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NON-ANTICIPATIVE STRATEGIES IN GUARANTEE OPTIMIZATION PROBLEMS UNDER FUNCTIONAL CONSTRAINTS ON DISTURBANCES

For a dynamical system controlled under conditions of disturbances, a problem of optimizing the guaranteed result is considered. A feature of the problem is the presence of functional constraints on disturbances, under which, in general, the set of admissible disturbances is not closed with respect to the operation of "gluing up" of two of its elements. This circumstance does not allow to apply directly the methods developed within the differential games theory for studying the problem and, thus, leads to the necessity of modifying them appropriately. The paper provides a new notion of a non-anticipative control strategy. It is proved that the corresponding functional of the optimal guaranteed result satisfies the dynamic programming principle. As a consequence, so-called properties of u- and v-stability of this functional are established, which may allow, in the future, to obtain a constructive solution of the problem in the form of feedback (positional) controls.

Keywords: guarantee optimization, functional constraints, non-anticipative strategies, dynamic programming principle.

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Introduction

The paper deals with a problem of optimizing the guaranteed result for a dynamical system controlled under conditions of disturbances. A motion of the system is considered on a finite time interval and is described by an ordinary differential equation. Admissible controls are measurable functions with values in a given compact set. The goal of control is to minimize a cost functional, which evaluates system's motions.

In a classical formulation of such problems, admissible disturbances are usually assumed to satisfy two conditions similar to those imposed on admissible controls, i.e., instantaneous (or geometric) constraints coupled with the claim of measurability. In this case, the theory of zero-sum differential games can be involved to study and solve the problem. In particular, in order to obtain a constructive solution in the form of feedback (positional) controls, we can use, for example, the method of extremal aiming, developed within the positional approach (see, e.g., [1,2]). At the same time, it should be noted that a direct application of many basic results of the differential games theory, including the method of extremal aiming, requires (often implicitly) the following "gluing up" property: an admissible disturbance extended by any other one from any time composes a new admissible disturbance.

On the other hand, there is a wide range of applied control problems in which the standard two assumptions on the set of admissible disturbances are supplemented by constraints of a functional nature. For example, they can be determined by such conditions as constancy, continuity, or Lipschitz continuity, as well as boundedness of a number of discontinuity points and compactness in some topology (see, e.g., [3–10]). Usually, by taking this additional information into account, we can significantly improve the value of optimal guaranteed result. However, under such functional constraints, the property of "gluing up" may no longer be satisfied. In other words, in general, the "continuation" of an admissible disturbance depends on its "beginning" (here, it seems appropriate to designate this property of the set of admissible disturbances as "heredity"). This circumstance does not allow to apply directly the results and methods developed within the differential games theory. So, the presence of a functional constraint is the main feature of the considered guarantee optimization problem. Note that we do not fix a specific form of this constraint, and, in particular, the set of admissible disturbances may turn out to be hereditary.

The aim of this paper is to construct a functional in the space of positions of the dynamical system that possesses suitable properties of u- and v-stability (see, e.g., [1, Sect. 4.2] and [2, Sect. 8]), which may allow, in the future, to use this functional as a basis for obtaining optimal feedback controls via the extremal aiming technique. For these purposes, generalized solutions of the corresponding Hamilton–Jacobi equations, constructions of program iterations, or the apparatus of non-anticipative control strategies are traditionally involved. We follow the latter option and consider the functional of the optimal guaranteed result with respect to non-anticipative control strategies. This choice is prompted by the classical result (see, e.g., [2, Sect. 9]) about the coincidence of the optimal guaranteed results in the classes of positional strategies and non-anticipative strategies under Isaacs' condition and similar results (see, e.g., [5, 11, 12]) under some other conditions imposed on the control problem. Nevertheless, dealing with the functional constraints leads to a notion of a non-anticipative control strategy that differs from the standard ones (see, e.g., [13–16]). Namely, the new class of strategies depends on the system's dynamics and the current history of the control process. An additional argument in favor of the proposed framework is provided by analyzing simple examples, one of which is presented in the motivation part of the paper. Thus, the main result of the paper is the proof of the fact that the introduced functional satisfies the dynamic programming principle and, as a consequence, possesses the properties of u- and v-stability.

Finally, we note that the necessity of modifying the standard notion of a non-anticipative control strategy also arises, for example, in differential games with phase constraints (see, e.g., [17, 18]), and with constraints on players' controls given by graphs (see, e.g., [19]).

§ 1. Problem statement

This section is devoted to the statement of a guarantee optimization problem studied in the paper. In particular, we describe a dynamical system and a cost functional, give a definition of a functional constraint on disturbance, and introduce a new notion of a non-anticipative control strategy.

1.1. Dynamical system

We consider a dynamical system whose motion is described by the differential equation

$$\dot{x}(t) = f(t, x(t), u(t), v(t)), \quad t \in T := [t_0, \vartheta], \tag{1.1}$$

and the initial condition

$$x(t_0) = x_0. \tag{1.2}$$

Here, t is the time; $x(t) \in \mathbb{R}^n$ is the current value of the state vector; $\dot{x}(t) := dx(t)/dt$; $u(t) \in \mathbb{R}^p$ and $v(t) \in \mathbb{R}^q$ are respectively the current values of control and disturbance; t_0 and ϑ are the initial and terminal times; $x_0 \in \mathbb{R}^n$ is the initial value of the state vector, fixed throughout the paper. Control and disturbance are subject to instantaneous constraints

$$u(t) \in P, \quad v(t) \in Q, \quad t \in T, \tag{1.3}$$

where $P \subset \mathbb{R}^p$ and $Q \subset \mathbb{R}^q$ are given compact sets. The function $f : T \times \mathbb{R}^n \times P \times Q \to \mathbb{R}^n$ is continuous, local Lipschitz continuous with respect to the second variable, and satisfies a sublinear growth condition with respect to the second variable (see, e.g., [2, § 3]).

1.2. Admissible controls and disturbances

Let $\mathcal{U}$ and $\mathcal{V}$ be the sets of all (Lebesgue) measurable functions $u : T \to P$ and $v : T \to Q$, respectively. In accordance with (1.3), by an admissible control, we mean a function $u(\cdot) \in \mathcal{U}$. On the other hand, we assume that a non-empty set $V \subset \mathcal{V}$ is specified, and only the functions

$$v(\cdot) \in V \tag{1.4}$$

are considered as admissible disturbances. This additional constraint is called "functional". Note that we do not restrict a way of how disturbance is formed (in particular, it can be generated by some feedback rule). It is only required that the realized by the terminal time ϑ disturbance belongs to V. So, relations (1.1)–(1.3), describing the dynamical system, are complemented by (1.4), which is a distinctive feature of the problem studied in the paper.

Problems with constraints of this kind arise when the admissible disturbances as functions of time have some additional properties and these properties are known. Below, we present some typical examples.

$i)$ Only constant disturbances are admissible. It means that disturbance can be considered as an unknown parameter (see, e.g., [8]). In this case, the set V consists of functions $v : T \to Q$ such that $v(t) = v'$, $t \in T$, for some $v' \in Q$.

$ii)$ A finite number of certain admissible disturbances $v_i(\cdot) \in \mathcal{V}$, $i \in \overline{1,k}$, is specified (see, e.g., [9], where such disturbances are called possible scenarios).

$iii)$ It is known that admissible disturbances are piecewise constant functions $v : T \to Q$ with the prescribed number of possible discontinuity points. For example, if only one discontinuity point is allowed, the corresponding set V consists of functions $v : T \to Q$ such that $v(t) = v'$, $t \in [t_0, t')$, and $v(t) = v''$, $t \in [t', \vartheta]$, for some v', $v'' \in Q$ and $t' \in T$. As another possible variant of such constraints, we point out the case when v', $v'' \in Q$ are fixed and only the time t' is unknown, which is close to so-called problems with a possible breakdown (see, e.g., [6, 7, 10]).

$iv)$ Admissible disturbances are continuous functions $v : T \to Q$ that meet a given restriction on the modulus of continuity. In particular, one can consider the set V of functions $v : T \to Q$ satisfying a Lipschitz condition with a prescribed constant (see, e.g., [3]).

$v)$ Only functions $v : T \to Q$ that possess certain coordinatewise monotonicity properties are admissible disturbances (see, e.g., [4]).

$vi)$ It is known that admissible disturbances are piecewise constant (or piecewise continuous) functions $v : T \to Q$ with an arbitrary number of discontinuity points.

Note that the proposed notion of a functional constraint on disturbances substantially differs from the one developed in [5, 11, 12]. Namely, in these studies, dealing, for example, with a kind of constraints described in item $iii)$, it is supposed that a number of possible discontinuity points that admissible disturbances may have is bounded from above by some number, but this number itself is unknown. Thus, the setting considered in the present paper allows to cover the case when this number is given or estimated from above, which often seems more reasonable from the practical viewpoint.

1.3. Cost functional

By virtue of the conditions imposed on f, for every admissible control $u(\cdot) \in \mathcal{U}$ and disturbance $v(\cdot) \in V$, there exists a unique system's motion $x(\cdot) := x(\cdot; u(\cdot), v(\cdot))$, which is an absolutely continuous function $x : T \to \mathbb{R}^n$ that meets initial condition (1.2) and, together with $u(\cdot)$ and $v(\cdot)$, satisfies dynamic equation (1.1) for almost every (a.e.) $t \in T$.

We assume that a cost functional has the following form:

$$J(u(\cdot), v(\cdot)) = \sigma\big(x(\cdot; u(\cdot), v(\cdot))\big), \quad u(\cdot) \in \mathcal{U}, \quad v(\cdot) \in V. \tag{1.5}$$

Here, $\sigma : C(T; \mathbb{R}^n) \to \mathbb{R}$ is a given function, and $C(T; \mathbb{R}^n)$ is the set of continuous functions from T to $\mathbb{R}^n$. Note that, for validity of the results presented in the paper, we do not require any properties of the function σ. The goal of control is to minimize the value of cost functional (1.5) against the worst case disturbance. Thus, we formulate control problem under conditions of disturbances (1.1)–(1.5) as a guarantee optimization problem.

A usual approach to investigate this problem is to look at it as a zero-sum differential game in which the first player (by choosing control actions) wants to minimize the cost functional, while the second player (by choosing disturbance actions) wants to maximize it. However, the classical theory of differential games deals only with the case when $V = \mathcal{V}$, i.e., every function $v(\cdot) \in \mathcal{V}$ is considered as an admissible disturbance. Studying this issue in more detail, we observe that the following decomposability property of the set $\mathcal{V}$ is crucial: for any $v(\cdot)$, $v'(\cdot) \in \mathcal{V}$ and $t' \in T$, the function $v''(\cdot)$ defined by $v''(t) = v(t)$ for $t \in [t_0, t')$ and $v''(t) = v'(t)$ for $t \in [t', \vartheta]$ belongs to $\mathcal{V}$. In other words, if we take any two admissible disturbances $v(\cdot)$ and $v'(\cdot)$ and "glue them up" at an arbitrary time t', we obtain the function $v''(\cdot)$, which is also an admissible disturbance. On the other hand, if $V \neq \mathcal{V}$, the case which the present paper is focused on, this property may be violated (see typical examples $i)$ to $v)$ above), which does not allow to apply directly the classical approaches developed in the theory of differential games to study the problem under consideration.

Moreover, let us note that the absence of the decomposability property of a set $V \subset \mathcal{V}$ may result in the fact that V is hereditary. Namely, given a time $t' \in T$ and an admissible disturbance $v(\cdot) \in V$, it may happen that the future values $v(t)$, $t \in [t', \vartheta]$, depend on the past values $v(t)$, $t \in [t_0, t')$, or even are uniquely determined by them.

1.4. Non-anticipative control strategies

The need to correctly take functional constraint (1.4) into account leads to changes in a mathematical formalization of the guarantee optimization problem and, in particular, affects the choice of the class of non-anticipative control strategies.

Let us introduce some notation. For each $t \in T$, $x \in \mathbb{R}^n$, and $v \in Q$, define

$$\mathbf{q}(t, x, v) := \big\{v' \in Q \mid f(t, x, u, v') = f(t, x, u, v),\ u \in P\big\}, \tag{1.6}$$

where f is the function from (1.1). Note that, for every $t \in T$ and $x \in \mathbb{R}^n$, the relation $\{(v, v') \in Q \times Q \mid v' \in \mathbf{q}(t, x, v)\}$ is an equivalence relation on Q. Hence, for any v, $v' \in Q$,

$$\big(v' \in \mathbf{q}(t, x, v)\big) \Leftrightarrow \big(v \in \mathbf{q}(t, x, v')\big) \Leftrightarrow \big(\mathbf{q}(t, x, v) = \mathbf{q}(t, x, v')\big). \tag{1.7}$$

A function $\alpha : V \to \mathcal{U}$ is called a non-anticipative control strategy if it satisfies the following property (P): for every $v_1(\cdot)$, $v_2(\cdot) \in V$ and $\tau \in (t_0, \vartheta]$, the inclusion

$$v_1(s) \in \mathbf{q}\big(s, x(s; \alpha(v_2)(\cdot), v_2(\cdot)), v_2(s)\big), \quad s \in [t_0, \tau), \tag{1.8}$$

implies the equality

$$x(s; \alpha(v_1)(\cdot), v_1(\cdot)) = x(s; \alpha(v_2)(\cdot), v_2(\cdot)), \quad s \in [t_0, \tau],$$

where $x(\cdot; \alpha(v_i)(\cdot), v_i(\cdot))$ is the system's motion that corresponds to the control $u_i(\cdot) = \alpha(v_i)(\cdot)$ and the disturbance $v_i(\cdot)$, $i \in \{1, 2\}$. Let $\mathcal{A}$ be the set of non-anticipative control strategies.

In view of (1.6), the inclusion (1.8) means that, for every $s \in [t_0, \tau)$ and $u \in P$,

$$f\big(s, x(s; \alpha(v_2)(\cdot), v_2(\cdot)), u, v_1(s)\big) = f\big(s, x(s; \alpha(v_2)(\cdot), v_2(\cdot)), u, v_2(s)\big). \tag{1.9}$$

In this sense, in premise (1.8) of non-anticipativity property (P), we identify admissible disturbances that can not be distinguished when substituted into dynamic equation (1.1).

An example of a non-anticipative control strategy is a function $\alpha_*(v)(\cdot) := u_*(\cdot)$, $v(\cdot) \in V$, where $u_*(\cdot) \in \mathcal{U}$ is fixed (see the proof of Lemma 4 below). In particular, we have $\mathcal{A} \neq \varnothing$.

We define the optimal guaranteed result in problem (1.1)–(1.5) as follows:

$$\Gamma := \inf_{\alpha \in \mathcal{A}} \sup_{v(\cdot) \in V} \sigma\big(x(\cdot; \alpha(v)(\cdot), v(\cdot))\big). \tag{1.10}$$

Below, in § 2 and § 3, we discuss the introduced new property of non-anticipation. In particular, we study a connection between property (P) and a standard property of non-anticipation in the context of the problem under consideration and present a simple motivating example. Further, in § 4, we introduce a suitable notion of a position of the dynamical system, which takes into account functional constraint (1.4), and define the value of the optimal guaranteed result for each position in § 5. After that, in § 6, we prove that the functional of the optimal guaranteed result satisfies the dynamic programming principle, which is the main result of the paper. As a consequence, in § 7, we derive that the functional of the optimal guaranteed result possesses so-called u- and v-stability properties, which seem to be important for further research in the direction of constructing optimal feedback controls by suitably modifying the methods developed in the positional differential games theory.

§ 2. Standard property of non-anticipation

Starting from the seminal works [13–16], it is widely accepted to call a function $\alpha_* : V \to \mathcal{U}$ non-anticipative if it satisfies the following property (P_*): if $v_1(\cdot)$, $v_2(\cdot) \in V$, $\tau \in (t_0, \vartheta]$, and

$$v_1(s) = v_2(s), \quad s \in [t_0, \tau), \tag{2.1}$$

then

$$\alpha_*(v_1)(s) = \alpha_*(v_2)(s) \text{ for a.e. } s \in [t_0, \tau]. \tag{2.2}$$

The only difference here is that we restrict the domain of α_* to V in accordance with (1.4). Let $\mathcal{A}_*$ be the set of all such functions α_*. Note that, during this section, with some abuse of terminology, any function from V to $\mathcal{U}$ that has this or that property of non-anticipation is treated as a control strategy. So, by analogy with (1.10), we define the value of the optimal guaranteed result in the set of strategies $\mathcal{A}_*$ as follows:

$$\Gamma_* := \inf_{\alpha_* \in \mathcal{A}_*} \sup_{v(\cdot) \in V} \sigma\big(x(\cdot; \alpha_*(v)(\cdot), v(\cdot))\big). \tag{2.3}$$

Let us study the relationship between the sets of strategies $\mathcal{A}$ and $\mathcal{A}_*$ and the values of the optimal guaranteed results Γ and Γ_*. We observe that non-anticipativity properties (P) and (P_*) differ both in premise and conclusion. In this regard, as an intermediate step, it is convenient to introduce one more property of non-anticipation with the premise as in (P_*) and the conclusion as in (P). Namely, we say that a function $\alpha^* : V \to \mathcal{U}$ possesses property (P^*) if, for any $v_1(\cdot)$, $v_2(\cdot) \in V$ and $\tau \in (t_0, \vartheta]$ such that (2.1) holds, we have

$$x(s; \alpha^*(v_1)(\cdot), v_1(\cdot)) = x(s; \alpha^*(v_2)(\cdot), v_2(\cdot)), \quad s \in [t_0, \tau].$$

Respectively, we consider the set $\mathcal{A}^*$ of all such functions α^* and define the corresponding value of the optimal guaranteed result by

$$\Gamma^* := \inf_{\alpha^* \in \mathcal{A}^*} \sup_{v(\cdot) \in V} \sigma\big(x(\cdot; \alpha^*(v)(\cdot), v(\cdot))\big).$$

First of all, we note that $\mathcal{A}_* \subset \mathcal{A}^*$ since, for every $v_1(\cdot)$, $v_2(\cdot) \in V$ and $\tau \in (t_0, \vartheta]$, equalities (2.1) and (2.2) imply that $x(s; \alpha_*(v_1)(\cdot), v_1(\cdot)) = x(s; \alpha_*(v_2)(\cdot), v_2(\cdot))$, $s \in [t_0, \tau]$. Hence, $\Gamma^* \leq \Gamma_*$, and, in order to conclude that $\Gamma^* = \Gamma_*$, we prove the proposition below.

Proposition 1. *For every $\alpha^* \in \mathcal{A}^*$, there exists $\alpha_* \in \mathcal{A}_*$ such that $x(\cdot; \alpha_*(v)(\cdot), v(\cdot)) = x(\cdot; \alpha^*(v)(\cdot), v(\cdot))$ for any $v(\cdot) \in V$.*

P r o o f. **1.** For every $v(\cdot) \in V$, denote

$$\mathfrak{U}(v) := \big\{u(\cdot) \in \mathcal{U} \mid x(\cdot; u(\cdot), v(\cdot)) = x(\cdot; \alpha^*(v)(\cdot), v(\cdot))\big\}.$$

One can show that the set-valued function $V \ni v(\cdot) \mapsto \mathfrak{U}(v) \subset \mathcal{U}$ meets the following property of non-anticipation, corresponding to (P_*): if $v_1(\cdot)$, $v_2(\cdot) \in V$ and $\tau \in (t_0, \vartheta]$ are such that $v_1(s) = v_2(s)$, $s \in [t_0, \tau)$, then, for any $u_1(\cdot) \in \mathfrak{U}(v_1)$, there exists $u_2(\cdot) \in \mathfrak{U}(v_2)$ satisfying $u_1(s) = u_2(s)$, $s \in [t_0, \tau)$. Indeed, in view of property (P^*) of α^*, the required function $u_2(\cdot)$ can be defined by $u_2(s) := u_1(s)$ for $s \in [t_0, \tau)$ and $u_2(s) := \alpha^*(v_2)(s)$ for $s \in [\tau, \vartheta]$. Thus, in order to complete the proof, it is sufficient to verify that the set-valued function $\mathfrak{U}$ admits a selection α_* that is non-anticipative in the sense of (P_*).

2. For $t \in [t_0, \vartheta]$, $x \in \mathbb{R}^n$, $u \in P$ and $v \in Q$, by analogy with (1.6), consider the set

$$\mathbf{p}(t, x, u, v) := \big\{u' \in P \mid f(t, x, u, v) = f(t, x, u', v)\big\}.$$

Taking into account compactness of P and continuity of f, we obtain that the set-valued function $T \times \mathbb{R}^n \times P \times Q \ni (t, x, u, v) \mapsto \mathbf{p}(t, x, u, v) \subset P$ has non-empty and compact values and is upper semicontinuous. Hence, due to, e.g., [20, Corollary III.3], the function $\mathbf{p} : T \times \mathbb{R}^n \times P \times Q \to \mathbf{comp}(P)$ is Borel measurable, where $\mathbf{comp}(P)$ is the set of all non-empty compact subsets of P endowed with the Hausdorff metric.

Further, introduce the set-valued function $\mathbf{comp}(P) \ni D \mapsto \mathrm{Id}(D) = D \subset P$. Based on, e.g., [20, Theorem III.6], we conclude that this set-valued function admits a Borel measurable selection $\mathbf{comp}(P) \ni D \mapsto \pi(D) \in P$.

Now, let $\mathbf{u} : T \times \mathbb{R}^n \times P \times Q \to P$ be the composition of $\mathbf{p}$ and π, i.e.,

$$\mathbf{u}(t, x, u, v) := \pi(\mathbf{p}(t, x, u, v)), \quad t \in T, \quad x \in \mathbb{R}^n, \quad u \in P, \quad v \in Q.$$

Note that $\mathbf{u}$ is Borel measurable and, for every $t \in T$, $x \in \mathbb{R}^n$, $u \in P$ and $v \in Q$, we have

$$f(t, x, u, v) = f(t, x, \mathbf{u}(t, x, u, v), v), \tag{2.4}$$

and, moreover, if $u' \in \mathbf{p}(t, x, u, v)$, then

$$\mathbf{u}(t, x, u, v) = \mathbf{u}(t, x, u', v). \tag{2.5}$$

3. Given $v(\cdot) \in V$, consider the function $u_v(t) := \mathbf{u}\big(t, x(t; \alpha^*(v)(t), v(t)), \alpha^*(v)(t), v(t)\big)$, $t \in T$. Note that $u_v(\cdot) \in \mathcal{U}$. Due to (2.4), for the motion $x(\cdot) := x(\cdot; \alpha^*(v)(\cdot), v(\cdot))$, we have

$$\dot{x}(t) = f(t, x(t), \alpha^*(v)(t), v(t)) = f(t, x(t), u_v(t), v(t)) \text{ for a.e. } t \in T,$$

wherefrom it follows that $x(\cdot) = x(\cdot; u_v(\cdot), v(\cdot))$ and, consequently, $u_v(\cdot) \in \mathfrak{U}(v)$.

Then, the function $\alpha_*(v)(\cdot) := u_v(\cdot)$, $v(\cdot) \in V$, is a selection of the set-valued function $\mathfrak{U}$, and it remains to establish that α_* possesses property (P_*). Fix $v_1(\cdot)$, $v_2(\cdot) \in V$ and $\tau \in (t_0, \vartheta]$ such that $v_1(s) = v_2(s)$, $s \in [t_0, \tau]$, and denote $u_1(\cdot) := \alpha_*(v_1)(\cdot) := u_{v_1}(\cdot)$, $u_2(\cdot) := \alpha_*(v_2)(\cdot) := u_{v_2}(\cdot)$. We need to show that $u_1(s) = u_2(s)$ for a.e. $s \in [t_0, \tau]$. Since α^* satisfies property (P^*), we obtain $x(s; \alpha^*(v_1)(\cdot), v_1(\cdot)) = x(s; \alpha^*(v_2)(\cdot), v_2(\cdot))$, $s \in [t_0, \tau]$. Therefore, for a.e. $s \in [t_0, \tau]$, we derive

$$f\big(s, x(s; \alpha^*(v_1)(\cdot), v_1(\cdot)), \alpha^*(v_1)(s), v_1(s)\big) = f\big(s, x(s; \alpha^*(v_2)(\cdot), v_2(\cdot)), \alpha^*(v_2)(s), v_2(s)\big),$$

and, hence,

$$f\big(s, x(s; \alpha^*(v_1)(\cdot), v_1(\cdot)), \alpha^*(v_1)(s), v_1(s)\big) = f\big(s, x(s; \alpha^*(v_1)(\cdot), v_1(\cdot)), \alpha^*(v_2)(s), v_1(s)\big).$$

This means that $\alpha^*(v_2)(s) \in \mathbf{p}(s, x(s; \alpha^*(v_1)(\cdot), v_1(\cdot)), \alpha^*(v_1)(s), v_1(s))$ for a.e. $s \in [t_0, \tau]$. So, owing to (2.5), we derive

$$\mathbf{u}\big(s, x(s; \alpha^*(v_1)(\cdot), v_1(\cdot)), \alpha^*(v_1)(s), v_1(s)\big) = \mathbf{u}\big(s, x(s; \alpha^*(v_1)(\cdot), v_1(\cdot)), \alpha^*(v_2)(s), v_1(s)\big)$$

for a.e. $s \in [t_0, \tau]$. Thus, taking into account that, for $s \in [t_0, \tau)$,

$$\mathbf{u}\big(s, x(s; \alpha^*(v_1)(\cdot), v_1(\cdot)), \alpha^*(v_2)(s), v_1(s)\big) = \mathbf{u}\big(s, x(s; \alpha^*(v_2)(\cdot), v_2(\cdot)), \alpha^*(v_2)(s), v_2(s)\big),$$

we conclude $u_1(s) = u_2(s)$ for a.e. $s \in [t_0, \tau]$. The proposition is proved. □

Remark 1. The main part of the proof above consists in constructing a non-anticipative selection of a given non-anticipative set-valued function. Note that existence of such a selection can be obtained on the basis of general results from, e.g., [21–23] (see also [24]) and the corresponding technique developed in these works. However, the presented proof seems more straightforward, and, in particular, it does not require the use of Zorn's lemma.

Finally, taking into account that, in accordance with (1.6), equality (2.1) implies inclusion (1.8), we derive $\mathcal{A} \subset \mathcal{A}^*$, and, thus, we get

$$\Gamma_* = \Gamma^* \le \Gamma. \tag{2.6}$$

This means that the value of the optimal guaranteed result Γ_* in the set $\mathcal{A}_*$ of the standard non-anticipative control strategies is always not greater than that in the set $\mathcal{A}$ of the non-anticipative control strategies considered in the paper.

The example below shows that the inequality in (2.6) can be strict. Moreover, it illustrates the fact that, in general, the value Γ_* can not be ensured by any feedback control rule, and, in this sense, the use of the standard non-anticipative control strategies does not meet possibilities that are available from a practical viewpoint. These circumstances are the motivation for the approach proposed in the paper.

§ 3. Motivating example

Consider a dynamical system whose motion is described by the differential equation

$$\dot{x}(t) = u(t) + b(t)v(t), \quad t \in [0, 2], \tag{3.1}$$

and the initial condition $x(0) = 0$. Here, $x(t)$, $u(t)$, $v(t) \in \mathbb{R}$ and $b(t) = \max\{0, 2(t-1)\}$, $t \in [0, 2]$. Further, the geometric constraints on control and disturbance are as follows:

$$u(t) \in [-1, 1], \quad v(t) \in \{-1, 1\}, \quad t \in [0, 2]. \tag{3.2}$$

The set V describing the functional constraint on disturbance consists of two functions

$$v_+(t) = 1, \quad v_-(t) = -1, \quad t \in [0, 2], \tag{3.3}$$

and, so, it is hereditary. The goal of control is to minimize the cost functional

$$J(u(\cdot), v(\cdot)) = -|x(2; u(\cdot), v(\cdot))|, \quad u(\cdot) \in \mathcal{U}, \quad v(\cdot) \in V. \tag{3.4}$$

In other words, the problem is to ensure the maximum possible deviation of a system's motion at the final time $\vartheta = 2$ from the origin against the admissible disturbances $v_+(\cdot)$ and $v_-(\cdot)$.

Note that $b(t) = 0$ for $t \in [0, 1]$, and, therefore, disturbance does not affect the dynamical system on the initial time interval $[0, 1]$. From the practical viewpoint, this means that, during this time interval, there is no possibility to find out the disturbance involved in the system. Hence, we have to choose control actions $u(t)$ for $t \in [0, 1]$ without knowing which one of the two disturbances will act on the future time interval $(1, 2]$, since $b(t) > 0$ for $t \in (1, 2]$. In particular, this fact substantially distinguishes the considered example from the problems studied in [8], where it is assumed that, at the first time step, it is possible to evaluate or reconstruct the disturbance (the unknown parameter).

Below, for the considered problem, we describe the sets of non-anticipative control strategies $\mathcal{A}$ and $\mathcal{A}_*$, compute the corresponding values of the optimal guaranteed results Γ and Γ_*, and study the question of whether or not these results can be ensured by feedback controls.

3.1. Standard non-anticipative strategies

In the considered example, due to a special form of the set V (see (3.3)), property (P_*) degenerates, and any function $\alpha_* : \mathcal{U} \to V$ belongs to $\mathcal{A}_*$. In particular, the following function is a standard non-anticipative control strategy:

$$\alpha_*^\circ(v)(\cdot) = v(\cdot), \quad v(\cdot) \in V. \tag{3.5}$$

Then, for the corresponding motions $x_\pm(\cdot) := x(\cdot; \alpha_*^\circ(v_\pm)(\cdot), v_\pm(\cdot))$, in view of (3.2), we have

$$|x_\pm(2)| = \Big| \int_0^2 \big(v_\pm(t) + b(t)v_\pm(t)\big)\, dt \Big| = 2 + \int_1^2 2(t-1)\, dt = 3,$$

wherefrom, according to (2.3) and (3.4), we obtain $\Gamma_* \leq -3$. On the other hand, for every $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in V$, we derive

$$|x(2; u(\cdot), v(\cdot))| \leq \int_0^2 |u(t)|\, dt + \int_1^2 2(t-1)|v(t)|\, dt \leq 3, \tag{3.6}$$

and, therefore, $\Gamma_* \geq -3$. Thus, we come to the equality

$$\Gamma_* = -3. \tag{3.7}$$

Note that, by virtue of (3.6), this result can not be improved.

By construction, the presented optimal strategy α_*° assigns control actions $u(t)$ depending on the exact values of $v(t)$, which can not be recognized by observing the system during the initial time interval $[0, 1]$. In this sense, we see that the result $\Gamma_* = -3$ is achieved by using "insider" information.

3.2. Non-anticipative strategies from $\mathcal{A}$

According to (1.6), (3.1), and (3.2), for every $t \in [0, 2]$, $x \in \mathbb{R}$, and $v \in \{-1, 1\}$, we have

$$\mathbf{q}(t, x, v) = \big\{ v' \in \{-1, 1\} \mid b(t)v' = b(t)v \big\}.$$

Then, taking into account that $b(t) = 0$ for $t \in [0,1]$ and $b(t) > 0$ for $t \in (1,2]$, we derive

$$\mathbf{q}(t,x,1) = \begin{cases} \{-1,1\}, & \text{if } t \in [0,1], \\ 1, & \text{if } t \in (1,2], \end{cases} \quad \mathbf{q}(t,x,-1) = \begin{cases} \{-1,1\}, & \text{if } t \in [0,1], \\ -1, & \text{if } t \in (1,2], \end{cases} \quad x \in \mathbb{R}.$$

Therefore, a function $\alpha : V \to \mathcal{U}$ possesses property (P) if and only if

$$x(t;\alpha(v_+)(\cdot), v_+(\cdot)) = x(t;\alpha(v_-)(\cdot), v_-(\cdot)), \quad t \in [0,1]. \tag{3.8}$$

In view of (3.1), the last equality is equivalent to the following:

$$\alpha(v_+)(t) = \alpha(v_-)(t) \text{ for a.e. } t \in [0,1]. \tag{3.9}$$

Thus, we see that this property of non-anticipation corresponds to the fact that the control can not react to the disturbance on the initial time interval $[0,1]$. Note that the function α_*° from (3.5) does not satisfy property (P), and, hence, does not belong to $\mathcal{A}$.

Let us consider the function $\alpha^\circ : V \to \mathcal{U}$ defined for every $v(\cdot) \in V$ by

$$\alpha^\circ(v)(t) = 0, \quad t \in [0,1]; \quad \alpha^\circ(v)(t) = v(t), \quad t \in (1,2].$$

We have $\alpha^\circ \in \mathcal{A}$. Further, for the motions $x_\pm(\cdot) := x(\cdot;\alpha^\circ(v_\pm)(\cdot), v_\pm(\cdot))$, we derive

$$|x_\pm(2)| = \Big| \int_1^2 \big(v_\pm(t) + b(t)v_\pm(t)\big)\,dt \Big| = 1 + \int_1^2 2(t-1)\,dt = 2,$$

wherefrom, according to (1.10) and (3.4), we obtain $\Gamma \le -2$. Let us prove that

$$\Gamma = -2. \tag{3.10}$$

Arguing by contradiction, suppose that $\Gamma = -2 - \xi$ for some $\xi > 0$. Then, there exists $\widetilde{\alpha} \in \mathcal{A}$ such that the motions $\widetilde{x}_\pm(\cdot) := x(\cdot;\widetilde{\alpha}(v_\pm)(\cdot), v_\pm(\cdot))$ satisfy the inequalities

$$|\widetilde{x}_\pm(2)| \ge 2 + \xi/2. \tag{3.11}$$

Note that $\widetilde{x}_+(2) \ge 0$. Indeed, if we assume that $\widetilde{x}_+(2) < 0$, then, owing to (3.2), we obtain

$$|\widetilde{x}_+(2)| = -\widetilde{x}_+(2) = -\int_0^2 \widetilde{\alpha}(v_+)(t)\,dt - \int_1^2 2(t-1)v_+(t)\,dt = -\int_0^2 \widetilde{\alpha}(v_+)(t)\,dt - 1 \le 1,$$

which contradicts (3.11). In a similar way, one can prove that $\widetilde{x}_-(2) \le 0$. Hence, for the motion $\widetilde{x}_+(\cdot)$, in view of (3.11), we have

$$|\widetilde{x}_+(2)| = \widetilde{x}_+(2) = \int_0^2 \widetilde{\alpha}(v_+)(t)\,dt + 1 \ge 2 + \xi/2,$$

and, therefore,

$$\widetilde{x}_+(1) = \int_0^1 \widetilde{\alpha}(v_+)(t)\,dt = \int_0^2 \widetilde{\alpha}(v_+)(t)\,dt - \int_1^2 \widetilde{\alpha}(v_+)(t)\,dt \ge 1 + \xi/2 - 1 = \xi/2.$$

Since $\widetilde{x}_-(1) = \widetilde{x}_+(1)$ due to (3.8), we get

$$|\widetilde{x}_-(2)| = -\widetilde{x}_-(2) = -\widetilde{x}_-(1) - \int_1^2 \widetilde{\alpha}(v_-)(t)\,dt - \int_1^2 2(t-1)v_-(t)\,dt \le -\xi/2 + 2,$$

which contradicts (3.11). Thus, equality (3.10) is proved.

As a result, according to (3.7) and (3.10), we conclude that, in the considered example, the inequality in (2.6) is strict. Below, we show that, in contrast to the value $\Gamma_* = -3$, the value $\Gamma = -2$ can be ensured by feedback controls.

3.3. Feedback controls

Following the positional approach (see, e.g. [1, 2]), let us assume that a control $u(\cdot) \in \mathcal{U}$ is formed in a feedback way by some positional control strategy $U : [0, 2] \times \mathbb{R} \to [-1, 1]$. For brevity, we omit a rigorous mathematical description of how such a strategy U generates a control $u(\cdot)$ and refer the reader to the cited monographs for details.

Let us take $\varepsilon \in (0, 1)$ and consider the positional control strategy U_ε defined by

$$U_\varepsilon(t, x) = \begin{cases} 0, & \text{if } t \in [0, 1+\varepsilon], \quad x \in \mathbb{R}, \\ 1, & \text{if } t \in (1+\varepsilon, 2], \quad x \geq 0, \\ -1, & \text{if } t \in (1+\varepsilon, 2], \quad x < 0. \end{cases}$$

This strategy U_ε relies on the following ideas. Note that the problem is symmetric with respect to the line $\ell = \{(t, x) \in [0, 2] \times \mathbb{R} \mid x = 0\}$, and, therefore, since we can not get any information about the disturbance on the initial time interval $[0, 1]$, the optimal behaviour is to stay at this line ℓ until $t = 1$. On the next time interval $(1, 1+\varepsilon]$, by setting $u(t) = 0$, we obtain that, in the case of the disturbance $v_+(\cdot)$, the motion of the system goes above the line ℓ, and, otherwise, in the case of the disturbance $v_-(\cdot)$, the motion goes below the line ℓ. This allows us to find out the disturbance acting in the system. On the last time interval $(1+\varepsilon, 1]$, if $x(t) > 0$ at some t, then the disturbance is $v_+(\cdot)$ and we should choose $u(t) = 1$. Similarly, if $x(t) \leq 0$, then the disturbance is $v_-(\cdot)$ and we should take $u(t) = -1$.

Thus, in the case of the disturbance $v_+(\cdot)$, the positional strategy U_ε generates the control

$$u_+^{(\varepsilon)}(t) = 0, \quad t \in [0, 1+\varepsilon]; \quad u_+^{(\varepsilon)}(t) = 1, \quad t \in (1+\varepsilon, 2],$$

and, otherwise, in the case of the disturbance $v_-(\cdot)$, it generates the control

$$u_-^{(\varepsilon)}(t) = 0, \quad t \in [0, 1+\varepsilon]; \quad u_-^{(\varepsilon)}(t) = -1, \quad t \in (1+\varepsilon, 2].$$

For the corresponding motions $x_\pm^{(\varepsilon)}(\cdot) := x(\cdot; u_\pm^{(\varepsilon)}(\cdot), v_\pm(\cdot))$, we have

$$|x_\pm^{(\varepsilon)}(2)| = \left| \int_{1+\varepsilon}^{2} u_\pm(t)\, dt + \int_{1}^{2} b(t) v_\pm(t)\, dt \right| = 2 - \varepsilon.$$

Since the number $\varepsilon \in (0, 1)$ can be taken arbitrarily small, we obtain that, by applying positional control strategies, we can ensure for cost functional (3.4) the value that is arbitrarily close to $\Gamma = -2$. On the other hand, we observe that, since the disturbance does not affect the system on the initial time interval $[0, 1]$, then, given any positional control strategy (or any other feedback control rule), the corresponding responses $u_+(\cdot)$ and $u_-(\cdot)$ to the disturbances $v_+(\cdot)$ and $v_-(\cdot)$, respectively, must satisfy the equality $u_+(t) = u_-(t)$, $t \in [0, 1]$. Therefore, according to (3.9), we can not ensure any value of the cost functional that is lower than $\Gamma = -2$. In particular, we can not guarantee the value $\Gamma_* = -3$.

§ 4. Positions of the dynamical system

Given $t_* \in T$, $z_*(\cdot) \in C(T;\mathbb{R}^n)$, $u(\cdot) \in \mathcal{U}$, and $v(\cdot) \in V$, let $x(\cdot; t_*, z_*(\cdot), u(\cdot), v(\cdot))$ denote the corresponding motion of the dynamical system, which is defined, similarly to the case of initial condition (1.2), as a unique function $x : T \to \mathbb{R}^n$ that meets the equality $x(t) = z(t)$ for $t \in [t_0, t_*]$, is absolutely continuous on $[t_*, \vartheta]$, and, together with $u(\cdot)$ and $v(\cdot)$, satisfies dynamic equation (1.1) for a.e. $t \in [t_*, \vartheta]$. In accordance with this definition, the following semigroup property of motions holds:

$$x(\cdot; u(\cdot), v(\cdot)) = x\big(\cdot; t, x(\cdot; u(\cdot), v(\cdot)), u(\cdot), v(\cdot)\big), \quad t \in T, \quad u(\cdot) \in \mathcal{U}, \quad v(\cdot) \in V. \tag{4.1}$$

We call a triple $(t, z(\cdot), v(\cdot)) \in T \times C(T;\mathbb{R}^n) \times V$ a position of the dynamical system provided that there exists a control $u(\cdot) \in \mathcal{U}$ such that $(z(\cdot)\,|\,[t_0, t]) = (x(\cdot; u(\cdot), v(\cdot))\,|\,[t_0, t])$. Here and below, the symbol $(z(\cdot)\,|\,[t_0, t])$ stands for the restriction of the function $z(\cdot)$ to the segment $[t_0, t]$. Let G be set of all positions.

Further, we introduce the sets of all controls and all disturbances that are compatible with a position $(t, z(\cdot), v(\cdot)) \in G$ in the following way (in this connections, see, e.g., [25, Ch. VI]):

$$\mathcal{U}(t, z(\cdot), v(\cdot)) := \big\{u(\cdot) \in \mathcal{U} \mid (z(\cdot)\,|\,[t_0, t]) = (x(\cdot; u(\cdot), v(\cdot))\,|\,[t_0, t])\big\}, \tag{4.2}$$

$$V(t, z(\cdot), v(\cdot)) := \big\{v'(\cdot) \in V \mid v'(s) \in \mathbf{q}(s, z(s), v(s)),\ s \in [t_0, t)\big\}, \tag{4.3}$$

where $\mathbf{q}(s, z(s), v(s))$ is defined by (1.6). Note that $\mathcal{U}(t, z(\cdot), v(\cdot)) \neq \varnothing$ by construction, and $V(t, z(\cdot), v(\cdot)) \neq \varnothing$ in view of the inclusion

$$v(\cdot) \in V(t, z(\cdot), v(\cdot)). \tag{4.4}$$

In addition, we have

$$(x(\cdot; \overline{u}(\cdot), \overline{v}(\cdot))\,|\,[t_0, t]) = (z(\cdot)\,|\,[t_0, t]), \quad \overline{u}(\cdot) \in \mathcal{U}(t, z(\cdot), v(\cdot)), \quad \overline{v}(\cdot) \in V(t, z(\cdot), v(\cdot)), \tag{4.5}$$

and, for every $v'(\cdot) \in V$ such that $(t, z(\cdot), v'(\cdot)) \in G$, due to (1.7), we obtain

$$\begin{gathered}\big(v'(\cdot) \in V(t, z(\cdot), v(\cdot))\big) \Leftrightarrow \big(v(\cdot) \in V(t, z(\cdot), v'(\cdot))\big) \Leftrightarrow \\ \Leftrightarrow \big(V(t, z(\cdot), v(\cdot)) = V(t, z(\cdot), v'(\cdot))\big),\end{gathered} \tag{4.6}$$

which means that the disturbances $v(\cdot)$ and $v'(\cdot)$ can not be distinguished until time t by impacting on the dynamical system along the history $z(\cdot)$.

Below, we establish some basic properties of sets $V(t, z(\cdot), v(\cdot))$ and $\mathcal{U}(t, z(\cdot), v(\cdot))$, which play an important role in the proof of Theorem 1 in § 6. Namely, Lemma 1 concerns the evolution of these sets along a given function $z(\cdot)$. It is most transparent when $z'(\cdot) = z(\cdot)$ and $v'(\cdot) = v(\cdot)$. Lemma 2 presents a peculiar semigroup property of $V(t, z(\cdot), v(\cdot))$.

Lemma 1. *If $(t, z(\cdot), v(\cdot))$, $(t', z'(\cdot), v'(\cdot)) \in G$ satisfy $t \leq t'$, $(z'(\cdot)\,|\,[t_0, t]) = (z(\cdot)\,|\,[t_0, t])$, and $v'(\cdot) \in V(t, z(\cdot), v(\cdot))$, then the following inclusions hold:*

$$V(t', z'(\cdot), v'(\cdot)) \subset V(t, z(\cdot), v(\cdot)), \quad \mathcal{U}(t', z'(\cdot), v'(\cdot)) \subset \mathcal{U}(t, z(\cdot), v(\cdot)).$$

P r o o f. Let $(t, z(\cdot), v(\cdot))$, $(t', z'(\cdot), v'(\cdot)) \in G$ be as in the statement of the lemma. In view of (1.7), we have

$$\mathbf{q}(s, z(s), v(s)) = \mathbf{q}(s, z(s), v'(s)) = \mathbf{q}(s, z'(s), v'(s)), \quad s \in [t_0, t).$$

Therefore, for every $\overline{v}(\cdot) \in V(t', z'(\cdot), v'(\cdot))$, we get $\overline{v}(s) \in \mathbf{q}(s, z(s), v(s))$, $s \in [t_0, t)$, which yields $\overline{v}(\cdot) \in V(t, z(\cdot), v(\cdot))$. Further, for each $\overline{u}(\cdot) \in \mathcal{U}(t', z'(\cdot), v'(\cdot))$, taking (1.6) into account, we derive

$$z(s) = z'(s) = x(s; \overline{u}(\cdot), v'(\cdot)) = x_0 + \int_{t_0}^{s} f(\xi, z'(\xi), \overline{u}(\xi), v'(\xi))\, d\xi =$$
$$= x_0 + \int_{t_0}^{s} f(\xi, z(\xi), \overline{u}(\xi), v'(\xi))\, d\xi = x_0 + \int_{t_0}^{s} f(\xi, z(\xi), \overline{u}(\xi), v(\xi))\, d\xi = x(s; \overline{u}(\cdot), v(\cdot))$$

for any $s \in [t_0, t]$, which implies the inclusion $\overline{u}(\cdot) \in \mathcal{U}(t, z(\cdot), v(\cdot))$. The lemma is proved. □

Lemma 2. *Let $(t, z(\cdot), v(\cdot)) \in G$, $t' \in [t, \vartheta]$, and $\mu : V(t, z(\cdot), v(\cdot)) \to \mathcal{U}(t, z(\cdot), v(\cdot))$. Then, the following equality holds:*

$$V(t, z(\cdot), v(\cdot)) = \bigcup_{v'(\cdot) \in V(t, z(\cdot), v(\cdot))} V\big(t', x(\cdot; \mu(v')(\cdot), v'(\cdot)), v'(\cdot)\big). \tag{4.7}$$

P r o o f. Let V_* be the set from the right-hand side of equality (4.7). Note that this set is well-defined since $(t', x(\cdot; \mu(v')(\cdot), v'(\cdot)), v'(\cdot)) \in G$ for any $v'(\cdot) \in V(t, z(\cdot), v(\cdot))$.

For every $\overline{v}(\cdot) \in V(t, z(\cdot), v(\cdot))$, we have $(t', x(\cdot; \mu(\overline{v})(\cdot), \overline{v}(\cdot)), \overline{v}(\cdot)) \in G$, and, hence, owing to (4.4), we obtain $\overline{v}(\cdot) \in V(t', x(\cdot; \mu(\overline{v})(\cdot), \overline{v}(\cdot)), \overline{v}(\cdot)) \subset V_*$. Thus, $V(t, z(\cdot), v(\cdot)) \subset V_*$. On the other hand, take arbitrarily $v'(\cdot) \in V(t, z(\cdot), v(\cdot))$. Then, in view of (4.5), the inclusion $\mu(v')(\cdot) \in \mathcal{U}(t, z(\cdot), v(\cdot))$ yields $(x(\cdot; \mu(v')(\cdot), v'(\cdot)) \mid [t_0, t]) = (z(\cdot) \mid [t_0, t])$. Therefore, applying Lemma 1, we derive $V(t', x(\cdot; \mu(v')(\cdot), v'(\cdot)), v'(\cdot)) \subset V(t, z(\cdot), v(\cdot))$. So, $V_* \subset V(t, z(\cdot), v(\cdot))$, which implies the required equality $V(t, z(\cdot), v(\cdot)) = V_*$. The lemma is proved. □

§ 5. Functional of the optimal guaranteed result

By a non-anticipative control strategy compatible with a position $(t, z(\cdot), v(\cdot)) \in G$, we mean a function $\alpha : V(t, z(\cdot), v(\cdot)) \to \mathcal{U}(t, z(\cdot), v(\cdot))$ satisfying the following property: for every $v_1(\cdot)$, $v_2(\cdot) \in V(t, z(\cdot), v(\cdot))$ and $\tau \in (t, \vartheta]$, the inclusion

$$v_1(s) \in \mathbf{q}\big(s, x(s; \alpha(v_2)(\cdot), v_2(\cdot)), v_2(s)\big), \quad s \in [t, \tau),$$

implies the equality

$$x(s; \alpha(v_1)(\cdot), v_1(\cdot)) = x(s; \alpha(v_2)(\cdot), v_2(\cdot)), \quad s \in [t, \tau].$$

Below, by analogy with § 1, this property of non-anticipation is referred to as property (P). Let $\mathcal{A}(t, z(\cdot), v(\cdot))$ be the set of all such control strategies. We emphasize that, compared to the usual case, this set $\mathcal{A}(t, z(\cdot), v(\cdot))$ depends on the system dynamics and the history of the control process (in this connection, see also [18]).

Note that, for any $u_*(\cdot) \in \mathcal{U}(t, z(\cdot), v(\cdot))$, the function $\alpha_* : V(t, z(\cdot), v(\cdot)) \to \mathcal{U}(t, z(\cdot), v(\cdot))$ defined by $\alpha_*(v)(\cdot) := u_*(\cdot)$, $v(\cdot) \in V(t, z(\cdot), v(\cdot))$, belongs to $\mathcal{A}(t, z(\cdot), v(\cdot))$ (see the proof of Lemma 4 below). In particular, we have $\mathcal{A}(t, z(\cdot), v(\cdot)) \neq \varnothing$.

Observe also that, for every $\alpha \in \mathcal{A}(t, z(\cdot), v(\cdot))$ and $v'(\cdot) \in V(t, z(\cdot), v(\cdot))$, the following equality holds due to (4.5):

$$x(s; \alpha(v')(\cdot), v'(\cdot)) = z(s), \quad s \in [t_0, t]. \tag{5.1}$$

The guaranteed result of a non-anticipative control strategy $\alpha \in \mathcal{A}(t, z(\cdot), v(\cdot))$ and the optimal guaranteed result are defined respectively by (see also (1.10)):

$$\Gamma(t, z(\cdot), v(\cdot), \alpha) := \sup_{v'(\cdot) \in V(t, z(\cdot), v(\cdot))} \sigma\big(x(\cdot; \alpha(v')(\cdot), v'(\cdot))\big), \tag{5.2}$$

$$\Gamma(t, z(\cdot), v(\cdot)) := \inf_{\alpha \in \mathcal{A}(t, z(\cdot), v(\cdot))} \Gamma(t, z(\cdot), v(\cdot), \alpha). \tag{5.3}$$

Further, assume that $(t, z'(\cdot), v'(\cdot)) \in G$ satisfies the relations $(z'(\cdot) \,|\, [t_0, t]) = (z(\cdot) \,|\, [t_0, t])$ and $v'(\cdot) \in V(t, z(\cdot), v(\cdot))$. Then, in view of Lemma 1 and (4.6), we obtain

$$V(t, z'(\cdot), v'(\cdot)) = V(t, z(\cdot), v(\cdot)), \quad \mathcal{U}(t, z'(\cdot), v'(\cdot)) = \mathcal{U}(t, z(\cdot), v(\cdot)). \tag{5.4}$$

Hence,

$$\mathcal{A}(t, z'(\cdot), v'(\cdot)) = \mathcal{A}(t, z(\cdot), v(\cdot)), \tag{5.5}$$

and, moreover, for every $\alpha \in \mathcal{A}(t, z(\cdot), v(\cdot))$,

$$\Gamma(t, z'(\cdot), v'(\cdot), \alpha) = \Gamma(t, z(\cdot), v(\cdot), \alpha), \quad \Gamma(t, z'(\cdot), v'(\cdot)) = \Gamma(t, z(\cdot), v(\cdot)). \tag{5.6}$$

These equalities mean that the considered positions $(t, z(\cdot), v(\cdot))$ and $(t, z'(\cdot), v'(\cdot))$ are equivalent with respect to the guarantee optimization problem.

The main point of the definitions given in this section is that the statement of the guarantee optimization problem for an arbitrary position from G agrees with the original statement from § 1. Namely, at the initial time t_0, every triple $(t_0, z(\cdot), v(\cdot))$ such that $z(\cdot) \in C(T; \mathbb{R}^n)$, $z(t_0) = x_0$, and $v(\cdot) \in V$ belongs to G. Moreover, $\mathcal{U}(t_0, z(\cdot), v(\cdot)) = \mathcal{U}$ and $V(t_0, z(\cdot), v(\cdot)) = V$, and, hence, $\mathcal{A}(t_0, z(\cdot), v(\cdot)) = \mathcal{A}$ and $\Gamma(t_0, z(\cdot), v(\cdot)) = \Gamma$.

§ 6. Dynamic programming principle

In this section, we prove that the functional of the optimal guaranteed result Γ defined by (5.3) satisfies the dynamic programming principle.

Theorem 1. *For every $(t, z(\cdot), v(\cdot)) \in G$ and $t' \in [t, \vartheta]$, the equality below holds:*

$$\Gamma(t, z(\cdot), v(\cdot)) = \inf_{\alpha \in \mathcal{A}(t, z(\cdot), v(\cdot))} \sup_{v'(\cdot) \in V(t, z(\cdot), v(\cdot))} \Gamma\big(t', x(\cdot; \alpha(v')(\cdot), v'(\cdot)), v'(\cdot)\big).$$

P r o o f. **1.** Given $\varepsilon > 0$, choose $\alpha \in \mathcal{A}(t, z(\cdot), v(\cdot))$ from the condition (see (5.2))

$$\Gamma(t, z(\cdot), v(\cdot), \alpha) \leq \Gamma(t, z(\cdot), v(\cdot)) + \varepsilon. \tag{6.1}$$

Take $v'(\cdot) \in V(t, z(\cdot), v(\cdot))$ and denote $x'(\cdot) := x(\cdot; \alpha(v')(\cdot), v'(\cdot))$. Let us prove that

$$\Gamma(t', x'(\cdot), v'(\cdot)) \leq \Gamma(t, z(\cdot), v(\cdot)) + \varepsilon. \tag{6.2}$$

Since $\alpha(v')(\cdot) \in \mathcal{U}(t, z(\cdot), v(\cdot))$, then $(x'(\cdot) \,|\, [t_0, t]) = (z(\cdot) \,|\, [t_0, t])$ due to (4.5), and, in addition, applying Lemma 1, we get

$$V(t', x'(\cdot), v'(\cdot)) \subset V(t, z(\cdot), v(\cdot)). \tag{6.3}$$

Taking this into account, define a function $\alpha' : V(t', x'(\cdot), v'(\cdot)) \to \mathcal{U}(t, z(\cdot), v(\cdot))$ as the restriction of α to $V(t', x'(\cdot), v'(\cdot))$:

$$\alpha'(v'')(\cdot) := \alpha(v'')(\cdot), \quad v''(\cdot) \in V(t', x'(\cdot), v'(\cdot)). \tag{6.4}$$

Let us verify the inclusion $\alpha' \in \mathcal{A}(t', x'(\cdot), v'(\cdot))$. Note that α' inherits the corresponding non-anticipativity property (P) from α, and, therefore, in order to obtain this inclusion, it remains to show that $\alpha'(v'')(\cdot) \in \mathcal{U}(t', x'(\cdot), v'(\cdot))$ for every $v''(\cdot) \in V(t', x'(\cdot), v'(\cdot))$. According to (4.3), the inclusion $v''(\cdot) \in V(t', x'(\cdot), v'(\cdot))$ means that

$$v''(s) \in \mathbf{q}(s, x'(s), v'(s)), \quad s \in [t_0, t'). \tag{6.5}$$

Hence, since $x'(\cdot) := x(\cdot; \alpha(v')(\cdot), v'(\cdot))$, then, by virtue of (4.4), the non-anticipativity of α and (5.1) imply that $(x'(\cdot) \,|\, [t_0, t']) = (x(\cdot; \alpha(v'')(\cdot), v''(\cdot)) \,|\, [t_0, t'])$. Moreover, it follows from (6.5) (see also (1.9)) that

$$f(s, x'(s), \alpha(v'')(s), v'(s)) = f(s, x'(s), \alpha(v'')(s), v''(s)), \quad s \in [t_0, t'].$$

Consequently, we have

$$x'(s) = x_0 + \int_{t_0}^{s} f(\xi, x'(\xi), \alpha(v'')(\xi), v''(\xi))\, d\xi = x_0 + \int_{t_0}^{s} f(\xi, x'(\xi), \alpha(v'')(\xi), v'(\xi))\, d\xi$$

for any $s \in [t_0, t']$, which yields $(x'(\cdot) \,|\, [t_0, t']) = (x(\cdot; \alpha(v'')(\cdot), v'(\cdot)) \,|\, [t_0, t'])$. Thus, in view of (4.2), we obtain $\alpha'(v'')(\cdot) := \alpha(v'')(\cdot) \in \mathcal{U}(t', x'(\cdot), v'(\cdot))$, and, so, $\alpha' \in \mathcal{A}(t', x'(\cdot), v'(\cdot))$.

Further, owing to (6.3) and (6.4), we get

$$\Gamma(t', x'(\cdot), v'(\cdot), \alpha') := \sup_{v''(\cdot) \in V(t', x'(\cdot), v'(\cdot))} \sigma\big(x(\cdot; \alpha'(v'')(\cdot), v''(\cdot))\big) \leq$$
$$\leq \sup_{v''(\cdot) \in V(t, z(\cdot), v(\cdot))} \sigma\big(x(\cdot; \alpha(v'')(\cdot), v''(\cdot))\big) := \Gamma(t, z(\cdot), v(\cdot), \alpha),$$

wherefrom, based on (5.3) and (6.1), we derive the required inequality (6.2).

Finally, we observe that (6.2) implies

$$\overline{\Gamma} := \inf_{\alpha \in \mathcal{A}(t, z(\cdot), v(\cdot))} \sup_{v'(\cdot) \in V(t, z(\cdot), v(\cdot))} \Gamma\big(t', x(\cdot; \alpha(v')(\cdot), v'(\cdot)), v'(\cdot)\big) \leq \Gamma(t, z(\cdot), v(\cdot)).$$

2. Hence, in order to complete the proof, it is sufficient to verify the inequality

$$\Gamma(t, z(\cdot), v(\cdot)) \leq \overline{\Gamma}. \tag{6.6}$$

Given $\varepsilon > 0$, choose $\beta \in \mathcal{A}(t, z(\cdot), v(\cdot))$ from the condition

$$\sup_{v'(\cdot) \in V(t, z(\cdot), v(\cdot))} \Gamma\big(t', x(\cdot; \beta(v')(\cdot), v'(\cdot)), v'(\cdot)\big) \leq \overline{\Gamma} + \varepsilon. \tag{6.7}$$

For any $v'(\cdot) \in V(t, z(\cdot), v(\cdot))$, denote $z_{v'}(\cdot) := x(\cdot; \beta(v')(\cdot), v'(\cdot))$ and consider the set

$$X_{v'} := \big\{(t', \overline{z}(\cdot), \overline{v}(\cdot)) \in G \mid (\overline{z}(\cdot) \,|\, [t_0, t']) = (z_{v'}(\cdot) \,|\, [t_0, t']),\ \overline{v}(\cdot) \in V(t', z_{v'}(\cdot), v'(\cdot))\big\}.$$

Note that $X_{v'} \neq \varnothing$ since $(t', z_{v'}(\cdot), v'(\cdot)) \in X_{v'}$, and, moreover, in view of (5.4), the family $\mathcal{X} := \{X_{v'} \mid v'(\cdot) \in V(t, z(\cdot), v(\cdot))\}$ is disjoint, i.e., if X, $X' \in \mathcal{X}$ and $X \cap X' \neq \varnothing$, then $X = X'$. Further, for every $X \in \mathcal{X}$, take $(t', \overline{z}_X(\cdot), \overline{v}_X(\cdot)) \in X$ and, in accordance with (5.3), choose $\eta_X \in \mathcal{A}(t', \overline{z}_X(\cdot), \overline{v}_X(\cdot))$ such that

$$\Gamma(t', \overline{z}_X(\cdot), \overline{v}_X(\cdot), \eta_X) \leq \Gamma(t', \overline{z}_X(\cdot), \overline{v}_X(\cdot)) + \varepsilon. \tag{6.8}$$

Let $v'(\cdot) \in V(t, z(\cdot), v(\cdot))$ be fixed. Since $(t', \overline{z}_{X_{v'}}(\cdot), \overline{v}_{X_{v'}}(\cdot)) \in X_{v'}$ by construction, then we have $(\overline{z}_{X_{v'}}(\cdot) \,|\, [t_0, t']) = (z_{v'}(\cdot) \,|\, [t_0, t'])$, and, therefore, it follows from (5.5) that

$$\eta_{X_{v'}} \in \mathcal{A}(t', z_{v'}(\cdot), v'(\cdot)), \tag{6.9}$$

and, by (5.6) and (6.8), the inequality $\Gamma(t', z_{v'}(\cdot), v'(\cdot), \eta_{X_{v'}}) \leq \Gamma(t', z_{v'}(\cdot), v'(\cdot)) + \varepsilon$ holds. In particular, we conclude

$$\sup_{v'(\cdot) \in V(t, z(\cdot), v(\cdot))} \Gamma(t', z_{v'}(\cdot), v'(\cdot), \eta_{X_{v'}}) \leq \sup_{v'(\cdot) \in V(t, z(\cdot), v(\cdot))} \Gamma(t', z_{v'}(\cdot), v'(\cdot)) + \varepsilon. \tag{6.10}$$

Taking (6.9) into account, define a function $\omega : V(t, z(\cdot), v(\cdot)) \to \mathcal{U}$ by

$$\omega(v')(\cdot) := \eta_{X_{v'}}(v')(\cdot), \quad v'(\cdot) \in V(t, z(\cdot), v(\cdot)). \tag{6.11}$$

Let us prove the inclusion $\omega \in \mathcal{A}(t, z(\cdot), v(\cdot))$. For every $v'(\cdot) \in V(t, z(\cdot), v(\cdot))$, recalling that $z_{v'}(\cdot) := x(\cdot; \beta(v')(\cdot), v'(\cdot))$, we get $(z_{v'}(\cdot) \,|\, [t_0, t]) = (z(\cdot) \,|\, [t_0, t])$ due to (5.1), and, hence, Lemma 1 yields $\mathcal{U}(t', z_{v'}(\cdot), v'(\cdot)) \subset \mathcal{U}(t, z(\cdot), v(\cdot))$, wherefrom, in accordance with (6.9), we derive $\omega(v')(\cdot) \in \mathcal{U}(t, z(\cdot), v(\cdot))$. Thus, it remains to show that ω has non-anticipativity property (P). Namely, given $v_1(\cdot)$, $v_2(\cdot) \in V(t, z(\cdot), v(\cdot))$ and $\tau \in (t, \vartheta]$ satisfying

$$v_2(s) \in \mathbf{q}(s, x(s; \omega(v_1)(\cdot), v_1(\cdot)), v_1(s)), \quad s \in [t, \tau), \tag{6.12}$$

it is required to verify the inequality

$$(x(\cdot; \omega(v_1)(\cdot), v_1(\cdot)) \,|\, [t, \tau]) = (x(\cdot; \omega(v_2)(\cdot), v_2(\cdot)) \,|\, [t, \tau]). \tag{6.13}$$

In the case $\tau \leq t'$, for $i \in \{1, 2\}$, since $\omega(v_i)(\cdot) \in \mathcal{U}(t', x(\cdot; \beta(v_i)(\cdot), v_i(\cdot)), v_i(\cdot))$ owing to (6.9), then, by virtue of (4.2), we have

$$(x(\cdot; \omega(v_i)(\cdot), v_i(\cdot)) \,|\, [t_0, \tau]) = (x(\cdot; \beta(v_i)(\cdot), v_i(\cdot)) \,|\, [t_0, \tau]). \tag{6.14}$$

So, it follows from (6.12) that $v_2(s) \in \mathbf{q}(s, x(s; \beta(v_1)(\cdot), v_1(\cdot)), v_1(s))$, $s \in [t, \tau)$, and, by non-anticipativity property (P) of β, we conclude

$$(x(\cdot; \beta(v_1)(\cdot), v_1(\cdot)) \,|\, [t, \tau]) = (x(\cdot; \beta(v_2)(\cdot), v_2(\cdot)) \,|\, [t, \tau]). \tag{6.15}$$

Equalities (6.14) and (6.15) yield (6.13).

Assume that $\tau > t'$. Taking into account that $v_2(\cdot) \in V(t, z(\cdot), v_1(\cdot))$ in view of (4.6), we derive from (5.4) that $v_2(\cdot) \in V(t, z_{v_1}(\cdot), v_1(\cdot))$. Further, arguing as above, we obtain

$$\begin{aligned} &(x(\cdot; \omega(v_1)(\cdot), v_1(\cdot)) \,|\, [t, t']) = (x(\cdot; \beta(v_1)(\cdot), v_1(\cdot)) \,|\, [t, t']) = (z_{v_1}(\cdot) \,|\, [t, t']) = \\ &= (z_{v_2}(\cdot) \,|\, [t, t']) = (x(\cdot; \beta(v_2)(\cdot), v_2(\cdot)) \,|\, [t, t']) = (x(\cdot; \omega(v_2)(\cdot), v_2(\cdot)) \,|\, [t, t']). \end{aligned} \tag{6.16}$$

Hence, we get $v_2(\cdot) \in V(t', z_{v_1}(\cdot), v_1(\cdot))$ due to (6.12), and, moreover, $(t', z_{v_2}(\cdot), v_2(\cdot)) \in X_{v_1}$. Therefore, recalling that $(t', z_{v_2}(\cdot), v_2(\cdot)) \in X_{v_2}$ and the family $\mathcal{X}$ is disjoint, we conclude $X_{v_1} = X_{v_2} := X_*$ and, as a consequence, $\eta_{X_{v_1}} = \eta_{X_{v_2}} = \eta_{X_*}$.

Now, by virtue of (6.11) and (6.12), we have $v_2(s) \in \mathbf{q}(s, x(s; \eta_{X_*}(v_1)(\cdot), v_1(\cdot)), v_1(s))$ for $s \in [t', \tau)$, and, in accordance with non-anticipativity property (P) of η_{X_*}, we derive

$$\begin{aligned} &(x(\cdot; \omega(v_1)(\cdot), v_1(\cdot)) \,|\, [t', \tau]) = (x(\cdot; \eta_{X_*}(v_1)(\cdot), v_1(\cdot)) \,|\, [t', \tau]) = \\ &= (x(\cdot; \eta_{X_*}(v_2)(\cdot), v_2(\cdot)) \,|\, [t', \tau]) = (x(\cdot; \omega(v_2)(\cdot), v_2(\cdot)) \,|\, [t', \tau]). \end{aligned} \tag{6.17}$$

Equalities (6.16) and (6.17) imply (6.13), and, so, the inclusion $\omega \in \mathcal{A}(t, z(\cdot), v(\cdot))$ is proved.

Thus, it follows from (5.2), (5.3), and (6.11) that

$$\Gamma(t, z(\cdot), v(\cdot), \omega) \leq \Gamma(t, z(\cdot), v(\cdot), \omega) = \sup_{v'(\cdot) \in V(t, z(\cdot), v(\cdot))} \sigma\big(x(\cdot; \eta_{X_{v'}}(v')(\cdot), v'(\cdot))\big). \tag{6.18}$$

For any $v'(\cdot) \in V(t, z(\cdot), v(\cdot))$, we get $(x(\cdot; \eta_{X_{v'}}(v')(\cdot), v'(\cdot)) \mid [t_0, t']) = (z_{v'}(\cdot) \mid [t_0, t'])$ owing to (5.1) and (6.9). Then, based on (4.1), we obtain

$$\begin{aligned} x(\cdot; \eta_{X_{v'}}(v')(\cdot), v'(\cdot)) = x\big(\cdot; t', x(\cdot; \eta_{X_{v'}}(v')(\cdot), v'(\cdot)), \eta_{X_{v'}}(v')(\cdot), v'(\cdot)\big) = \\ = x(\cdot; t', z_{v'}(\cdot), \eta_{X_{v'}}(v')(\cdot), v'(\cdot)). \end{aligned} \tag{6.19}$$

Hence,

$$\begin{aligned} &\sup_{v'(\cdot) \in V(t, z(\cdot), v(\cdot))} \sigma\big(x(\cdot; \eta_{X_{v'}}(v')(\cdot), v'(\cdot))\big) = \\ &= \sup_{v'(\cdot) \in V(t, z(\cdot), v(\cdot))} \sigma\big(x(\cdot; t', z_{v'}(\cdot), \eta_{X_{v'}}(v')(\cdot), v'(\cdot))\big) = \\ &= \sup_{v'(\cdot) \in V(t, z(\cdot), v(\cdot))} \sup_{\overline{v}(\cdot) \in V(t', z_{v'}(\cdot), v'(\cdot))} \sigma\big(x(\cdot; t', z_{\overline{v}}(\cdot), \eta_{X_{\overline{v}}}(\overline{v})(\cdot), \overline{v}(\cdot))\big), \end{aligned} \tag{6.20}$$

where the last equality follows from Lemma 2. Note that, for every $v'(\cdot) \in V(t, z(\cdot), v(\cdot))$, the inclusion $\overline{v}(\cdot) \in V(t', z_{v'}(\cdot), v'(\cdot))$ means that $\overline{v}(s) \in \mathbf{q}(s, x(s; \beta(v')(\cdot), v'(\cdot)), v'(s))$, $s \in [t_0, t']$. Therefore, due to non-anticipativity property (P) of β and remark (5.1), we have

$$(z_{\overline{v}}(\cdot) \mid [t_0, t']) := (x(\cdot; \beta(\overline{v})(\cdot), \overline{v}(\cdot)) \mid [t_0, t']) = (x(\cdot; \beta(v')(\cdot), v'(\cdot)) \mid [t_0, t']) := (z_{v'}(\cdot) \mid [t_0, t']).$$

So, we conclude $X_{\overline{v}} = X_{v'}$ and, consequently, $\eta_{X_{\overline{v}}} = \eta_{X_{v'}}$. Then, by analogy with (6.19), we derive $x(\cdot; t', z_{\overline{v}}(\cdot), \eta_{X_{\overline{v}}}(\overline{v})(\cdot), \overline{v}(\cdot)) = x(\cdot; \eta_{X_{v'}}(\overline{v})(\cdot), \overline{v}(\cdot))$. Thus,

$$\begin{aligned} &\sup_{v'(\cdot) \in V(t, z(\cdot), v(\cdot))} \sup_{\overline{v}(\cdot) \in V(t', z_{v'}(\cdot), v'(\cdot))} \sigma\big(x(\cdot; t', z_{\overline{v}}(\cdot), \eta_{X_{\overline{v}}}(\overline{v})(\cdot), \overline{v}(\cdot))\big) = \\ &= \sup_{v'(\cdot) \in V(t, z(\cdot), v(\cdot))} \sup_{\overline{v}(\cdot) \in V(t', z_{v'}(\cdot), v'(\cdot))} \sigma\big(x(\cdot; t', z_{v'}(\cdot), \eta_{X_{v'}}(\overline{v})(\cdot), \overline{v}(\cdot))\big) = \\ &= \sup_{v'(\cdot) \in V(t, z(\cdot), v(\cdot))} \Gamma(t', z_{v'}(\cdot), v'(\cdot), \eta_{X_{v'}}). \end{aligned} \tag{6.21}$$

From (6.18), (6.20), and (6.21), in view of (6.7) and (6.10), we get $\Gamma(t, z(\cdot), v(\cdot)) \leq \overline{\Gamma} + 2\varepsilon$. Since this estimate holds for every $\varepsilon > 0$, we obtain (6.6). The theorem is proved. □

§ 7. Properties of u- and v-stability

In the motivating example in § 3, it is shown that the value of the optimal guaranteed result Γ can be ensured with the help of positional control strategies, which provides a constructive solution of the problem. In order to obtain an analogue of this result in the general case, bearing in mind the method of extremal aiming developed within the positional differential games theory, we establish the following properties of u- and v-stability of the functional Γ (see, e.g., [1, Sect. 4.2] and [2, Sect. 8]).

Lemma 3 (Property of u-stability). *For every $(t, z(\cdot), v(\cdot)) \in G$, $v'(\cdot) \in V(t, z(\cdot), v(\cdot))$, $t' \in [t, \vartheta]$, and $\varepsilon > 0$, there exists a control $u'(\cdot) \in \mathcal{U}(t, z(\cdot), v(\cdot))$ such that*

$$\Gamma\big(t', x(\cdot; u'(\cdot), v'(\cdot)), v'(\cdot)\big) \leq \Gamma(t, z(\cdot), v(\cdot)) + \varepsilon.$$

The validity of this lemma follows directly from relations (6.3) and (6.2).

Lemma 4 (Property of v-stability). *For every $(t, z(\cdot), v(\cdot)) \in G$, $u'(\cdot) \in \mathcal{U}(t, z(\cdot), v(\cdot))$, $t' \in [t, \vartheta]$, and $\varepsilon > 0$, there exists a disturbance $v'(\cdot) \in V(t, z(\cdot), v(\cdot))$ such that*

$$\Gamma\big(t', x(\cdot; u'(\cdot), v'(\cdot)), v'(\cdot)\big) \geq \Gamma(t, z(\cdot), v(\cdot)) - \varepsilon.$$

P r o o f. Let a function $\alpha : V(t, z(\cdot), v(\cdot)) \to \mathcal{U}(t, z(\cdot), v(\cdot))$ be defined by $\alpha(\overline{v})(\cdot) := u'(\cdot)$, $\overline{v}(\cdot) \in V(t, z(\cdot), v(\cdot))$. Let us show that $\alpha \in \mathcal{A}(t, z(\cdot), v(\cdot))$. Take $v_1(\cdot)$, $v_2(\cdot) \in V(t, z(\cdot), v(\cdot))$ and $\tau \in (t, \vartheta]$, denote $x_i(\cdot) := x(\cdot; u'(\cdot), v_i(\cdot))$, $i \in \{1, 2\}$, and assume that

$$v_2(s) \in \mathbf{q}(s, x(s; \alpha(v_1)(\cdot), v_1(\cdot)), v_1(s)) = \mathbf{q}(s; x_1(s), v_1(s)), \quad s \in [t, \tau).$$

Then, we have $(x_1(\cdot) \,|\, [t_0, t]) = (z(\cdot) \,|\, [t_0, t]) = (x_2(\cdot) \,|\, [t_0, t])$ due to (4.5), and, taking into account that, for any $s \in [t, \tau]$,

$$x_1(s) = x_2(t) + \int_t^s f(\xi, x_1(\xi), u'(\xi), v_1(\xi))\, d\xi = x_2(t) + \int_t^s f(\xi, x_1(\xi), u'(\xi), v_2(\xi))\, d\xi,$$

we get $(x_1(\cdot) \,|\, [t_0, \tau]) = (x_2(\cdot) \,|\, [t_0, \tau])$. Hence, α has property (P), and, so, $\alpha \in \mathcal{A}(t, z(\cdot), v(\cdot))$.

Therefore, according to (6.6), we obtain the inequality

$$\Gamma(t, z(\cdot), v(\cdot)) \le \sup_{v'(\cdot) \in V(t, z(\cdot), v(\cdot))} \Gamma\big(t', x(\cdot; u'(\cdot), v'(\cdot)), v'(\cdot)\big).$$

which implies the claim of the lemma. □

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Неупреждающие стратегии в задачах оптимизации гарантии при функциональных ограничениях на помехи

Ключевые слова: оптимизация гарантии, функциональные ограничения, неупреждающие стратегии, принцип динамического программирования.

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Для динамической системы, управляемой в условиях помех, рассматривается задача оптимизации гарантированного результата. Особенностью задачи является наличие функциональных ограничений на помехи, при которых свойство замкнутости множества допустимых помех относительно операции «склейки» двух его элементов, вообще говоря, отсутствует. Это обстоятельство препятствует непосредственному применению методов теории дифференциальных игр для исследования задачи и тем самым приводит к необходимости их походящей модификации. В работе предложено новое понятие неупреждающей стратегии управления (квазистратегии). Доказано, что соответствующий функционал оптимального гарантированного результата удовлетворяет принципу динамического программирования. Как следствие, установлены так называемые свойства u- и v-стабильности этого функционала, которые в дальнейшем позволят построить конструктивное решение задачи в позиционных стратегиях.

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