+1})-X(\tau\wedge t_k)=\int_{\tau\wedge t_k}^{\tau\wedge t_{k+1}}f(t,X(t),u(t),v(t))dt.$$

Notice that by construction $u(t)$ on $[t_k,t_{k+1}]$ is equal to $u(t_k)$. Further,

$$\begin{aligned}\|f(t,X(t),&u(t),v(t))-f(t_k,X(t_k),u(t_k),v(t)\|\\ &\leq(\alpha_2(t_{k+1}-t_k)+L\|X(t_{k+1})-X(t_k)\|(t_{k+1}-t_k).\end{aligned}$$

Hence, using the inequality $\|X(t)-X(t_k)\|\leq R(t-t_k)$, we obtain that

$$\begin{aligned}\Big|\langle X(\tau\wedge t_k)-&Y^1(\tau\wedge t_k),X(\tau\wedge t_{k+1})-X(\tau\wedge t_k)\rangle\\ &-\Big\langle X(\tau\wedge t_k)-Y^1(\tau\wedge t_k),\int_{\tau\wedge t_k}^{\tau\wedge t_{k+1}}f(\tau\wedge t_k,X(\tau\wedge t_k),u(\tau\wedge t_k),v(t))dt\Big\rangle\Big|\\ \leq[\alpha_2(t_k-&t_{k+1})]^2\cdot(t_{k+1}-t_k)+\frac{1}{2}\|X(\tau\wedge t_k)-Y^1(\tau\wedge t_k)\|^2(t_{k+1}-t_k).\end{aligned}\tag{3.14}$$

Finally, let us evaluate the term

$$\begin{aligned}\langle X(\tau\wedge t_k)-Y^1(\tau\wedge t_k),Y^1(\tau\wedge t_{k+1})-Y^1(\tau\wedge t_k)\rangle\\ =\langle X(t_k)-Y^1(t_k),Y^1((\tau\wedge t_{k+1})\vee t_k)-Y^1(t_k)\rangle.\end{aligned}$$

By Lemma 1, we have that the process

$$l_{X(t_k),Y^1(t_k)}(Y^1(t))-\int_{t_k}^{t}\int_U L^2_\theta[u,v(\theta)]l_{X(t_k),Y^1(t_k)}(Y^1(\theta))\hat{\xi}^1(\theta,du)d\theta$$

is a $\{\mathcal{F}_t\}_{t\in[t_k,t_{k+1}]}$-martingale. Here

$$l_{y_1,y_2}(y)\triangleq\langle y_1-y_2,y-y_2\rangle.$$

Using the Doob sampling option theorem [15, Theorem 7.12] for the stopping times $(\tau \wedge t_{k+1}) \vee t_k$ and t_k, we conclude that

$$\begin{aligned}
&\mathbb{E}\langle X(\tau\wedge t_k) - Y^1(\tau\wedge t_k), Y^1(\tau\wedge t_{k+1}) - Y^1(\tau\wedge t_k)\rangle\\
&\quad = \mathbb{E} l_{X(t_k),Y^1(t_k)}(Y^1((\tau\wedge t_{k+1})\vee t_k))\\
&\quad = \mathbb{E}\int_{t_k}^{(\tau\wedge t_{k+1})\vee t_k}\int_U L_t^2[u,\hat{v}^1(t)] l_{X(t_k),Y^1(t_k)}(Y^1(t))\hat{\xi}^1(t,du)dt\\
&\quad = \mathbb{E}\int_{t_k}^{(\tau\wedge t_{k+1})\vee t_k}\int_U \langle X(\tau\wedge t_k) - Y^1(\tau\wedge t_k), b(t,Y^1(t),u,\hat{v}^1(t))\rangle\hat{\xi}^1(t,du)dt.
\end{aligned}$$

In the last equality we use the definition of the function b (see (2.4)). Evaluating the right-hand side of this formula as above (see (3.14)) and using definitions (3.3), (3.4), (3.5), we conclude that

$$\begin{aligned}
&\Big|\mathbb{E}\langle X(\tau\wedge t_k) - Y_1(\tau\wedge t_k), Y^1(\tau\wedge t_{k+1}) - Y^1(\tau\wedge t_k)\rangle\\
&\qquad -\mathbb{E}\int_{\tau\wedge t_k}^{(\tau\wedge t_{k+1})}\int_U\langle X(\tau\wedge t_k) - Y^1(\tau\wedge t_k),\\
&\qquad\quad b(\tau\wedge t_k, Y^1(\tau\wedge t_k), u, \hat{v}(\tau\wedge t_k, X(\tau\wedge t_k), Y^1(\tau\wedge t_k)))\rangle\hat{\xi}^{\tau\wedge t_k,t_{k+1}}_{Y^1(\tau\wedge t_k),\hat{v}^1(\tau\wedge t_k)}(t,du)dt\Big|\\
&\le \alpha_3(t_{k+1}-t_k)\cdot(t_{k+1}-t_k) + \frac{1}{2}\|X(\tau\wedge t_k) - Y^1(\tau\wedge t_k)\|^2(t_{k+1}-t_k)
\end{aligned} \tag{3.15}$$

Here α_3 is a vanishing at zero continuous function.

Combining (3.11), (3.12), (3.13), (3.14) and (3.15), we obtain the following estimate:

$$\begin{aligned}
&\mathbb{E}\|X(\tau\wedge t_{k+1}) - Y^1(\tau\wedge t_{k+1})\|^2 \le \mathbb{E}\|X(\tau\wedge t_k) - Y^1(\tau\wedge t_k)\|^2 + 2\varepsilon^2(t_{k+1}-t_k)\\
&+2\mathbb{E}\int_{\tau\wedge t_k}^{\tau\wedge t_{k+1}}\langle X(\tau\wedge t_k) - Y^1(\tau\wedge t_k),\\
&\qquad f(\tau\wedge t_k, X(\tau\wedge t_k), \hat{u}(\tau\wedge t_k, X(\tau\wedge t_k), Y^1(\tau\wedge t_k)), v(t))\rangle dt\\
&\quad -2\mathbb{E}\int_{\tau\wedge t_k}^{(\tau\wedge t_{k+1})}\int_U\langle X(\tau\wedge t_k) - Y^1(\tau\wedge t_k),\\
&\qquad b(\tau\wedge t_k, Y^1(\tau\wedge t_k), u, \hat{v}(\tau\wedge t_k, X(\tau\wedge t_k), Y^1(\tau\wedge t_k)))\rangle\hat{\xi}^{\tau\wedge t_k,t_{k+1}}_{Y^1(\tau\wedge t_k),\hat{v}^1(\tau\wedge t_k)}(du)dt\\
&\qquad\quad +4\|X(\tau\wedge t_k) - Y^1(\tau\wedge t_k)\|^2(t_{k+1}-t_k) + \alpha_4(t_{k+1}-t_k)\cdot(t_{k+1}-t_k).
\end{aligned} \tag{3.16}$$

In the previous formula, we denote

$$\alpha_4(\epsilon) \triangleq \alpha_1(\epsilon) + [\alpha_2(\epsilon)]^2 + \alpha_3(\epsilon).$$

Further, we have that

$$\begin{aligned}
&\int_{\tau\wedge t_k}^{\tau\wedge t_{k+1}}\langle X(\tau\wedge t_k) - Y^1(\tau\wedge t_k),\\
&\qquad f(\tau\wedge t_k, X(\tau\wedge t_k), \hat{u}(\tau\wedge t_k, X(\tau\wedge t_k), Y^1(\tau\wedge t_k)), v(t))\rangle dt\\
&\quad -\int_{\tau\wedge t_k}^{(\tau\wedge t_{k+1})}\int_U\langle X(\tau\wedge t_k) - Y^1(\tau\wedge t_k),\\
&\qquad\quad b(\tau\wedge t_k, Y^1(\tau\wedge t_k), u, \hat{v}(\tau\wedge t_k, X(\tau\wedge t_k), Y^1(\tau\wedge t_k)))\rangle\hat{\xi}^{\tau\wedge t_k,t_{k+1}}_{Y^1(\tau\wedge t_k),\hat{v}^1(\tau\wedge t_k)}(t,du)dt
\end{aligned} \tag{3.17}$$

$$\leq \int_{\tau\wedge t_k}^{\tau\wedge t_{k+1}} \langle X(\tau\wedge t_k) - Y^1(\tau\wedge t_k),$$
$$f(\tau\wedge t_k, X(\tau\wedge t_k), \hat{u}(\tau\wedge t_k, X(\tau\wedge t_k), Y^1(\tau\wedge t_k)), v(t))\rangle dt$$
$$-\mathbb{E}\int_{\tau\wedge t_k}^{(\tau\wedge t_{k+1})}\int_U \langle X(\tau\wedge t_k) - Y^1(\tau\wedge t_k),$$
$$f(\tau\wedge t_k, Y^1(\tau\wedge t_k), u, \hat{v}(\tau\wedge t_k, X(\tau\wedge t_k), Y^1(\tau\wedge t_k)))\rangle \hat{\xi}^{\tau\wedge t_k, t_{k+1}}_{Y^1(\tau\wedge t_k), \hat{v}^1(\tau\wedge t_k)}(t, du)dt$$
$$+ (K+1/2)\|X(\tau\wedge t_k) - Y^1(\tau\wedge t_k)\|^2(t_{k+1}-t_k) + \varepsilon^2/2(t_{k+1}-t_k).$$

Here K is a Lipschitz constant for the function f.

Moreover, the construction of the functions $\hat{u}$ and $\hat{v}$ implies that, for any $t \in [0,T]$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$, $u \in U$ and $v \in V$,

$$\langle x - y, f(t,x,\hat{u}(t,x,y),v)\rangle - \langle x - y, f(t,x,u,\hat{v}(t,x,y))\rangle \leq 0.$$

This together with inequalities (3.16), (3.17) yield the following estimate

$$\begin{aligned}\mathbb{E}\|X(\tau\wedge t_{k+1}) - Y^1(\tau\wedge t_{k+1})\|^2 \\ \leq \mathbb{E}\|X(\tau\wedge t_k) - Y^1(\tau\wedge t_k)\|^2(1+\varkappa(t_{k+1}-t_k)) \\ + 3\varepsilon^2(t_{k+1}-t_k) + \alpha_4(t_{k+1}-t_k)\cdot(t_{k+1}-t_k).\end{aligned} \tag{3.18}$$

Here we put

$$\varkappa \triangleq 2K+5.$$

Applying inequality (3.18) sequentially, we conclude that

$$\begin{aligned}\mathbb{E}\|X(\tau\wedge t_{k+1}) - Y^1(\tau\wedge t_{k+1})\|^2 \\ \leq e^{\varkappa T}\mathbb{E}\|X(\tau\wedge t_0) - Y^1(\tau\wedge t_0)\|^2 + 2e^{\varkappa T}T\varepsilon + e^{\varkappa T}T\alpha_4(d(\Delta)).\end{aligned}$$

Taking into account (3.9), (3.10), we obtain the statement of the lemma for

$$C_* \triangleq 3e^{\varkappa T}T = 2e^{(2K+5)T}T, \tag{3.19}$$

and the function $\alpha_*(\epsilon) \triangleq e^{\varkappa T}T\alpha_4(\epsilon)$. □

P r o o f of Theorem 1. The proof of the theorem directly follows from Lemmas 2 and 3. □

§ 4. The second player's strategy

The aim of the section is to prove Theorem 2. Assume that the function φ^- is v-stable. The strategy of the second player does not require the stopping time and realizes the Krasovskii–Subbotin extremal shift rule with the stochastic guide governed by the Lévy–Khintchine generator [2]. For the sake of completeness, we briefly describe the construction of the suboptimal strategy of the second player based on the function φ^-.

Let t_0 be an initial time and let $\Delta = \{t_i\}_{i=0}^N$ be a partition of the time interval $[t_0, T]$. We will construct the second player's strategy $\mathfrak{v}^\Delta = (\Omega^{V,\Delta}, \mathcal{F}^{V,\Delta}, \{\mathcal{F}_t^{V,\Delta}\}_{t\in[0,T]}, P^{V,\Delta}_{x(\cdot)}, v^\Delta)$ that provides the inequality formulated in the Theorem 2. To this end, we will use the functions $\hat{u}$ and $\hat{v}$ introduced by (3.1), (3.2).

Further, let filtered measurable spaces $(\widetilde{\Omega}^{t_k,t_{k+1}}, \widetilde{\mathcal{F}}^{t_k,t_{k+1}}, \{\widetilde{\mathcal{F}}_t^{t_k,t_{k+1}}\}_{t\in[t_k,t_{k+1}]})$ be introduced in Definition 7 for the function φ^- and each time interval $[t_k, t_{k+1}]$. As above, we assume that $\widetilde{\mathcal{F}}^{t_k,t_{k+1}} = \widetilde{\mathcal{F}}^{t_k,t_{k+1}}_{t_{k+1}}$.

Set,

$$\Omega^{V,\Delta} \triangleq \bigtimes_{k=0}^{N-1} \widetilde{\Omega}^{t_k,t_{k+1}}.$$

For $k = 0, \ldots, N-1$, $t \in [t_k, t_{k+1}]$, define

$$\mathcal{F}_t^{V,\Delta} \triangleq \bigotimes_{i=0}^{k-1} \widetilde{\mathcal{F}}^{t_i,t_{i+1}} \otimes \widetilde{\mathcal{F}}_t^{t_k,t_{k+1}} \otimes \bigotimes_{i=k+1}^{N-1} \{\varnothing, \widetilde{\Omega}^{t_i,t_{i+1}}\}.$$

Denote

$$\mathcal{F}^{V,\Delta} \triangleq \mathcal{F}_T^{V,\Delta}.$$

Now, let $x(\cdot) \in C([t_0, T]; \mathbb{R}^d)$. We will define $P_{x(\cdot)}^{V,\Delta}(A)$, $v_{x(\cdot)}^{\Delta}(t)$ using an auxiliary stochastic process $Y_{x(\cdot)}^2(\cdot)$ inductively.

First, we put

$$Y_{x(\cdot)}^2(t_0) \triangleq x(t_0).$$

Now assume that the auxiliary motion $Y_{x(\cdot)}^2(t)$ is defined for all $t \in [t_0, t_l]$. Denote

$$x_l \triangleq x(t_l), \quad y_l \triangleq Y_{x(\cdot)}^2(t_l).$$

Set

$$v_{x(\cdot)}^{\Delta}(t) \triangleq \hat{v}(t_l, x_l, y_l).$$

Furthermore, denote

$$u_l \triangleq \hat{u}(t_l, x_l, y_l).$$

Since φ^- is v-stable, there exists a probability $\widetilde{P}_{y_l,u_l}^{t_l,t_{l+1}}$, a motion $\widetilde{Y}_{y_l,u_l}^{t_l,t_{l+1}}$ and a relaxed control of the first player $\hat{\zeta}_{y_l,u_l}^{t_l,t_{l+1}}$ such that conditions (L1)–(L3) hold for $s = t_l$, $r = t_{l+1}$, $y = y_l$, $u = u_l$. If $t \in [t_l, t_{l+1}]$, we put

$$Y_{x(\cdot)}^2(t) \triangleq \widetilde{Y}_{y_l,u_l}^{t_l,t_{l+1}},$$

$$v_{x(\cdot)}^{\Delta}(t) \triangleq v_l = \hat{v}(t_l, x_l, y_l).$$

Further, denote by $\nu_{x(\cdot),l}(dy)$ the distribution of $y_l = Y_{x(\cdot)}^2(t_l)$. For $A_k \in \widetilde{\mathcal{F}}_{t_{k+1}}^{t_k,t_{k+1}}$, $k = 0, \ldots, l$, we set

$$P_{x(\cdot)}^{V,\Delta}(A_0 \times \ldots \times A_{l-1} \times A_l \times \widehat{\Omega}^{t_{l+1},t_{l+2}} \times \ldots \times \widetilde{\Omega}^{t_{N-1},t_N}) \triangleq$$
$$\int_{\mathbb{R}^d} P_{x(\cdot)}^{U,\Delta}(A_0 \times \ldots \times A_{l-1} \times \widetilde{\Omega}^{t_l,t_{l+1}} \times \widetilde{\Omega}^{t_{l+1},t_{l+2}} \times \ldots \times \widetilde{\Omega}^{t_{N-1},t_N} | Y_{x(\cdot)}^1(t_l) = y)$$
$$\widetilde{P}_{y_l,v_l}^{t_l,t_{l+1}}(A_l) \nu_{x(\cdot),l}(dy).$$

Now we choose $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[s,r]}, P, X, u, v, \tau)$ to be a motion corresponding to the strategy $\mathfrak{v}^{\Delta}$. When $t \in [t_l, t_{l+1})$, set

$$Y^2(t) \triangleq Y_{X(t_l)}^2(t),$$

$$\hat{u}^2(t) \triangleq v_l = \hat{u}(t_l, X(t_l), Y^1(t_l)),$$

$$\hat{\zeta}^1(t) \triangleq \xi_{Y^2(t_l),\hat{u}^2(t_l)}^{t_l,t_{l+1}}.$$

Lemma 4. *The following statements hold.*

1. *For each $\phi \in \mathcal{D}^1$, the process*

$$\phi(X(t)) - \int_{t_0}^{t} L_1[u(\tau), v(\tau)]\phi(X(\tau))dt$$

is a $\{\mathcal{F}_t\}_{t\in[t_0,T]}$-martingale.

2. For each $\phi \in \mathcal{D}^2$, the process

$$\phi(Y^2(t)) - \int_{t_0}^{t} \int_U L_1[u, \hat{v}^1(\theta)]\phi(Y^2(\theta))\hat{\xi}^1(\theta, du)dt$$

is a $\{\mathcal{F}_t\}_{t\in[t_0,T]}$-martingale.

The proof directly follows from the construction of the motion $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[s,r]}, P, X, u, v, \tau)$ and the process Y^2.

Lemma 5. $\varphi^-(t_0, x_0) \leq \mathbb{E}g(\tau, Y^1(\tau))$.

The proof of this lemma is similar to the proof of Lemma 2.

Analogously, changing the players in the proof of Lemma 3, we obtain the following.

Lemma 6. $\mathbb{E}\|X(\tau) - Y^2(\tau)\|^2 \leq C_*\varepsilon^2 + \alpha_*(d(\Delta))$.

Using this lemmas, one can prove Theorem 2 repeating the proof of Theorem 1.

§ 5. Equivalence between differential game with minimal cost and differential game with stopping controlled by the first player

In this section we prove Theorem 3. Recall (see [24, 29]) that the value function of differential game with the minimal cost (1.1), (1.2) is simultaneously u- and v-stable. The following definitions hold [24, 29].

Definition 8. We say that the lower semicontinuous function $\varphi\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}$ is u-stable for differential game with minimal cost (1.1), (1.2) if $\varphi(T,x) = g(T,x)$ and, for every $s, r \in [0,T]$, $s < r$, $v \in V$, there exists a solution of the differential inclusion

$$\frac{d}{dt}x(t) \in \mathrm{co}\{f(t, x(t), u, v)\colon u \in U\}, \quad x(s) = y \tag{5.1}$$

such that

$$\varphi(s, y) \geq \varphi(h, x(h)),$$

where

$$h = \min\{t \in [s, r]\colon \varphi(t, x(t)) = g(t, x(t))\}$$

provided that this minimum is achieved and r otherwise.

Definition 9. An upper semicontinuous function $\varphi\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}$ is v-stable for differential game with minimal cost (1.1), (1.2) provided that $\varphi(t,x) \leq g(t,x)$ on $[0,T]\times\mathbb{R}^d$ and, for each $s, r \in [0,T]$, $s < r$, $y \in \mathbb{R}^d$, one can find a function $x(\cdot)$ on $[s, r]$ taking values in $\mathbb{R}^d$ satisfying the differential inclusion

$$\frac{d}{dt}x(t) \in \mathrm{co}\{f(t, x, u, v)\colon v \in V\}, \quad x(s) = y$$

such that

$$\varphi(s, y) \leq \varphi(r, x(r)).$$

The link between two definitions of u–stability presented in Definitions 6, 8 is given in the following.

Proposition 1. *If a function φ is u-stable for differential game with minimal cost* (1.1)*,* (1.2)*, then it is u-stable for the generator L^1.*

P r o o f. To show the u-stability in the sense of Definition 6, it suffices to choose

$$\widehat{\Omega}^{s,r} \triangleq C([s,r];\mathbb{R}^d),$$

set $\widehat{\mathcal{F}}_t^{s,r}$ to be equal to the minimal σ-algebra generated by the cylindrical sets

$$\mathcal{A}^{t_1,\dots,t_n}_{\Gamma_1,\dots,\Gamma_n} \triangleq \{x(\cdot)\in C([s,r];\mathbb{R}^d)\colon x(t_k)\in\Gamma_k,\ k=1,\dots,n\},$$

where $t_1,\dots,t_n\in[s,t]$, and $\Gamma_1,\dots,\Gamma_n$ are Borel subsets of $\mathbb{R}^d$. Certainly, in this case we put $\widehat{\mathcal{F}}^{s,r}\triangleq\widehat{\mathcal{F}}_r^{s,r}$. Further, let $x^{s,r}_{y,v}(\cdot)$ satisfy (5.1), and let $\xi^{s,r}_{y,v}$ be a relaxed control of the first player such that

$$\frac{d}{dt}x^{s,r}_{y,v}(t) = \int_U f(t,x^{s,r}_{y,v}(t),u,v)\xi^{s,r}_{y,v}(t,du),\quad x^{s,r}_{y,v}(s)=y.$$

Recall that $\Omega = C([s,r];\mathbb{R}^d)$. We set

$$\widehat{Y}^{s,r}_{y,v}(t,\omega)\triangleq\omega(t),$$

choose $\hat{\xi}^{s,r}_{y,v}(t,\omega)$ be equal to $\xi^{s,r}_{y,v}(t)$ when $\omega(\cdot)=x^{s,r}_{y,v}(\cdot)$ and put

$$\widehat{P}^{s,r}_{y,v}\triangleq\delta_{x^{s,r}_{y,v}(\cdot)}.$$

Finally, let

$$\hat{\tau}^{s,r}_{y,v}(\omega)\triangleq\min\{t\in[s,r]\colon \varphi(t,\omega(t)) = g(t,\omega(t))\}\wedge r.$$

Clearly, $(\widehat{\Omega}^{s,r},\widehat{\mathcal{F}}^{s,r},\{\widehat{\mathcal{F}}_t^{s,r}\}_{t\in[s,r]},\widehat{P}^{s,r}_{y,v},\widehat{Y}^{s,r}_{y,v},\hat{\xi}^{s,r}_{y,v},\delta_v)$ is a control process admissible for the generator L^1. Furthermore, using Definition 8, we obtain that

$$\widehat{\mathbb{E}}^{s,r}_{y,v}\varphi(\hat{\tau}^{s,r}_{y,v},\widehat{Y}^{s,r}_{y,v}(\hat{\tau}^{s,r}_{y,v})) = \varphi(h,x^{s,r}_{y,v}(h))\le\varphi(s,y),$$

where h is defined by the rule

$$h=\min\{t\in[s,r]\colon \varphi(t,x^{s,r}_{y,v}) = g(t,x^{s,r}_{y,v})\}\wedge r.$$

Hence, the function φ is u-stable for the generator L^1 in the sense of Definition 6. □

The following statement provides the link between two definitions of v-stability (Definitions 7, 9).

Proposition 2. *If the function φ is v-stable for the differential game with minimal cost* (1.1)*,* (1.2)*, then it is v-stable for the generator L^1 in the sense of Definition* 7*.*

The proof of this statement is similar to the proof of Proposition 1 and, thus, omitted.

P r o o f of Theorem 3. Recall that the function Val_0 is u- and v-stable for differential game with minimal cost (1.1), (1.2). By Propositions 1 and 2 the function Val_0 is u- and v-stable for the generator L^1. Letting the generator L^2 be equal to the generator L^1, we have the constant ε equal to 0. Therefore, Corollary 1 gives that

$$\mathrm{Val}^+(t_0,x_0)\le\mathrm{Val}_0(t_0,x_0). \tag{5.2}$$

Analogously, Corollary 2 yields the inequality

$$\mathrm{Val}_0(t_0,x_0)\le\mathrm{Val}^-(t_0,x_0). \tag{5.3}$$

Since $\mathrm{Val}^-\le\mathrm{Val}^+$, inequalities (5.2), (5.3) imply the equality:

$$\mathrm{Val}^+(t_0,x_0)=\mathrm{Val}_0(t_0,x_0)=\mathrm{Val}^-(t_0,x_0).$$

This completes the proof. □

§ 6. Isaacs–Bellman equation with additional inequality constraints

This section deals with sufficient condition on a function $\varphi(t,x)$ to be a u- or v-stable function.

Let $\rho\in\{0,1,2\}$ be the maximal order of the derivative involved in the definition of the generator L^2.

Definition 10. We say that the first player's feedback strategy $u^0(t,x)$ is feasible for the generator L^2 if, given $s,r\in[0,T]$, $s<r$, one can find a filtered measurable space $(\widehat{\Omega}^{s,r},\widehat{\mathcal{F}}^{s,r},\{\widehat{\mathcal{F}}_t^{s,r}\}_{t\in[s,r]})$ satisfying the following property: for each $y\in\mathbb{R}^d$, $v\in V$, there exists an admissible for the generator L^2 control process $(\widehat{\Omega}^{s,r},\widehat{\mathcal{F}}^{s,r},\{\widehat{\mathcal{F}}_t^{s,r}\}_{t\in[s,r]},\widehat{P}^{s,r}_{y,v},\widehat{Y}^{s,r}_{y,v},\hat{\xi}^{s,r}_{y,v},\delta_v)$ such that $\widehat{Y}^{s,r}_{y,v}(s)=y$, $\widehat{P}^{s,r}_{y,v}$-a.s., and $\hat{\xi}^{s,r}_{y,v}(t,du)=\delta_{u^0(t,X(t))}(du)$.

Theorem 4. *Assume that $\varphi\colon[0,T]\times\mathbb{R}^d\to\mathbb{R}$ is differentiable w. r. t. t and ρ times differentiable w. r. t. x and satisfies the following conditions*

- (*boundary condition*)
$$\varphi(T,x)=g(T,x);$$
- *if $\varphi(t,x)<g(t,x)$,*
$$\frac{\partial\varphi}{\partial t}+\min_{u\in U}\max_{v\in V}L^2_t[u,v]\varphi\le 0;\tag{6.1}$$
- *the strategy $u^\natural$ defined by the rule*
$$\min_{u\in U}\max_{v\in V}L^2_t[u,v]\varphi(t,x)=\max_{v\in V}L^2_t[u^\natural(t,x),v]\varphi(t,x)\tag{6.2}$$
is feasible.

Then, the function φ is u-stable.

P r o o f. The proof is by the verification arguments.

Let $s,r\in[0,T]$, $s<r$. Using the assumption that $u^\natural$ is feasible (see Definition 10), we find a filtered measurable space $(\widehat{\Omega}^{s,r},\widehat{\mathcal{F}}^{s,r},\{\widehat{\mathcal{F}}_t^{s,r}\}_{t\in[s,r]})$ providing the following property, for any $y\in\mathbb{R}^d$, $v_*\in V$, there exists a controlled process $(\widehat{\Omega}^{s,r},\widehat{\mathcal{F}}^{s,r},\{\widehat{\mathcal{F}}_t^{s,r}\}_{t\in[s,r]},\widehat{P}^{s,r}_{y,v},\widehat{Y}^{s,r}_{y,v_*},\hat{\xi}^{s,r}_{y,v_*},\delta_{v_*})$ such that $\widehat{Y}^{s,r}_{y,v_*}(s)=y$, $\widehat{P}^{s,r}_{y,v_*}$-a.s., and $\hat{\xi}^{s,r}_{y,v_*}(t,du)=\delta_{u^\natural(t,X(t))}(du)$. Set
$$\hat{\tau}^{s,r}_{y,v_*}\triangleq\inf\{t\in[s,r]\colon\varphi(t,\widehat{Y}^{s,r}_{y,v}(t))=g(t,\widehat{Y}^{s,r}_{y,v_*}(t))\}.$$
Further, denote
$$\hat{\vartheta}^{s,r}_{y,v_*}\triangleq\hat{\tau}^{s,r}_{y,v_*}\wedge r.$$
Using the optional sampling theorem [15, Theorem 7.12], (6.1) and (6.2), we obtain that
$$\begin{aligned}
&\varphi(s,y)\\
&\quad=\mathbb{E}\Big[\hat{\varphi}(\hat{\vartheta}^{s,r}_{y,v_*},\widehat{Y}^{s,r}_{y,v_*}(\hat{\vartheta}^{s,r}_{y,v_*}))\\
&\qquad\qquad-\int_s^{\hat{\vartheta}^{s,r}_{y,v_*}}\Big(\frac{\partial\varphi}{\partial t}(t,\widehat{Y}^{s,r}_{y,v_*}(t))+\int_U L_t[u,v_*]\varphi(t,\widehat{Y}^{s,r}_{y,v_*}(t))\hat{\xi}^{s,r}_{y,v_*}(t,du)\Big)dt\Big]\\
&\quad=\mathbb{E}\Big[\varphi(\hat{\vartheta}^{s,r}_{y,v_*},\widehat{Y}^{s,r}_{y,v_*}(\hat{\vartheta}^{s,r}_{y,v_*}))\\
&\qquad\qquad-\int_s^{\hat{\vartheta}^{s,r}_{y,v_*}}\Big(\frac{\partial\varphi}{\partial t}(t,\widehat{Y}^{s,r}_{y,v}(t))+L^2_t[u^\natural(t,\widehat{Y}^{s,r}_{y,v_*}(t)),v]\varphi(t,\widehat{Y}^{s,r}_{y,v_*}(t))\Big)dt\Big]\\
&\quad\ge\mathbb{E}\Big[\varphi(\hat{\vartheta}^{s,r}_{y,v_*},\widehat{Y}^{s,r}_{y,v_*}(\hat{\tau}^{s,r}_{y,v_*}))-\int_s^{\hat{\vartheta}^{s,r}_{y,v_*}}\Big(\frac{\partial\varphi}{\partial t}(t,\widehat{Y}^{s,r}_{y,v_*}(t))+\min_{u\in U}\max_{v\in V}L^2_t[u,v]\varphi(t,\widehat{Y}^{s,r}_{y,v_*}(t))\Big)dt\Big]\\
&\quad\ge\mathbb{E}\varphi(\hat{\vartheta}^{s,r}_{y,v_*},\widehat{Y}^{s,r}_{y,v_*}(\hat{\tau}^{s,r}_{y,v_*})).
\end{aligned}$$

Since, $\varphi(\hat{\vartheta}^{s,r}_{y,v_*}, \widehat{Y}^{s,r}_{y,v_*}(\hat{\vartheta}^{s,r}_{y,v_*})) = \hat{\varphi}(\hat{\tau}^{s,r}_{y,v_*}, \widehat{Y}^{s,r}_{y,v_*}(\hat{\tau}^{s,r}_{y,v_*}))$ (see (2.3)), we conclude that the function φ is u-stable. □

The sufficient condition for v-stability is given in the same way.

First, we introduce the definition of feasible feedback strategy of the second player.

Definition 11. We say that the second player's feedback strategy $v^0(t,x)$ is feasible for the generator L^2 provided that, given $s,r \in [0,T]$, $s < r$, one can find a filtered measurable space $(\widetilde{\Omega}^{s,r}, \widetilde{\mathcal{F}}^{s,r}, \{\widetilde{\mathcal{F}}^{s,r}_t\}_{t\in[s,r]})$ satisfying the following property: for each $y \in \mathbb{R}^d$, $u \in U$, there exists an admissible for the generator L^2 control process $(\widetilde{\Omega}^{s,r}, \widetilde{\mathcal{F}}^{s,r}, \{\widetilde{\mathcal{F}}^{s,r}_t\}_{t\in[s,r]}, \widetilde{P}^{s,r}_{y,v}, \widetilde{Y}^{s,r}_{y,u}, \delta_u, \tilde{\zeta}^{s,r}_{y,u})$ such that $\widetilde{Y}^{s,r}_{y,u}(s) = y$, $\widetilde{P}^{s,r}_{y,u}$-a.s., and $\tilde{\zeta}^{s,r}_{y,u}(t,dv) = \delta_{v^0(t,X(t))}(dv)$.

Theorem 5. *Assume that $\varphi\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}$ is differentiable w. r. t. t and ρ times differentiable w. r. t. x and satisfies the following conditions*

- (*boundary condition*)
$$\varphi(T,x) = g(T,x);$$
- *for any $t \in [0,T]$, $x \in \mathbb{R}^d$,*
$$\varphi(t,x) \le g(t,x);$$
- *on $[0,T]\times\mathbb{R}^d$,*
$$\frac{\partial\varphi}{\partial t}(t,x) + \max_{v\in V}\min_{u\in U} L^2_t[u,v]\varphi(t,x) \ge 0;$$
- *the strategy $v^\natural$ defined by the rule*
$$\max_{u\in U}\min_{u\in U} L^2_t[u,v]\varphi(t,x) = \min_{u\in U} L^2_t[u,v^\natural(t,x)]\varphi(t,x)$$
is feasible.

Then, the function φ is v-stable.

The proof of this theorem is entirely similar to the proof of Theorem 4 and, thus, omitted.

We illustrate Theorems 1, 2 by two examples dealing with the auxiliary continuous time stochastic games with stopping time determined by the first player.

First, let the dynamics be given by the stochastic differential equation

$$dX(t) = f^a(t,X(t),u(t),v(t))dt + \sigma(t,X(t))dW(t). \tag{6.3}$$

Here $W(t)$ stands for the Wiener process. This dynamics corresponds to the generator L^2 with $G^a(t,x,u,v) \triangleq \sigma^T(t,x)\sigma(t,x)$ and $\nu^a \equiv 0$. Notice that the closeness conditions take the form

$$|f(t,x,u,v) - f^a(t,x,u,v)| \le \varepsilon, \quad \|\sigma(t,x)\| \le \varepsilon.$$

Additionally, we assume Isaacs condition for system (6.3). This means that, for every $t \in [0,T]$, $x,y \in \mathbb{R}^d$,

$$\min_{u\in U}\max_{v\in V}\langle y, f^a(t,x,u,v)\rangle = \max_{v\in V}\min_{u\in U}\langle y, f^a(t,x,u,v)\rangle$$

Using Theorems 1, 2, we arrive at the following.

Proposition 3. *If a function φ is differentiable w. r. t. t and twice differentiable w. r. t. x and satisfies the following conditions*

- $\varphi(T,x) = g(T,x)$;
- $\varphi(t,x) \leq g(t,x)$ *on* $[0,T] \times \mathbb{R}^d$;
- *on* $[0,T] \times \mathbb{R}^d$

$$\frac{\partial \varphi(t,x)}{\partial t} + \min_{u\in U}\max_{v\in V}\langle \nabla\varphi(t,x), f^a(t,x,u,v)\rangle + \frac{1}{2}\langle G(t,x)\nabla,\nabla\rangle\varphi(t,x) \geq 0;$$

- *if* $\varphi(t,x) < g(t,x)$,

$$\frac{\partial \varphi(t,x)}{\partial t} + \min_{u\in U}\max_{v\in V}\langle \nabla\varphi(t,x), f^a(t,x,u,v)\rangle + \frac{1}{2}\langle G(t,x)\nabla,\nabla\rangle\varphi(t,x) \leq 0;$$

- *the strategies* $u^\natural(t,x)$ *and* $v^\natural(t,x)$ *defined by the rules*

$$\min_{u\in U}\max_{v\in V}\langle \nabla\varphi(t,x), f^a(t,x,u,v)\rangle = \max_{v\in V}\langle \nabla\varphi(t,x), f^a(t,x,u^\natural(t,x),v)\rangle,$$

$$\max_{v\in V}\min_{u\in U}\langle \nabla\varphi(t,x), f^a(t,x,u,v)\rangle = \min_{u\in U}\langle \nabla\varphi(t,x), f^a(t,x,u,v^\natural(t,x))\rangle$$

  are feasible.

Then, the function φ *is* u*- and* v*-stable simultaneously.*

The second example is concerned with the dynamics given by the Markov chain defined on at most countable state space $\mathcal{S} \subset \mathbb{R}^d$ determined by the controlled Kolmogorov matrix $Q(t,u,v) \triangleq (Q_{\bar{x},\bar{y}}(t,u,v))_{\bar{x},\bar{y}\in\mathcal{S}}$. As above, we impose the Isaacs' condition, that now takes the form: for every $t \in [0,T]$, $\bar{x} \in \mathcal{S}$, $w \in \mathbb{R}^d$,

$$\min_{u\in U}\max_{v\in V}\left\langle w, \sum_{\bar{y}\in\mathcal{S}}(\bar{y}-\bar{x})Q_{\bar{x},\bar{y}}(t,u,v)\right\rangle = \max_{v\in V}\min_{u\in U}\left\langle w, \sum_{\bar{y}\in\mathcal{S}}(\bar{y}-\bar{x})Q_{\bar{x},\bar{y}}(t,u,v)\right\rangle$$

This Markov chain corresponds to the generator

$$L_t^2[u,v]\phi(\bar{x}) = \sum_{\bar{y}\in\mathcal{S}}(\phi(\bar{y})-\phi(\bar{x}))Q_{\bar{x},\bar{y}}(t,u,v). \tag{6.4}$$

Thus, the closeness conditions take the form

$$\left|\sum_{\bar{y}\in\mathcal{S}}(\bar{y}-\bar{x})Q_{\bar{x},\bar{y}}(t,u,v) - f(t,\bar{x},u,v)\right| \leq \varepsilon, \quad \sum_{\bar{y}\in\mathcal{S}}\|\bar{y}-\bar{x}\|^2 Q_{\bar{x},\bar{y}}(t,u,v) \leq \varepsilon^2.$$

For this case, Theorems 1, 2 can be reformulated as follows.

Proposition 4. *Assume that the function* $\varphi\colon [0,T]\times\mathcal{S} \to \mathbb{R}$ *satisfies the following conditions:*

- $\varphi(T,\bar{x}) = g(T,\bar{x})$, $\bar{x} \in \mathcal{S}$;
- $\varphi(t,\bar{x}) \leq g(t,\bar{x})$, $t \in [0,T]$, $\bar{x} \in \mathcal{S}$;
- *for every* $t \in [0,T]$ *and* $\bar{x} \in \mathcal{S}$,

$$\frac{d}{dt}\varphi(t,\bar{x}) + \min_{u\in U}\max_{v\in V}\sum_{\bar{y}\in\mathcal{S}} Q_{\bar{x},\bar{y}}\varphi(t,\bar{y}) \geq 0;$$

- *for* $(t,\bar{x}) \in [0,T] \times \mathcal{S}$ *such that* $\varphi(t,\bar{x}) < g(t,\bar{x})$,

$$\frac{d}{dt}\varphi(t,\bar{x}) + \min_{u\in U}\max_{v\in V}\sum_{\bar{y}\in\mathcal{S}} Q_{\bar{x},\bar{y}}\varphi(t,\bar{y}) \leq 0.$$

Then φ *is* u*- and* v*-stable simultaneously.*

Notice that we do not add the assumption that for the case of generator given by (6.4) the strategy is feasible. This fact directly follows from the Measurable Maximum Theorem [1, Theorem 18.19].

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Аппроксимация функции цены дифференциальной игры с критерием, задаваемым условием минимизации

Ключевые слова: дифференциальные игры, стохастический поводырь, апрокимация функции цены, уравнение Айзекса–Беллмана.

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В статье рассматривается аппроксимация функции цены антагонистической дифференциальной игры с критерием, задаваемым условием минимизации некоторой величины вдоль реализовавшейся траектории, решениями стохастических игр с непрерывным временем и моментом остановки, управляемым одним из игроков. Отметим, что если в качестве вспомогательной игры выбрана стохастическая дифференциальная игра, то ее функция цены задается параболическим уравнением второй степени в частных производных с дополнительными ограничениями в форме неравенств, в то время как для случая вспомогательной игры с динамикой, задаваемой марковской цепью, функция цены определяется системой обыкновенных дифференциальных уравнений с дополнительными ограничениями. Развиваемый в статье метод аппроксимации основан на концепции стохастического поводыря, впервые предложенном в работах Н. Н. Красовского и А. Н. Котельниковой.

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