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ON INVESTIGATION OF THE INVERSE PROBLEM FOR A PARABOLIC INTEGRO-DIFFERENTIAL EQUATION WITH A VARIABLE COEFFICIENT OF THERMAL CONDUCTIVITY

The inverse problem of determining a multidimensional kernel of an integral term depending on a time variable t and $(n-1)$-dimensional spatial variable $x' = (x_1, \dots, x_{n-1})$ in the n-dimensional heat equation with a variable coefficient of thermal conductivity is investigated. The direct problem is the Cauchy problem for this equation. The integral term has the time convolution form of kernel and direct problem solution. As additional information for solving the inverse problem, the solution of the direct problem on the hyperplane $x_n = 0$ is given. At the beginning, the properties of the solution to the direct problem are studied. For this, the problem is reduced to solving an integral equation of the second kind of Volterra-type and the method of successive approximations is applied to it. Further the stated inverse problem is reduced to two auxiliary problems, in the second one of them an unknown kernel is included in an additional condition outside integral. Then the auxiliary problems are replaced by an equivalent closed system of Volterra-type integral equations with respect to unknown functions. Applying the method of contraction mappings to this system in the Hölder class of functions, we prove the main result of the article, which is a local existence and uniqueness theorem of the inverse problem solution.

Keywords: integro-differential equation, inverse problem, kernel, contraction mapping principle.

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Introduction

The theory and applications of integro-differential equations with an integral term of convolution-type with respect to a time variable play an important role in the mathematical modeling of heat and mass transfer. In these processes the convolution kernel describes the relaxation times of the heat flux. In the general case, in integro-differential equations of such kind, the convolution kernel takes into account the memory affects as usual. The fundamental point, when dealing with memory effects, is that the kernel cannot be considered a known function, since there are no ways to measure it directly. What we do is to reconstruct this kernel by additional measurements of a physical field, taken on a suitable subset of the body. Thus, the inverse problem arises, the study of solvability (the theorem of existence and uniqueness) of which is of great interest in the theory of inverse problems.

The dynamical representation of heat transfer processes is modeled quite accurately by including the past history of one or more time and space variables through memory kernel [1–4]. This phenomenon is governed by parabolic integro-differential equations with a time dependent memory kernel when the medium is homogeneous and a time-space dependent memory kernel when the medium is heterogeneous.

In the last years, various approaches have been applied for identification of heat source terms, of unknown boundary conditions, of memory kernel or spatially varying coefficients. The determining of the convective coefficients using different kinds of temperature measurements by means of inverse heat conduction were investigated in [5–10] (see also references therein). Authors in these works discussed the issues of existence, uniqueness and stability estimates of solution. Here a numerical approach for solving such problems has also been applied.

In papers [11–13] the inverse problems of finding the time-dependent heat source with non-local boundary and integral additional conditions were investigated. The existence, uniqueness and continuous dependence of the solution of the inverse problem upon the data were established. Also, numerical results were presented and discussed. In the work [14] the inverse boundary value problem of heat conduction with a piecewise constant coefficient was considered. This problem was investigated using the Fourier series in eigenfunctions for an equation with a discontinuous coefficient.

Among the works devoted to the kernel determination problems in heat conduction equations, one-dimensional problems are most encountered, that is the problems of memory kernel determining depending only on a time variable. For example, in [1,2,15] (see also the references therein) these problems were investigated based on a fixed point argument and derived the local in time existence and uniqueness of inverse problems. Additionally, for problem of [15] a numerical experiment was conducted.

The results on multidimensional problems of determining the kernel in parabolic integro-differential equations are very rare. In this direction we only note the works [16–19]. In [16] author deals with the problem of recovering a memory kernel $k(t,\eta)$, depending on time t and on an angular variable η, in a parabolic integro-differential equation related to a toric domain involved in $\mathbb{R}^2$. In the works [17,18] the problems of determining a kernel depending on a time variable t and $(n-1)$-dimensional spatial variable $x'=(x_1,\dots,x_{n-1})$ entering into in integral term of a n-dimensional heat equation were investigated. The local unique solvability result was obtained . In [19] the uniqueness theorem of recovering kernel depending on all variables t and $x=(x_1,\dots,x_n)$ was proved for special case of kernels.

In the present paper we study an inverse problem to determine the temporal and $(n-1)$-dimensional spatially varying memory kernel for a parabolic integro-differential equation with a variable coefficient of thermal conductivity.

Consider the problem of determining functions $u(x,t)$, $k(x',t)$, $x=(x_1,x_2,\dots,x_{n-1},x_n)=$ $=(x',x_n)\in\mathbb{R}^n$, $t>0$ from the following equations:

$$u_t-a(t)\Delta u=\int_0^t k(x',\tau)u(x,t-\tau)\,d\tau,\quad (x,t)\in\mathbb{R}^n_T, \tag{0.1}$$

$$u|_{t=0}=\varphi(x),\quad x\in\mathbb{R}^n, \tag{0.2}$$

$$u|_{x_n=0}=f(x',t),\quad 0\leqslant t\leqslant T,\quad f(x',0)=\varphi(x',0), \tag{0.3}$$

where $\triangle$ is the Laplace operator with respect to spatial variables $x=(x_1,\dots,x_n)$, $\mathbb{R}^n_T=\{(x,t)|x=(x',x_n)\in\mathbb{R}^n,\ 0<t<T\}$ is a strip with thickness T, $T>0$ is an arbitrary fixed number; $a(t)\in C^1[0,T]$, $0<a_0\leqslant a(t)\leqslant a_1<\infty$, a_0 and a_1 are given numbers.

To the best of our knowledge, the problem (0.1)–(0.3) have not been studied earlier. Our result generalizes the work [17] to the case of an integro-differential heat equation with a variable coefficient of thermal conductivity.

In the sequel,we will use the Hölder space H^α for functions depending on spatial variables and for functions depending on spatial and time variables — Hölder space $H^{\alpha,\alpha/2}$ with exponents α and $\alpha/2$, α is non-negative integer. Starting with Section 2, we will assume that $\varphi(x)\in H^{l+6}(\mathbb{R}^n)$, $\varphi(x)\geqslant\varphi_0=\text{const}>0$, $f(x',t)\in H^{l+4,(l+4)/2}\left(\overline{\mathbb{R}}^{n-1}_T\right)$,

$$\overline{\mathbb{R}}^{n-1}_T=\left\{(x',t)|x'\in\mathbb{R}^{n-1},\quad 0\leqslant t\leqslant T\right\},\quad l\in(0,1),$$

Spaces $H^l(Q)$, $H^{l,l/2}(Q_T)$ and norms in them are defined in [20, pp. 16–27]. In what follows, for norm of functions in the space $H^{l,l/2}(Q_T)$ (in concrete cases $Q_T=\mathbb{R}^n_T$ or $Q_T=\mathbb{R}^{n-1}_T$) depending on spatial and time variables we will use notation $|\cdot|_T^{l,l/2}$, while for functions depending only on spatial variables we will use $|\cdot|^l$ (in this case $Q=\mathbb{R}^n$ or $Q=\mathbb{R}^{n-1}$).

As a rule, inverse problems are ill-posed. According to [21, pp. 31–44] problems that are ill-posed some spaces but can be well-posed for another choice of spaces, are called weakly ill-posed. In this work, functional spaces for given and unknown functions are indicated, in which the problem (0.1)–(0.3) is well-posed. In this case, we restrict ourselves to proving the existence and uniqueness theorem for the solution of the problem (Section 4).

Note that the equation (0.1) is an integro-differential equation with a heat operator on the left and a Volterra-convolutional integral on the right sides. To get acquainted with the issues of solvability of various problems for integro-differential equations with Volterra-non-convolutional integrals, we refer the reader to the papers [22, 23] (also see the literature in them).

The article is organized as follows. In Section 1, we investigate the direct problem (0.1) and (0.2). In Section 2, we transform the given problem into an auxiliary problem where the additional condition contains the unknown k outside integral. In Section 3, we reduce the auxiliary problem to a system of integral equations with respect to unknown functions. In Section 4, we prove the main result which states the existence and uniqueness of solution of the given problem by a fixed point argument.

§ 1. Direct problem

For the given functions a and k, the problem of determining the solution to the integro–differential equation (0.1) under the initial condition (0.2) is the Cauchy problem. This problem in the theory of inverse problems is called the direct problem.

The solution of the Cauchy problem (0.1) and (0.2) can be reduced to the solution of an integral equation of the Volterra type. To do this we use the formula

$$\begin{aligned} p(x,t) &= \int_{\mathbb{R}^n} \varphi(\xi) G\big(x-\xi;\theta(t)\big)\, d\xi + \\ &+ \int_0^{\theta(t)} \frac{d\tau}{a\left(\theta^{-1}(\tau)\right)} \int_{\mathbb{R}^n} F\left(\xi, \theta^{-1}(\tau)\right) G\big(x-\xi;\theta(t)-\tau\big)\, d\xi, \end{aligned} \tag{1.1}$$

which provides the solution of the following Cauchy problem for the heat equation with time-variable thermal conductivity:

$$p_t - a(t)\Delta p = F(x,t), \quad x\in\mathbb{R}^n, \quad t>0,$$
$$p(x,0)=\varphi(x), \quad x\in\mathbb{R}^n.$$

In (1.1) $\theta(t) = \displaystyle\int_0^t a(\tau)\, d\tau$ and $\theta^{-1}(t)$ is the inverse function to $\theta(t)$;

$$G\left(x-\xi;\theta(t)-\tau\right) = \frac{1}{\left(2\sqrt{\pi(\theta(t)-\tau)}\right)^n} e^{\frac{-|x-\xi|^2}{4(\theta(t)-\tau)}}$$

is the fundamental solution of the heat operator with the time-dependent coefficient of thermal conductivity $\frac{\partial}{\partial t} - a(t)\triangle$, $\xi=(\xi_1,\dots,\xi_n)$, $\xi'=(\xi_1,\dots,\xi_{n-1})$, $d\xi = d\xi_1\dots d\xi_n$, $|x|^2 = x_1^2 + {} + \dots + x_n^2$.

Taking this into account, we obtain the following integral equation to determine $u(x,t)$:

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^n} \varphi(\xi) G\big(x-\xi;\theta(t)\big)\, d\xi + \\ &+ \int_0^{\theta(t)} \frac{d\tau}{a\left(\theta^{-1}(\tau)\right)} \int_{\mathbb{R}^n} \int_0^{\theta^{-1}(\tau)} k(\xi',\alpha) u\left(\xi, \theta^{-1}(\tau)-\alpha\right) G\big(x-\xi;\theta(t)-\tau\big)\, d\alpha\, d\xi. \end{aligned} \tag{1.2}$$

Lemma 1. *If* $\varphi(x) \in H^{l+2}(\mathbb{R}^n)$, $k(x', t) \in H^{l,l/2}(\overline{\mathbb{R}}_T^{n-1})$, *and* $a(t)$ *satisfies the conditions of Introduction, then there exists a unique solution of integral equation* (1.2) *in the space* $u(x,t) \in \in H^{l+2,(l+2)/2}(\mathbb{R}_T^n)$.

P r o o f. To proof Lemma we use the method of successive approximations and consider the sequence of functions defined recursively by the formulas:

$$u_0(x,t) = \int_{\mathbb{R}^n} \varphi(\xi) G(x-\xi; \theta(t))\, d\xi,$$

$$u_n(x,t) =$$

$$= \int_0^{\theta(t)} \frac{d\tau}{a\left(\theta^{-1}(\tau)\right)} \int_{\mathbb{R}^n} \int_0^{\theta^{-1}(\tau)} k(\xi', \alpha) u_{n-1}\left(\xi, \theta^{-1}(\tau) - \alpha\right) G(x-\xi; \theta(t) - \tau)\, d\alpha\, d\xi, \tag{1.3}$$

$$(x,t) \in \mathbb{R}_T^n, \quad n = 1, 2, \ldots .$$

Under the assumptions of Lemma, one has the inclusion $u(x,t) \in H^{l+2,(l+2)/2}(\mathbb{R}_T^n)$. According to the general theory of Cauchy problems for equations of the parabolic type (see, [20]), this implies that the same property will be possessed by all functions $u_n(x,t)$ in $\mathbb{R}_T^n$ defined by relations (1.3). Set $\varphi_0 = |\varphi(x)|^{l+2}$. Using (1.3), we estimate the modulus of $u_n(x,t)$ in the domain $\mathbb{R}_T^n$ as

$$|u_0(x,t)|_T^{l+2,(l+2)/2} \leqslant \varphi_0,$$

$$|u_1(x,t)|_T^{l+2,(l+2)/2} \leqslant \int_0^{\theta(t)} \frac{d\tau}{a\left(\theta^{-1}(\tau)\right)} \int_{\mathbb{R}^n} \int_0^{\theta^{-1}(\tau)} |k(\xi',\alpha)|_T^{l,l/2} \left|u_0\left(\xi, \theta^{-1}(\tau) - \alpha\right)\right|_T^{l+2,(l+2)/2} \times$$

$$\times G(x-\xi; \theta(t) - \tau)\, d\alpha\, d\xi \leqslant \varphi_0 \frac{a_1 k_0 T}{a_0} \frac{t}{1!}, \quad (x,t) \in \mathbb{R}_T^n, \quad k_0 = |k(x',t)|_T^{l,l/2},$$

for arbitrary $n \geqslant 1$, we have

$$|u_n(x,t)|_T^{l+2,(l+2)/2} \leqslant \varphi_0 \left(\frac{a_1 k_0 T}{a_0}\right)^n \frac{t^n}{n!}.$$

In last estimates we have used the relation $\int_{\mathbb{R}^n} G(x-\xi, \theta(t)-\tau)\, d\xi = 1$. It follows from the above estimates that the series $\sum_{n=1}^{\infty} u_n(x,t)$ converges in $\mathbb{R}_T^n$ and its sum $u(x,t)$ belongs to the function space $H^{l+2,(l+2)/2}$. Since the sequence $u_n(x,t)$ defined by relations (1.3) converges to $u(x,t)$ uniformly in $\mathbb{R}_T^n$, we conclude that $u(x,t)$ is a solution of equation (1.2).

Now we show that this equation has a unique solution. Assume that there exist two solutions $u^1(x,t)$ and $u^2(x,t)$ of equation (1.2). Then their difference $Z(x,t) = u^2(x,t) - u^1(x,t)$ is a solution of the equation

$$Z(x,t) = \int_0^{\theta(t)} \frac{d\tau}{a\left(\theta^{-1}(\tau)\right)} \int_{\mathbb{R}^n} \int_0^{\theta^{-1}(\tau)} k(\xi', \alpha) Z(\xi, \theta^{-1}(\tau) - \alpha) G\left(x - \xi, \theta(t) - \tau\right)\, d\alpha\, d\xi.$$

Denote by $\tilde{Z}(t)$ the supremum of the modulus of the function $Z(x,t)$ over $x \in \mathbb{R}^n$ for each fixed $t \in [0,T]$. Then we have the inequality

$$\tilde{Z}(t) \leqslant \frac{a_1 k_0 T}{a_0} \int_0^{a_1 t} \tilde{Z}(\tau)\, d\tau, \quad t \in [0,T].$$

It is well known that there can be only one solution to this integral inequality: $\tilde{Z}(t) = 0$ for $t \in [0,T]$. Hence we also have $Z(x,t) = 0$ in $\mathbb{R}_T^n$; that is, $u^1(x,t) = u^2(x,t)$ in $\mathbb{R}_T^n$. Consequently, the equation (1.2) has a unique solution in $\mathbb{R}_T^n$. The proof of the lemma is complete. □

§ 2. Auxiliary problems

We introduce new function $\vartheta(x,t)$ by formula $\vartheta(x,t) = u_{x_n x_n}(x,t)$. Then the problem (0.1)–(0.3) leads us to the following relations:

$$\vartheta_t - a(t)\Delta\vartheta = \int_0^t k(x',\tau)\vartheta(x,t-\tau)\,d\tau, \tag{2.1}$$

$$\vartheta\Big|_{t=0} = \varphi_{x_n x_n}(x). \tag{2.2}$$

From the equalities (0.1) and (0.2) we obtain the additional condition for $\vartheta(x,t)$ by separating $a(t)u_{x_n x_n}$ in the term $a(t)\Delta u$ of (0.1) at $x_n = 0$ and using (2.2):

$$\vartheta|_{x_n=0} = \frac{1}{a(t)} f_t - \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} f - \frac{1}{a(t)} \int_0^t k(x',\tau) f(x',t-\tau)\,d\tau, \tag{2.3}$$

From this equality and (2.2) the matching condition of the given functions follows

$$\varphi_{x_n x_n}(x',0) = \frac{1}{a(0)} f_t(x',0) - \sum_{i=1}^{n-1} f_{x_i x_i}(x',0). \tag{2.4}$$

Now differentiate the equation (2.1) and (2.3) with respect to t and denote $\vartheta_t(x,t) = \omega(x,t)$. As result, one has problem of finding functions $\vartheta(x,t)$, $k(x',t)$, $\omega(x,t)$ from (2.1), (2.2) and the following equations:

$$\omega_t - a(t)\Delta\omega = (\ln a(t))'\omega - (\ln a(t))' \int_0^t k(x',\tau)\vartheta(x,t-\tau)\,d\tau +$$
$$+ \int_0^t k(x',\tau)\omega(x,t-\tau)\,d\tau + k(x',t)\varphi_{x_n x_n}(x), \tag{2.5}$$

$$\omega|_{t=0} = a(0)\Delta\varphi_{x_n x_n}(x), \tag{2.6}$$

$$\omega|_{x_n=0} = -\frac{a'(t)}{a^2(t)} f_t + \frac{1}{a(t)} f_{tt} - \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2} f_t + \frac{a'(t)}{a^2(t)} \int_0^t k(x',\tau) f(x',t-\tau)\,d\tau -$$
$$- \frac{1}{a(t)} \int_0^t k(x',\tau) f_t(x',t-\tau)\,d\tau - \frac{1}{a(t)} k(x',t)\varphi(x',0) \tag{2.7}$$

Here the initial condition was obtained by setting in (2.1) $t = 0$ and using (2.2). Note that the function $k(x',t)$ enters into the equation (2.7) without integral.

It is not difficult to show that at fulfilling of the matching conditions in (0.3), (2.4) and smoothness of given functions φ, f the inverse transformations are took place [17].

Thus, we proved the following assertion:

Lemma 2. *Suppose that all the conditions for the functions a, φ and f made in introduction are satisfied. Moreover, the matching conditions in (0.3) and (2.4) hold. Then, the problem (0.1)–(0.3) is equivalent to the auxiliary problems of determining functions $\vartheta(x,t)$, $k(x',t)$, $\omega(x,t)$ from (2.1), (1.2) and (2.5)–(2.7).*

§ 3. Reduction of the Auxiliary problems to the integral equations

Here we prove the main result of this Section, which is the following assertion:

Lemma 3. *The auxiliary problem (2.1), (2.2) and (2.5)–(2.7) is equivalent to the problem of determining functions $\vartheta(x,t)$, $k(x',t)$, $\omega(x,t)$ from the following system of integral equations:*

$$\vartheta(x,t) = \int_{\mathbb{R}^n} \varphi_{\xi_n\xi_n}(\xi) G(x-\xi,\theta(t))\,d\xi +$$

$$+\int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \int_0^{\theta^{-1}(\tau)} k(\xi',\alpha)\vartheta(\xi,\theta^{-1}(\tau)-\alpha) G(x-\xi,\theta(t)-\tau)\,d\alpha\,d\xi, \qquad (3.1)$$

$$\omega(x,t) = \int_{\mathbb{R}^n} a(0)\Delta\varphi_{\xi_n\xi_n}(\xi) G(x-\xi,\theta(t))\,d\xi +$$

$$+\int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \Big[(\ln a(\theta^{-1}(\tau)))'\omega(\xi,\theta^{-1}(\tau)) -$$

$$-(\ln a(\theta^{-1}(\tau)))' \int_0^{\theta^{-1}(\tau)} k(\xi',\alpha)\vartheta(\xi,\theta^{-1}(\tau)-\alpha)\,d\alpha +$$

$$+\int_0^{\theta^{-1}(\tau)} k(\xi',\alpha)\omega(\xi,\theta^{-1}(\tau)-\alpha)\,d\alpha + k(\xi',\theta^{-1}(\tau))\varphi_{\xi_n\xi_n}(\xi)\Big] G(x-\xi,\theta(t)-\tau)\,d\xi, \qquad (3.2)$$

$$k(x',t) = \frac{a(t)}{\varphi(x',0)} \Big[-\int_{R^n} a(0)\Delta\varphi_{\xi_n\xi_n}(\xi) G(x'-\xi',\xi_n,\theta(t))\,d\xi -$$

$$-\frac{a'(t)}{a^2(t)} f_t + \frac{1}{a(t)} f_{tt} - \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2} f_t \Big] +$$

$$+\frac{a(t)}{\varphi(x',0)} \Big(-\int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{R^n} \Big[(\ln a(\theta^{-1}(\tau)))'\omega(\xi,\theta^{-1}(\tau)) -$$

$$-(\ln a(\theta^{-1}(\tau))' \int_0^{\theta^{-1}(\tau)} k(\xi',\alpha)\vartheta(\xi,\theta^{-1}(\tau)-\alpha)\,d\alpha +$$

$$+\int_0^{\theta^{-1}(\tau)} k(\xi',\alpha)\omega(\xi,\theta^{-1}(\tau)-\alpha)\,d\alpha +$$

$$+k(\xi',\theta^{-1}(\tau))\varphi_{\xi_n\xi_n}(\xi)\Big] G(x'-\xi',\xi_n,\theta(t)-\tau)\,d\xi +$$

$$+\frac{a'(t)}{a^2(t)} \int_0^t k(x',\tau) f(x',t-\tau)\,d\tau - \frac{1}{a(t)} \int_0^t k(x',\tau) f_t(x',t-\tau)\,d\tau \Big). \qquad (3.3)$$

P r o o f. To prove Lemma it is not difficult to see that in view of (1.1) the integral equations (3.1), (3.2) are obtained from the problems (2.1), (2.2) and (2.5), (2.6), respectively. The equation (3.3) follows from (3.2) taking into account (2.7). □

§ 4. Existence and uniqueness

This section contains the main result of this paper. Here the existence and uniqueness result for the problem (2.1)–(2.3) is proved using the *contraction mapping principle* [24, pp. 87–97]. The idea is to write the integral equations for unknown functions $\vartheta(x,t),\ \omega(x,t),\ k(x',t)$ as a system with a nonlinear operator, and prove that this operator is a contraction mapping operator for sufficiently small T. The existence and uniqueness result then follows immediately.

Definition 1. Let F be an operator defined on a closed set Ω which is a subset of a Banach space. F is called *a contraction mapping operator* in Ω if it satisfies the following two properties:

(1) if $y \in \Omega$, then $Fx \in \Omega$ (that is F maps Ω into itself);

(2) if $y, z \in \Omega$, then $\|Fy - Fz\| \leqslant \rho\,\|y - z\|$ with $\rho < 1$ (ρ — is a constant independent of y and z).

Lemma 4 (contraction mapping principle [24, pp. 87–97]). *If F is a contraction mapping operator from Ω to Ω, then the equation $y = Fy$ has a unique solution $y_0 \in \Omega$.*

Theorem 1 (existence and uniqueness). *Suppose that all the conditions for the functions a, φ and f made in Introduction are satisfied. Moreover, the matching conditions in (0.3) and (2.4) are met. Then there exists a sufficiently small number $T > 0$ that the solution to the integral equations (3.1)–(3.3) in the class of functions $\{\vartheta(x,t),\ \omega(x,t)\} \in H^{l+2,(l+2)/2}(\bar{\mathbb{R}}^n_T)$, $k(x',t) \in H^{l,l/2}(\bar{\mathbb{R}}^{n-1}_T)$ exists and is unique. Thus, there is the unique classical solution to the problem (0.1)–(0.3).*

P r o o f. The system of equations (3.1), (3.2), (3.3) is a closed system for the unknown functions $\vartheta(x,t),\ \omega(x,t),\ k(x',t)$ in the domain $\mathbb{R}^n_T$. It can be rewritten in a nonlinear operator equation

$$\psi = A\psi, \tag{4.1}$$

where $\psi = (\psi_1,\ \psi_2,\ \psi_3)^* = (\vartheta(x,t), \omega(x,t),\ k(x',t))^*$, $*$ is the symbol of transposition, and according to the right sides of the equations (3.1)–(3.3), operator A has the form:

$A\psi = [(A\psi)_1,\ (A\psi)_2,\ (A\psi)_3]$, where

$$(A\psi)_1 = \psi_{01}(x,t) + \\ + \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \int_0^{\theta^{-1}(\tau)} \psi_3(\xi',\alpha)\psi_1(\xi,\theta^{-1}(\tau)-\alpha) G(x-\xi,\theta(t)-\tau)\, d\alpha\, d\xi, \tag{4.2}$$

$$(A\psi)_2 = \psi_{02}(x,t) + \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \Big[(\ln a(\theta^{-1}(\tau))'\psi_2(\xi,\theta^{-1}(\tau)) - \\ - (\ln a(\theta^{-1}(\tau))' \int_0^{\theta^{-1}(\tau)} \psi_3(\xi',\alpha)\psi_1(\xi,\tau-\alpha)\, d\alpha + \\ + \int_0^{\theta^{-1}(\tau)} \psi_3(\xi',\alpha)\psi_2(\xi,\theta^{-1}(\tau)-\alpha)\, d\alpha + \\ + \psi_3(\xi',\theta^{-1}(\tau))\varphi_{\xi_n\xi_n}(\xi)\Big] G(x-\xi,\theta(t)-\tau)\, d\xi, \tag{4.3}$$

$$(A\psi)_3 = \psi_{03}(x',t) + \frac{a(t)}{\varphi(x',0)} \Big(- \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \Big[(\ln a(\theta^{-1}(\tau)))'\psi_2(\xi,\theta^{-1}(\tau)) - \\ - (\ln a(\theta^{-1}(\tau)))' \int_0^{\theta^{-1}(\tau)} \psi_3(\xi',\alpha)\psi_1(\xi,\theta^{-1}(\tau)-\alpha)\, d\alpha + \\ + \int_0^{\theta^{-1}(\tau)} \psi_3(\xi',\alpha)\psi_2(\xi,\theta^{-1}(\tau)-\alpha)\, d\alpha + \\ + \psi_3(\xi',\theta^{-1}(\tau))\varphi_{\xi_n\xi_n}(\xi)\Big] G(x'-\xi',\xi_n,\theta(t)-\tau)\, d\xi \Big) + \\ + \frac{(\ln a(t))'}{\varphi(x',0)} \int_0^t \psi_3(x',\tau) f(x',t-\tau)\, d\tau - \frac{1}{\varphi(x',0)} \int_0^t \psi_3(x',\tau) f_t(x',t-\tau)\, d\tau. \tag{4.4}$$

In (4.2)–(4.4) we introduced notations:

$$\psi_{01}(x,t) = \int_{\mathbb{R}^n} \varphi_{\xi_n\xi_n}(\xi) G(x-\xi,\theta(t))\, d\xi,$$

$$\psi_{02}(x,t) = \int_{\mathbb{R}^n} a(0)\Delta\varphi_{\xi_n\xi_n}(\xi) G(x'-\xi',\xi_n,\theta(t))\, d\xi,$$

$$\psi_{03}(x',t) = \\ = \frac{a(t)}{\varphi(x',0)} \Big[- \int_{\mathbb{R}^n} a(0)\Delta\varphi_{\xi_n\xi_n}(\xi) G(x'-\xi',\xi_n,\theta(t))\, d\xi - \frac{a'(t)}{a^2(t)} f_t + \frac{1}{a(t)} f_{tt} - \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2} f_t \Big].$$

Denote $|\psi|_T^l = \max\left(|\psi_1|_T^{l/2}, \ |\psi_2|_T^{l/2}, \ |\psi_3|_T^{l/2}\right)$, $T < T_0$, T_0 is some positive number and consider in the space $H^{l,l/2}(\mathbb{R}_T^n)$ the set $S(T)$ of functions $\psi(x,t)$, which obey the inequality

$$|\psi - \psi_0|_T^l \leqslant |\psi_0|_{T_0}^l,$$

where $\psi_0 = (\psi_{01}, \ \psi_{02}, \ \psi_{03})$ and $|\psi_0|_{T_0}^l = \max(|\psi_{01}|_{T_0}^l, \ |\psi_{02}|_{T_0}^l, \ |\psi_{03}|_{T_0}^l)$.

It can be demonstrated that A satisfies the conditions of contraction mapping operator in $S(T)$, if T_0 is sufficiently small. Let $\psi \in S(T), \ T < T_0$. Then from the inequality (4.5), we have

$$|\psi_i|_T^l \leqslant 2\,|\psi_0|_{T_0}^l, \quad i = 1, 2, 3.$$

The Cauchy problem setting up the heat conduction equation has a solution, if initial condition, e. g., φ function is belong to the class H^{l+2} [20, p. 364]. In our case, φ is in H^{l+6}, because its fourth order derivatives are involved in the auxiliary problem.

Let us introduce the notations:

$$a_2 := \max_{t\in[0,T]} \left|(\ln a(t))'\right|, \quad \varphi_1 := |\varphi|^{l+6}, \quad f_0 := |f|_T^{l+4,(l+4)/2}.$$

First it is shown that A has the first property of a contraction mapping operator. Using the estimates of the thermal volume potentials [20, pp. 318–325] it is easy to obtain the following inequalities:

$$|(A\psi)_1 - \psi_{01}|_T^l =$$

$$= \Big|\int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \int_0^{\theta^{-1}(\tau)} \psi_3(\xi',\alpha)\psi_1(\xi,\theta^{-1}(\tau)-\alpha) G(x-\xi,\theta(t)-\tau)\,d\alpha\,d\xi\Big|_T^l \leqslant$$

$$\leqslant 4\beta_0(T)a_0^{-1}(|\psi_0|_{T_0}^l)^2,$$

$$|(A\psi)_2 - \psi_{02}|_T^l = \Big|\int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \Big[(\ln a(\theta^{-1}(\tau)))'\psi_2(\xi,\theta^{-1}(\tau)) -$$

$$- (\ln a(\theta^{-1}(\tau)))' \int_0^{\theta^{-1}(\tau)} \psi_3(\xi',\alpha)\psi_1(\xi,\theta^{-1}(\tau)-\alpha)\,d\alpha +$$

$$+ \int_0^{\theta^{-1}(\tau)} \psi_3(\xi',\alpha)\psi_2(\xi,\theta^{-1}(\tau)-\alpha)\,d\alpha + \psi_3(\xi',\tau)\varphi_{\xi_n\xi_n}(\xi)\Big] G(x-\xi,\theta(t)-\tau)\,d\xi\Big|_T^l \leqslant$$

$$\leqslant 2\beta_1(T)a_0^{-1}(a_2+\varphi_1)|\psi_0|_{T_0}^l + 4\beta_2(T)a_0^{-1}(a_2+1)\left(|\psi_0|_{T_0}^l\right)^2,$$

$$|(A\psi)_3 - \psi_{03}|_T^l = \Big|\frac{a(t)}{\varphi(x',0)}\Big(-\int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \Big[(\ln a(\theta^{-1}(\tau)))'\psi_2(\xi,\theta^{-1}(\tau)) -$$

$$- (\ln a(\theta^{-1}(\tau)))' \int_0^{\theta^{-1}(\tau)} \psi_3(\xi',\alpha)\psi_1(\xi,\theta^{-1}(\tau)-\alpha)\,d\alpha +$$

$$+ \int_0^{\theta^{-1}(\tau)} \psi_3(\xi',\alpha)\psi_2(\xi,\theta^{-1}(\tau)-\alpha)\,d\alpha + \psi_3(\xi',\theta^{-1}(\tau))\varphi_{\xi_n\xi_n}(\xi)\Big] G(x'-\xi',\xi_n,\theta(t)-\tau)\,d\xi\Big) +$$

$$+ \frac{(\ln a(t))'}{\varphi(x',0)} \int_0^t \psi_3(x',\tau)f(x',t-\tau)\,d\tau - \frac{1}{\varphi(x',0)} \int_0^t \psi_3(x',\tau)f_t(x',t-\tau)\,d\tau\Big|_T^l \leqslant$$

$$\leqslant 2\left[\beta_1(T)a_1a_0^{-1}\varphi_0^{-1}(a_2+\varphi_1) + f_0\varphi_0^{-1}T_0(a_2+1)\right]|\psi_0|_{T_0}^l + 4\beta_2(T)a_1a_0^{-1}\varphi_0^{-1}(a_2+1)\left(|\psi_0|_{T_0}^l\right)^2,$$

where $\beta_i(T)$ are increasing functions of T and $\beta_i(T) \to 0$ at $T \to 0$ for all $i = 0, 1, 2$. Therefore, if we choose T_0 so that the following inequalities should be satisfied:

$$\begin{gathered} 4\beta_0(T)a_0^{-1}|\psi_0|_{T_0}^l \leqslant 1, \\ 2\beta_1(T)a_0^{-1}(a_2+\varphi_1) + 4\beta_2(T)a_0^{-1}(a_2+1)|\psi_0|_{T_0}^l \leqslant 1, \\ 2\left[\beta_1(T)a_1a_0^{-1}\varphi_0^{-1}(a_2+\varphi_1) + f_0\varphi_0^{-1}T_0(a_2+1)\right] + 4\beta_2(T)a_1a_0^{-1}\varphi_0^{-1}(a_2+1)|\psi_0|_{T_0}^l \leqslant 1, \end{gathered} \tag{4.5}$$

then the operator A for $T < T_0$ has the first property of a contraction mapping operator, that is, $A\psi \in S(T)$.

Consider next the second property of a contraction mapping operator for A. Let $\psi^{(1)} = \left(\psi_1^{(1)},\ \psi_2^{(1)},\ \psi_3^{(1)}\right) \in S(T)$, $\psi^{(2)} = \left(\psi_1^{(2)},\ \psi_2^{(2)},\ \psi_3^{(2)}\right) \in S(T)$. Then we have

$$\big|\big((A\psi)^{(1)} - A\psi)^{(2)}\big)_1\big|_T^l = \Big|\int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \int_0^{\theta^{-1}(\tau)} \Big[\psi_3^{(1)}(\xi',\alpha)\psi_1^{(1)}(\xi,\theta^{-1}(\tau)-\alpha) - \\ - \psi_3^{(2)}(\xi',\alpha)\psi_1^{(2)}(\xi,\theta^{-1}(\tau)-\alpha)\Big] G(x-\xi,\theta(t)-\tau)\, d\alpha\, d\xi\Big|_T^l.$$

Here the integrand in the last integral can be estimated as follows

$$\Big|\psi_3^{(1)}\psi_1^{(1)} - \psi_3^{(2)}\psi_1^{(2)}\Big|_T^l = \Big|\left(\psi_3^{(1)} - \psi_2^{(2)}\right)\psi_1^{(1)} + \psi_3^{(2)}\left(\psi_1^{(1)} - \psi_1^{(2)}\right)\Big|_T^l \leqslant \\ \leqslant 2\Big|\psi^{(1)} - \psi^{(2)}\Big|_T^l \max\left(\Big|\psi_1^{(1)}\Big|_T^l,\ \Big|\psi_3^{(2)}\Big|_T^l\right) \leqslant 4\,|\psi_0|_T^l\,\Big|\psi^{(1)} - \psi^{(2)}\Big|_T^l.$$

Therefore

$$|(A\psi^{(1)} - A\psi^{(2)})_1|_T^l \leqslant 8\beta_0(T)a_0^{-1}|\psi_0|_{T_0}^l\,\left|\psi^{(1)} - \psi^{(2)}\right|_{T_0}^l.$$

The second and third components of $A\psi$ can be estimated in the analogous way:

$$\big|(A\psi^{(1)} - A\psi^{(2)})_2\big|_T^l = \\ = \Big|\int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \Big[(\ln a(\theta^{-1}(\tau)))'\Big(\psi_2^{(1)}(\xi,\theta^{-1}(\tau)) - \psi_2^{(2)}(\xi,\theta^{-1}(\tau))\Big) - \\ - (\ln a(\theta^{-1}(\tau)))' \int_0^{\theta^{-1}(\tau)} \Big(\psi_3^{(1)}(\xi',\alpha)\psi_1^{(1)}(\xi,\theta^{-1}(\tau)-\alpha) - \\ - \psi_3^{(2)}(\xi',\alpha)\psi_1^{(2)}(\xi,\theta^{-1}(\tau)-\alpha)\Big)\, d\alpha + \\ + \int_0^{\theta^{-1}(\tau)} \Big(\psi_3^{(1)}(\xi',\alpha)\psi_2^{(1)}(\xi,\theta^{-1}(\tau)-\alpha) - \psi_3^{(2)}(\xi',\alpha)\psi_2^{(2)}(\xi,\theta^{-1}(\tau)-\alpha)\Big)\, d\alpha + \\ + \Big(\psi_3^{(1)}(\xi',\theta^{-1}(\tau)) - \psi_3^{(2)}(\xi',\theta^{-1}(\tau))\Big)\varphi_{\xi_n\xi_n}(\xi)\Big] G(x-\xi,\theta(t)-\tau)\, d\xi\Big|_{T_0}^l \leqslant \\ \leqslant \big[2\beta_1(T)a_0^{-1}(a_2+\varphi_1) + 8\beta_2(T)a_0^{-1}(a_2+1)|\psi_0|_{T_0}^l\big]\,\left|\psi^{(1)} - \psi^{(2)}\right|_{T_0}^l.$$

$$\big|(A\psi^{(1)} - A\psi^{(2)})_3\big|_T^l = \\ = \Big|\frac{a(t)}{\varphi(x',0)}\Big(-\int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \Big[(\ln a(\theta^{-1}(\tau)))'\Big(\psi_2^{(1)}(\xi,\theta^{-1}(\tau)) - \psi_2^{(2)}(\xi,\theta^{-1}(\tau))\Big) - \\ - (\ln a(\theta^{-1}(\tau)))' \int_0^{\theta^{-1}(\tau)} \Big(\psi_3^{(1)}(\xi',\alpha)\psi_1^{(1)}(\xi,\theta^{-1}(\tau)-\alpha) - \\ - \psi_3^{(2)}(\xi',\alpha)\psi_1^{(2)}(\xi,\theta^{-1}(\tau)-\alpha)\Big)\, d\alpha +$$

$$+\int_0^{\theta^{-1}(\tau)}\Big(\psi_3^{(1)}(\xi',\alpha)\psi_2^{(1)}(\xi,\theta^{-1}(\tau)-\alpha)-\psi_3^{(2)}(\xi',\alpha)\psi_2^{(2)}(\xi,\theta^{-1}(\tau)-\alpha)\Big)\,d\alpha+$$
$$+\Big(\psi_3^{(1)}(\xi',\theta^{-1}(\tau))-\psi_3^{(2)}(\xi',\theta^{-1}(\tau))\Big)\varphi_{\xi_n\xi_n}(\xi)\Big]G(x'-\xi',\xi_n,\theta(t)-\tau)\,d\xi\Big)+$$
$$+\frac{(\ln a(t))'}{\varphi(x',0)}\int_0^t\Big(\psi_3^{(1)}(\xi',\tau)-\psi_3^{(2)}(\xi',\tau)\Big)f(x',t-\tau)\,d\tau-$$
$$-\frac{1}{\varphi(x',0)}\int_0^t\Big(\psi_3^{(1)}(\xi',\tau)-\psi_3^{(2)}(\xi',\tau)\Big)f_t(x',t-\tau)\,d\tau\Big|_T^l\leqslant$$
$$\leqslant\big[2\left[\beta_1(T)a_1a_0^{-1}\varphi_0^{-1}(a_2+\varphi_1)+f_0\varphi_0^{-1}T_0(a_2+1)\right]+8\beta_2(T)a_1a_0^{-1}\varphi_0^{-1}(a_2+1)|\psi_0|_{T_0}^l\big]\times$$
$$\times\left|\psi^{(1)}-\psi^{(2)}\right|_{T_0}^l.$$

Hence, $\left|\left(A\psi^{(1)}-A\psi^{(2)}\right)\right|_T^l<\rho\Big|\psi^{(1)}-\psi^{(2)}\Big|_{T_0}^l$, where $\rho<1$, if T_0 satisfies the conditions

$$\begin{gathered}8\beta_0(T)a_0^{-1}|\psi_0|_{T_0}^l\leqslant\rho<1,\\ 2\beta_1(T)a_0^{-1}(a_2+\varphi_1)+8\beta_2(T)a_0^{-1}(a_2+1)|\psi_0|_{T_0}^l\leqslant\rho<1,\\ 2\left[\beta_1(T)a_1a_0^{-1}\varphi_0^{-1}(a_2+\varphi_1)+f_0\varphi_0^{-1}T_0(a_2+1)\right]+8\beta_2(T)a_1a_0^{-1}\varphi_0^{-1}(a_2+1)|\psi_0|_{T_0}^l\leqslant\rho<1.\end{gathered}\tag{4.6}$$

It is not difficult to see that from fulfilling the inequalities (4.6) the inequalities (4.5) follows. This indicates that at T_0 satisfying the conditions (4.6), A satisfies both the properties of a contraction mapping operator for $T<T_0$, that is, A realizes contracted mapping of the set $S(T)$ onto itself. Then, according to Lemma 1, in the set $S(T)$ there exists only one fixed point of transformations, that is there exists only one solution to (4.1). Hence, solving the system of (3.1)–(3.3) for example, by the method of successive approximations, we uniquely find the functions $\vartheta(x,t)$, $k(x',t)$ which belong to $H^{l+2,(l+2)/2}\left(\mathbb{R}_T^n\right)$ and $H^{l,l/2}\left(\mathbb{R}_T^{n-1}\right)$, respectively. □

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Об исследовании обратной задачи для параболического интегро-дифференциального уравнения с переменным коэффициентом теплопроводности

Ключевые слова: интегро-дифференциальное уравнение, обратная задача, ядро, принцип сжимающих отображений.

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Исследуется обратная задача определения многомерного ядра интегрального члена, зависящего от временной переменной t и $(n-1)$-мерной пространственной переменной $x' = (x_1, \ldots, x_{n-1})$ из n-мерного уравнения теплопроводности с переменным коэффициентом теплопроводности. Прямую задачу представляет задача Коши для этого уравнения. Интегральный член имеет вид свертки по времени ядра и решения прямой задачи. Дополнительное условие для решения обратной задачи задается решение прямой задачи на гиперплоскости $x_n = 0$. В начале изучаются свойства решения прямой задачи. Для этого эта задача сводится к решению интегрального уравнения второго порядка вольтерровского типа и к нему применяется метод последовательных приближений. Далее поставленная обратная задача приводится к двум вспомогательным задачам, дополнительное условие второй из них содержит неизвестное ядро вне интеграла. Затем вспомогательные задачи заменяются эквивалентной замкнутой системой интегральных уравнений вольтерровского типа относительно неизвестных функций. Применяя метод сжатых отображений к этой системе в классе гёльдеровских функций доказываем основной результат статьи, который является теоремой локального существования и единственности решения обратной задачи.

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