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QUASI STATISTICALLY ROUGH CONVERGENCE OF SEQUENCES IN GRADUAL NORMED LINEAR SPACES

In the present article, we set forth with the new notion of quasi statistically rough convergence in the gradual normed linear spaces. We establish significant results that present several fundamental properties of this new notion. We also introduce the notion of $st_q^r(\mathcal{G})$-limit set and prove that it is gradually closed, convex, and plays an important role for the quasi statistically boundedness of a sequence in a gradual normed linear space.

Keywords: gradual number, gradual normed linear space, quasi density, $st_q^r(\mathcal{G})$-convergence, $st_q^r(\mathcal{G})$-limit set.

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Introduction and background

The concept of statistical convergence was developed by Fast [21] and Steinhaus [35] independently in the year 1951. The notion of natural density plays a prime role in the statistical convergence of sequences. If $A \subseteq \mathbb{N}$, then the natural density of A is denoted and defined by

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n}\Big|\{k \le n \colon k \in A\}\Big|,$$

where the vertical bars indicate the cardinality of the enclosed set. A real valued sequence $x = (x_k)$ is said to be statistically convergent to the number x_0 if for each $\eta > 0$,

$$\delta\Big(\{k \in \mathbb{N} \colon |x_k - x_0| \ge \eta\}\Big) = 0.$$

Later on, statistical convergence was further investigated and worked from the sequence space point of view by Fridy [23,24], Šalát [34], Tripathy [37,38], Connor [15], and many others [3–6, 26].

In an attempt to generalize the notion of statistical convergence, in 2012 Özgüc and Yurdakadim [30] generalized natural density to quasi density and statistical convergence to quasi statistical convergence as follows.

Let A be a subset of $\mathbb{N}$. The quasi-density of A is given by

$$\delta_c(A) = \lim_{n\to\infty} \frac{1}{c_n}\Big|\{k \le n \colon k \in A\}\Big|,$$

where $c = (c_n)$ is a sequence of real numbers satisfying the following properties:

$$c_n > 0 \quad \forall n \in \mathbb{N}, \qquad \lim_{n\to\infty} c_n = \infty \ \text{ and } \ \limsup_n \frac{c_n}{n} < \infty. \tag{0.1}$$

It is clear that, for any two subsets A and B of $\mathbb{N}$,

$$\delta_c(\mathbb{N} \setminus A) + \delta_c(A) = 1 \ \text{ and } \ A \subseteq B \ \text{ implies } \ \delta_c(A) \le \delta_c(B).$$

A real valued sequence $x = (x_k)$ is said to be quasi statistical convergent to a real number x_0 if for each $\varepsilon > 0$,

$$\delta_c\Big(\big\{k \in \mathbb{N}\colon |x_k - x_0| \geq \varepsilon\big\}\Big) = 0.$$

Here, x_0 is called the quasi statistical limit of the sequence x and symbolically it is expressed as $x_k \xrightarrow{st_q} l$. For $c_n = n$, quasi density reduces to natural density and quasi statistical convergence turns to statistical convergence. For more information on quasi statistical convergence, one may refer to [25, 29, 39].

In another direction, Phu [31] introduced and investigated the concept of rough convergence in finite dimensional normed spaces. It should be noted that the idea of rough convergence occurs quite naturally in numerical analysis and has interesting applications there. In 2003, Phu [32] further investigated the notion of rough convergence in infinite dimensional normed space setting. Combining the notion of rough convergence and statistical convergence, in 2008, Aytar [10] developed rough statistical convergence. But Akcay and Aytar [2] were the first who introduced and investigated the notion of rough convergence of a sequence of fuzzy numbers. For extensive study in this direction, one may refer to [7–9, 11, 16, 18, 28], where many more references can be found.

On the other hand, in 1965, the notion of fuzzy sets was introduced by Zadeh [40] as one of the extensions of the classical set-theoretical concept. These days, it has wide applications in different branches of science and engineering. The term “fuzzy number” is important in the study of fuzzy set theory. Fuzzy numbers were essentially the generalization of intervals, not numbers. Indeed fuzzy numbers do not obey a couple of algebraic properties of the classical numbers. So the term “fuzzy number” is debatable to many researchers due to its different behavior. The term “fuzzy intervals” is often used by many authors in place of fuzzy numbers. To overcome the confusion among the researchers, in 2008, Fortin et al. [22] introduced the notion of gradual real numbers as elements of fuzzy intervals. Gradual real numbers are mainly known by their respective assignment function whose domain is the interval $(0, 1]$. So, every real number can be thought of as a gradual number with a constant assignment function. The gradual real numbers also obey all the algebraic properties of the classical real numbers and have been used in computation and optimization problems.

In 2011, Sadeqi and Azari [33] were the first to introduce the concept of gradual normed linear space. They studied various properties from both the algebraic and topological points of view. Further development in this direction has been taken place due to Ettefagh et al. [19, 20], Choudhury and Debnath [12, 13], and many others. For an extensive study on gradual real numbers, one may refer to [1, 17, 27, 36].

§ 1. Definitions and preliminaries

In this section, we present some definitions, notions and results that will be exclusively used in the subsequent section. Throughout the paper, we use $c = (c_n)$ to denote a real valued sequence which satisfies (0.1).

Definition 1.1 (see [22]). A gradual real number $\tilde{s}$ is defined by an assignment function $\mathcal{R}_{\tilde{s}}\colon (0, 1] \to \mathbb{R}$. The set of all gradual real numbers is denoted by $\mathcal{G}(\mathbb{R})$. A gradual real number $\tilde{s}$ is said to be non-negative if, for every $0 < \vartheta \leq 1$, $\mathcal{R}_{\tilde{s}}(\vartheta) \geq 0$. The set of all non-negative gradual real numbers is denoted by $\mathcal{G}^*(\mathbb{R})$.

Definition 1.2 (see [22]). Let $*$ be any operation in $\mathbb{R}$ and suppose $\tilde{s}_1, \tilde{s}_2 \in \mathcal{G}(\mathbb{R})$ with assignment functions $\mathcal{R}_{\tilde{s}_1}$ and $\mathcal{R}_{\tilde{s}_2}$ respectively. Then, $\tilde{s}_1 * \tilde{s}_2 \in \mathcal{G}(\mathbb{R})$ is defined with the assignment function $\mathcal{R}_{\tilde{s}_1 * \tilde{s}_2}$ given by

$$\mathcal{R}_{\tilde{s}_1 * \tilde{s}_2}(\vartheta) = \mathcal{R}_{\tilde{s}_1}(\vartheta) * \mathcal{R}_{\tilde{s}_2}(\vartheta), \quad \forall\, 0 < \vartheta \leq 1.$$

In particular, the gradual addition $\tilde{s}_1 + \tilde{s}_2$ and the gradual scalar multiplication $c\tilde{s}$ ($c \in \mathbb{R}$) are defined as follows:

$$\mathcal{R}_{\tilde{s}_1 + \tilde{s}_2}(\vartheta) = \mathcal{R}_{\tilde{s}_1}(\vartheta) + \mathcal{R}_{\tilde{s}_2}(\vartheta) \;\text{ and }\; \mathcal{R}_{c\tilde{s}}(\vartheta) = c\mathcal{R}_{\tilde{s}}(\vartheta), \quad \forall\, 0 < \vartheta \leq 1.$$

Definition 1.3 (see [33]). Let X be a real vector space. The function $\|\cdot\|_{\mathcal{G}}\colon X \to \mathcal{G}^*(\mathbb{R})$ is said to be a gradual norm on X if, for every $0 < \vartheta \leq 1$, the following conditions are true for any $x_0, y_0 \in X$:

(1) $\mathcal{R}_{\|x_0\|_{\mathcal{G}}}(\vartheta) = \mathcal{R}_{\tilde{0}}(\vartheta)$ if and only if $x_0 = 0$;

(2) $\mathcal{R}_{\|\mu x_0\|_{\mathcal{G}}}(\vartheta) = |\mu|\mathcal{R}_{\|x_0\|_{\mathcal{G}}}(\vartheta)$ for any $\mu \in \mathbb{R}$;

(3) $\mathcal{R}_{\|x_0+y_0\|_{\mathcal{G}}}(\vartheta) \leq \mathcal{R}_{\|x_0\|_{\mathcal{G}}}(\vartheta) + \mathcal{R}_{\|y_0\|_{\mathcal{G}}}(\vartheta)$.

The pair $(X, \|\cdot\|_{\mathcal{G}})$ is called a gradual normed linear space (*GNLS*).

Example 1.1 (see [33]). Suppose $X = \mathbb{R}^n$ and for $x_0 = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $0 < \vartheta \leq 1$, define $\|\cdot\|_{\mathcal{G}}$ by

$$\mathcal{R}_{\|x_0\|_{\mathcal{G}}}(\vartheta) = e^{\vartheta} \sum_{i=1}^{n} |x_i|.$$

Then, $\|\cdot\|_{\mathcal{G}}$ is a gradual norm on $\mathbb{R}^n$ and $(\mathbb{R}^n, \|\cdot\|_{\mathcal{G}})$ is a GNLS.

Definition 1.4 (see [33]). Let $x = (x_k)$ be a sequence in the GNLS $(X, \|\cdot\|_{\mathcal{G}})$. Then, x is said to be gradually convergent to $x_0 \in X$ if, for every $0 < \vartheta \leq 1$ and $\eta > 0$, there exists $N(= N_\eta(\vartheta)) \in \mathbb{N}$ such that

$$\mathcal{R}_{\|x_k - x_0\|_{\mathcal{G}}}(\vartheta) < \eta, \quad \forall k \geq N.$$

Symbolically, $x_k \xrightarrow{\|\cdot\|_{\mathcal{G}}} x_0$.

Definition 1.5 (see [20]). Let $(X, \|\cdot\|_{\mathcal{G}})$ be a GNLS. Then, a sequence $x = (x_k)$ in X is said to be gradually bounded if, for every $0 < \vartheta \leq 1$, there exists $M = M(\vartheta) > 0$ such that

$$\mathcal{R}_{\|x_k\|_{\mathcal{G}}}(\vartheta) < M, \quad \forall k \in \mathbb{N}.$$

Definition 1.6 (see [31]). Let r be a non-negative real number. A sequence $x = (x_k)$ in a normed linear space $(X, \|\cdot\|)$ is said to be roughly convergent to $x_0 \in X$ with roughness degree r if, for every $\eta > 0$, there exists $N = (N_\eta)$ such that for all $k \geq N$,

$$\|x_k - x_0\| < r + \eta.$$

Symbolically, it is denoted as $x_k \xrightarrow{r-\|\cdot\|} x_0$.

Definition 1.7 (see [14]). Let $x = (x_k)$ be a sequence in the GNLS $(X, \|\cdot\|_{\mathcal{G}})$. Then, x is said to be gradually quasi statistically convergent (in short, $st_q(\mathcal{G})$-convergent) to $x_0 \in X$ if, for every $0 < \vartheta \leq 1$ and $\eta > 0$,

$$\delta_c\Big(\big\{k \in \mathbb{N}\colon \mathcal{R}_{\|x_k - x_0\|_{\mathcal{G}}}(\vartheta) \geq \eta\big\}\Big) = 0.$$

Symbolically, $x_k \xrightarrow{st_q(\mathcal{G})} x_0$.

Definition 1.8 (see [14]). Let $x = (x_k)$ be a sequence in the GNLS $(X, \|\cdot\|_{\mathcal{G}})$. Then, x is said to be gradually quasi statistically bounded if, for every $0 < \vartheta \leq 1$, there exists $M(= M(\vartheta)) > 0$ such that

$$\delta_c\Big(\big\{k \in \mathbb{N}\colon \mathcal{R}_{\|x_k\|_{\mathcal{G}}}(\vartheta) > M\big\}\Big) = 0.$$

§ 2. Main results

In this section, we present the main results of the paper. We begin with the following definition.

Definition 2.1. Let $x = (x_k)$ be a sequence in the GNLS $(X, \|\cdot\|_{\mathcal{G}})$ and r be a non-negative real number. Then, x is said to be gradually quasi statistically roughly convergent (in short, $st_q^r(\mathcal{G})$-convergent) to $x_0 \in X$, if for every $0 < \vartheta \leq 1$ and $\eta > 0$,

$$\delta_c\Big(\big\{k \in \mathbb{N}\colon \mathcal{R}_{\|x_k - x_0\|_{\mathcal{G}}}(\vartheta) \geq r + \eta\big\}\Big) = 0.$$

Symbolically, we write $x_k \xrightarrow{st_q^r(\mathcal{G})} x_0$.

Here, x_0 is called as the $st_q^r(\mathcal{G})$-limit of x, where r is the degree of roughness. For $r = 0$, the above definition turns to the Definition 1.7. But our main aim is to deal with the case $r > 0$. There are several reasons for such interest. Since a $st_q(\mathcal{G})$-convergent sequence $y = (y_k)$ with $y_k \xrightarrow{st_q(\mathcal{G})} x_0$ often cannot be measured or calculated accurately, one has to deal with a quasi statistically approximated sequence $x = (x_k)$ satisfying

$$\delta_c\Big(\big\{k \in \mathbb{N}\colon \mathcal{R}_{\|x_k - y_k\|_{\mathcal{G}}}(\vartheta) > r\big\}\Big) = 0.$$

Then, no one can assure the $st_q(\mathcal{G})$-convergence of x, but since for any $\eta > 0$, the following inclusion

$$\big\{k \in \mathbb{N}\colon \mathcal{R}_{\|y_k - x_0\|_{\mathcal{G}}}(\vartheta) \geq \eta\big\} \supseteq \big\{k \in \mathbb{N}\colon \mathcal{R}_{\|x_k - x_0\|_{\mathcal{G}}}(\vartheta) \geq r + \eta\big\}$$

holds, one can certainly assure the $st_q^r(\mathcal{G})-$convergence of x. We present the following example to illustrate the above fact more preciously.

Example 2.1. Let $X = \mathbb{R}^n$ and $\|\cdot\|_{\mathcal{G}}$ be the gradual norm defined in Example 1.1. Consider the sequence (c_n) defined by $c_n = \frac{n}{3}$. Suppose $y = (y_k)$ in $\mathbb{R}^n$ be defined as

$$y_k = \begin{cases} (0, 0, \ldots, 0, 0.5), & \text{if } k \text{ is not a perfect square,} \\ \Big(0, 0, \ldots, 0, 0.5 + 2 \cdot \frac{(-1)^k}{k}\Big), & \text{otherwise.} \end{cases}$$

Then, we have

$$\mathcal{R}_{\|y_k - (0,0,\ldots,0,0.5)\|_{\mathcal{G}}}(\vartheta) = \begin{cases} 0, & \text{if } k \text{ is not a perfect square,} \\ \frac{2e^{\vartheta}}{k}, & \text{otherwise.} \end{cases}$$

Therefore, for any $\eta > 0$, the following inclusion

$$\big\{k \in \mathbb{N}\colon \mathcal{R}_{\|y_k - (0,0,\ldots,0,0.5)\|_{\mathcal{G}}}(\vartheta) \geq \eta\big\} \subseteq \{1, 4, 9, \ldots\}$$

holds and eventually $y_k \xrightarrow{st_q(\mathcal{G})} (0, 0, \ldots, 0, 0.5)$. But, for sufficiently large k, it is impossible to calculate y_k exactly by computer but it is rounded to the nearest one. So, for the sake of simplicity, we approximate y_k by $x_k = (0, 0, \ldots, 0, z)$ at the perfect square positions where z is the integer satisfying $z - 0.5 < y_k < z + 0.5$. Then, the sequence $x = (x_k)$ does not $st_q(\mathcal{G})$-converge anymore. But, by definition, $x_k \xrightarrow{st_q^r(\mathcal{G})} (0, 0, \ldots, 0, 0.5)$ for $r = 0.5$. □

So our main interest is to investigate the case $r > 0$. Therefore, we construct $st_q^r(\mathcal{G})$-limit set of a sequence $x = (x_k)$ denoted and defined as follows:

$$st_q - LIM_x^r(\mathcal{G}) = \Big\{x_0 \in X \colon x_k \xrightarrow{st_q^r(\mathcal{G})} x_0\Big\}.$$

Theorem 2.1. *Let (x_k) and (y_k) be two sequences in the GNLS $(X, \|\cdot\|_{\mathcal{G}})$ such that $x_k \xrightarrow{st_q^{r_1}(\mathcal{G})} x_0$ and $y_k \xrightarrow{st_q^{r_2}(\mathcal{G})} y_0$. Then,*

(i) $x_k + y_k \xrightarrow{st_q^{(r_1+r_2)}(\mathcal{G})} x_0 + y_0$, *and*

(ii) $\mu x_k \xrightarrow{st_q^{|\mu|r_1}(\mathcal{G})} \mu x_0$ *for any* $\mu \in \mathbb{R}$.

P r o o f. (i) Since, $x_k \xrightarrow{st_q^{r_1}(\mathcal{G})} x_0$ and $y_k \xrightarrow{st_q^{r_2}(\mathcal{G})} y_0$, so for any $0 < \vartheta \leq 1$ and $\eta > 0$,

$$\delta_c(P) = \delta_c(Q) = 0,$$

where

$$P = \Big\{k \in \mathbb{N} \colon \mathcal{R}_{\|x_k - x_0\|_{\mathcal{G}}}(\vartheta) \geq r_1 + \frac{\eta}{2}\Big\} \text{ and } Q = \Big\{k \in \mathbb{N} \colon \mathcal{R}_{\|y_k - y_0\|_{\mathcal{G}}}(\vartheta) \geq r_2 + \frac{\eta}{2}\Big\}.$$

Now, as the inclusion

$$(\mathbb{N} \setminus P) \cap (\mathbb{N} \setminus Q) \subseteq \Big\{k \in \mathbb{N} \colon \mathcal{R}_{\|(x_k+y_k)-(x_0+y_0)\|_{\mathcal{G}}}(\vartheta) < r_1 + r_2 + \eta\Big\}$$

holds, so we must have

$$\delta_c\Big(\Big\{k \in \mathbb{N} \colon \mathcal{R}_{\|(x_k+y_k)-(x_0+y_0)\|_{\mathcal{G}}}(\vartheta) \geq r_1 + r_2 + \eta\Big\}\Big) \leq \delta_c(P \cup Q) = 0;$$

and consequently, $x_k + y_k \xrightarrow{st_q^{(r_1+r_2)}(\mathcal{G})} x_0 + y_0$.

(ii) If $\mu = 0$, then there is nothing to prove. So, let us assume that $\mu \neq 0$. Now as the conditions

$$\mathcal{R}_{\|x_k - x_0\|_{\mathcal{G}}}(\vartheta) \leq r_1 \text{ and } \mathcal{R}_{\|\mu x_k - \mu x_0\|_{\mathcal{G}}}(\vartheta) \leq |\mu| r_1$$

are equivalent in gradual normed algebras, so the result follows. □

Remark 2.1 (see [14]). Let (x_k) and (y_k) be two sequences in the GNLS $(X, \|\cdot\|_{\mathcal{G}})$ such that $x_k \xrightarrow{st_q(\mathcal{G})} x_0$ and $y_k \xrightarrow{st_q(\mathcal{G})} y_0$. Then,

(i) $x_k + y_k \xrightarrow{st_q(\mathcal{G})} x_0 + y_0$, and

(ii) $\mu x_k \xrightarrow{st_q(\mathcal{G})} \mu x_0$ for any $\mu \in \mathbb{R}$.

Theorem 2.2. *Let $x = (x_k)$ be a sequence in a GNLS $(X, \|\cdot\|_{\mathcal{G}})$. Then,*

$$\operatorname{diam}\big(st_q - LIM_x^r(\mathcal{G})\big) = \sup\big\{\mathcal{R}_{\|y-z\|_{\mathcal{G}}}(\vartheta) \colon y, z \in st_q - LIM_x^r(\mathcal{G}),\ \vartheta \in [0,1)\big\} \leq 2r.$$

In general, $\operatorname{diam}\big(st_q - LIM_x^r(\mathcal{G})\big)$ *has no smaller bound.*

P r o o f. If possible, let us assume that $\operatorname{diam}\big(st_q - LIM_x^r(\mathcal{G})\big) > 2r$. Then, there exists $y_0, z_0 \in$ $\in st_q - LIM_x^r(\mathcal{G})$ and $0 < \vartheta_0 \leq 1$ such that $\mathcal{R}_{\|y_0 - z_0\|_{\mathcal{G}}}(\vartheta_0) > 2r$. Choose $\eta > 0$ in such a manner that

$$\eta < \frac{\mathcal{R}_{\|y_0 - z_0\|_{\mathcal{G}}}(\vartheta_0)}{2} - r. \tag{2.1}$$

Since, $y_0, z_0 \in st_q - LIM_x^r(\mathcal{G})$, so for any $0 < \vartheta \leq 1$ and $\eta > 0$, $\delta_c(P) = 0$ and $\delta_c(Q) = 0$, where

$$P = \big\{k \in \mathbb{N} \colon \mathcal{R}_{\|x_k - y_0\|_{\mathcal{G}}}(\vartheta) \geq r + \eta\big\} \text{ and } Q = \big\{k \in \mathbb{N} \colon \mathcal{R}_{\|x_k - z_0\|_{\mathcal{G}}}(\vartheta) \geq r + \eta\big\}.$$

By the property of quasi density, it is clear that the set $(\mathbb{N} \setminus P) \cap (\mathbb{N} \setminus Q)$ is non-empty. Take $p \in (\mathbb{N} \setminus P) \cap (\mathbb{N} \setminus Q)$. Then, we have

$$\mathcal{R}_{\|y_0 - z_0\|_{\mathcal{G}}}(\vartheta_0) \leq \mathcal{R}_{\|x_p - y_0\|_{\mathcal{G}}}(\vartheta_0) + \mathcal{R}_{\|x_p - z_0\|_{\mathcal{G}}}(\vartheta_0) < 2(r + \eta),$$

which contradicts (2.1).

For the second part, suppose (x_k) is a sequence in a *GNLS* $(X, \|\cdot\|_{\mathcal{G}})$ such that $x_k \xrightarrow{st_q(\mathcal{G})} x_0$. Then, for any $0 < \vartheta \leq 1$ and $\eta > 0$,

$$\delta_c\Big(\big\{k \in \mathbb{N} \colon \mathcal{R}_{\|x_k - x_0\|_{\mathcal{G}}}(\vartheta) \geq \eta\big\}\Big) = 0.$$

Now, for each $y_0 \in \big(x_0 + \bar{N}(r, \vartheta)\big) = \big\{x \in X \colon \mathcal{R}_{\|x_0 - x\|_{\mathcal{G}}}(\vartheta) \leq r\big\}$, the following inequality holds:

$$\mathcal{R}_{\|x_k - y_0\|_{\mathcal{G}}}(\vartheta) \leq \mathcal{R}_{\|x_k - x_0\|_{\mathcal{G}}}(\vartheta) + \mathcal{R}_{\|x_0 - y_0\|_{\mathcal{G}}}(\vartheta) < r + \eta,$$

whenever $k \notin \big\{k \in \mathbb{N} \colon \mathcal{R}_{\|x_k - x_0\|_{\mathcal{G}}}(\vartheta) \geq \eta\big\}$. This shows that $y_0 \in st_q - LIM_x^r(\mathcal{G})$ and subsequently

$$st_q - LIM_x^r(\mathcal{G}) = \big(x_0 + \bar{N}(r, \vartheta)\big)$$

holds. Since, $\operatorname{diam}\big(x_0 + \bar{N}(r, \vartheta)\big) = 2r$, so, in general upper bound $2r$ of the gradual diameter of the set $st_q - LIM_x^r(\mathcal{G})$ cannot be decreased anymore. □

Remark 2.2 (see [14]). Let $x = (x_k)$ be a sequence in a GNLS $(X, \|\cdot\|_{\mathcal{G}})$ such that $x_k \xrightarrow{st_q(\mathcal{G})} x_0$. Then, x_0 is unique.

Theorem 2.3. *A sequence $x = (x_k)$ in a GNLS $(X, \|\cdot\|_{\mathcal{G}})$ is gradually quasi statistically bounded if and only if there exists some $r \geq 0$ such that $st_q - LIM_x^r(\mathcal{G}) \neq \emptyset$.*

P r o o f. Let $x = (x_k)$ be gradually quasi statistically bounded. Then, for every $\vartheta \in (0, 1]$, there exists $M(= M(\vartheta)) > 0$ such that

$$\delta_c(P) = 0, \quad \text{where} \quad P = \big\{k \in \mathbb{N} \colon \mathcal{R}_{\|x_k\|_{\mathcal{G}}}(\vartheta) > M\big\}.$$

Suppose

$$r' = \sup\big\{\mathcal{R}_{\|x_k\|_{\mathcal{G}}}(\vartheta) \colon k \in \mathbb{N} \setminus P,\ \vartheta \in [0, 1)\big\}.$$

Then, the set $st_q - LIM_x^{r'}(\mathcal{G})$ contains the zero vector of X and eventually

$$st_q - LIM_x^{r'}(\mathcal{G}) \neq \emptyset.$$

Conversely, suppose that $st_q - LIM_x^r(\mathcal{G}) \neq \emptyset$ for some $r \geq 0$. Then, for $x_0 \in st_q - LIM_x^r(\mathcal{G})$,

$$\delta_c\Big(\big\{k \in \mathbb{N} \colon \mathcal{R}_{\|x_k - x_0\|_{\mathcal{G}}}(\vartheta) \geq r + \eta\big\}\Big) = 0$$

holds for any $0 < \vartheta \leq 1$ and $\eta > 0$. This means that almost all x_k's are contained in some ball with any radius greater than r. Therefore, x is gradually quasi statistically bounded. □

Theorem 2.4. *Let $x = (x_k)$ be a sequence in a GNLS $(X, \|\cdot\|_{\mathcal{G}})$. Then, the set $st_q - LIM_x^r(\mathcal{G})$ is gradually closed.*

P r o o f. Let $y = (y_k)$ be a sequence in $st_q - LIM_x^r(\mathcal{G})$ such that

$$y_k \xrightarrow{\|\cdot\|_{\mathcal{G}}} y_0.$$

Then, for every $0 < \vartheta \le 1$ and $\eta > 0$, there exists $N(= N_\eta(\vartheta)) \in \mathbb{N}$ such that for all $k \ge N$,

$$\mathcal{R}_{\|y_k - y_0\|_{\mathcal{G}}}(\vartheta) < \frac{\eta}{2}.$$

Choose $k_0 \in \mathbb{N}$ such that $k_0 \ge N$. Then, $\mathcal{R}_{\|y_{k_0} - y_0\|_{\mathcal{G}}}(\vartheta) < \frac{\eta}{2}$. On the other hand, since $(y_k) \subseteq st_q - LIM_x^r(\mathcal{G})$, we must have

$$\delta_c\left(\left\{k \in \mathbb{N} \colon \mathcal{R}_{\|x_k - y_{k_0}\|_{\mathcal{G}}}(\vartheta) \ge r + \frac{\eta}{2}\right\}\right) = 0. \tag{2.2}$$

Suppose $p \notin \left\{k \in \mathbb{N} \colon \mathcal{R}_{\|x_k - y_{k_0}\|_{\mathcal{G}}}(\vartheta) \ge r + \frac{\eta}{2}\right\}$. Then, $\mathcal{R}_{\|x_p - y_{k_0}\|_{\mathcal{G}}}(\vartheta) < r + \frac{\eta}{2}$ and eventually

$$\mathcal{R}_{\|x_p - y_0\|_{\mathcal{G}}}(\vartheta) \le \mathcal{R}_{\|x_p - y_{k_0}\|_{\mathcal{G}}}(\vartheta) + \mathcal{R}_{\|y_{k_0} - y_0\|_{\mathcal{G}}}(\vartheta) < r + \eta.$$

This means that $p \notin \left\{k \in \mathbb{N} \colon \mathcal{R}_{\|x_k - y_0\|_{\mathcal{G}}}(\vartheta) \ge r + \eta\right\}$ and subsequently from (2.2) we obtain

$$\delta_c\left(\left\{k \in \mathbb{N} \colon \mathcal{R}_{\|x_k - y_0\|_{\mathcal{G}}}(\vartheta) \ge r + \eta\right\}\right) = 0.$$

Hence, $y_0 \in st_q - LIM_x^r(\mathcal{G})$ and the proof ends. □

Theorem 2.5. *Let $x = (x_k)$ be a sequence in a GNLS $(X, \|\cdot\|_{\mathcal{G}})$. If $y_0 \in st_q - LIM_x^{r_0}(\mathcal{G})$ and $y_1 \in st_q - LIM_x^{r_1}(\mathcal{G})$, then*

$$y_\tau = (1-\tau)y_0 + \tau y_1 \in st_q - LIM_x^{(1-\tau)r_0 + \tau r_1}(\mathcal{G}), \quad \textit{for} \ \ \tau \in [0, 1].$$

P r o o f. Since $y_0 \in st_q - LIM_x^{r_0}(\mathcal{G})$ and $y_1 \in st_q - LIM_x^{r_1}(\mathcal{G})$, so, for every $0 < \vartheta \le 1$ and $\eta > 0$, $\delta_c(P) = 0$ and $\delta_c(Q) = 0$, where

$$P = \left\{k \in \mathbb{N} \colon \mathcal{R}_{\|x_k - y_0\|_{\mathcal{G}}}(\vartheta) \ge r_0 + \eta\right\} \ \text{ and } \ Q = \left\{k \in \mathbb{N} \colon \mathcal{R}_{\|x_k - y_1\|_{\mathcal{G}}}(\vartheta) \ge r_1 + \eta\right\}.$$

Subsequently, for any $k \in (\mathbb{N} \setminus P) \cap (\mathbb{N} \setminus Q)$,

$$\begin{aligned}
\mathcal{R}_{\|x_k - y_\tau\|_{\mathcal{G}}}(\vartheta) &\le (1-\tau)\mathcal{R}_{\|x_k - y_0\|_{\mathcal{G}}}(\vartheta) + \tau\mathcal{R}_{\|x_k - y_1\|_{\mathcal{G}}}(\vartheta) \\
&< (1-\tau)(r_0 + \eta) + \tau(r_1 + \eta) \\
&= (1-\tau)r_0 + \tau r_1 + \eta.
\end{aligned}$$

This proves that

$$\left\{k \in \mathbb{N} \colon \mathcal{R}_{\|x_k - y_\tau\|_{\mathcal{G}}}(\vartheta) \ge (1-\tau)r_0 + \tau r_1 + \eta\right\} \subseteq P \cup Q.$$

Now, since the quasi density of the set in the right-hand side of the above inclusion is zero, so the quasi density of the set in the left-hand side is also zero. Hence, $y_\tau \in st_q - LIM_x^{(1-\tau)r_0 + \tau r_1}(\mathcal{G})$. □

Remark 2.3. Let $x = (x_k)$ be a sequence in a GNLS $(X, \|\cdot\|_{\mathcal{G}})$. Then, the set $st_q - LIM_x^r(\mathcal{G})$ is convex.

Theorem 2.6. *Let $r_1 \geq 0$ and $r_2 \geq 0$. A sequence $x = (x_k)$ in a GNLS $(X, \|\cdot\|_{\mathcal{G}})$ is $st_q^{(r_1+r_2)}(\mathcal{G})$-convergent to x_0 if and only if there exists a sequence $y = (y_k)$ such that*

$$y_k \xrightarrow{st_q^{r_1}(\mathcal{G})} x_0 \ \ \textit{and} \ \ \mathcal{R}_{\|x_k-y_k\|_{\mathcal{G}}}(\vartheta) \leq r_2$$

for all $k \in \mathbb{N}$.

P r o o f. Let us assume that $y_k \xrightarrow{st_q^{r_1}(\mathcal{G})} x_0$. Then, by definition, for any $0 < \vartheta \leq 1$ and $\eta > 0$,

$$\delta_c(P) = 0, \quad \text{where} \quad P = \big\{k \in \mathbb{N} \colon \mathcal{R}_{\|y_k-x_0\|_{\mathcal{G}}}(\vartheta) \geq r_1 + \eta\big\}.$$

Now, since $\mathcal{R}_{\|x_k-y_k\|_{\mathcal{G}}}(\vartheta) \leq r_2$ holds for all $k \in \mathbb{N}$, so for all $k \notin P$,

$$\mathcal{R}_{\|x_k-x_0\|_{\mathcal{G}}}(\vartheta) \leq \mathcal{R}_{\|x_k-y_k\|_{\mathcal{G}}}(\vartheta) + \mathcal{R}_{\|y_k-x_0\|_{\mathcal{G}}}(\vartheta) < r_1 + r_2 + \eta.$$

This implies that

$$\big\{k \in \mathbb{N} \colon \mathcal{R}_{\|x_k-x_0\|_{\mathcal{G}}}(\vartheta) \geq r_1 + r_2 + \eta\big\} \subseteq P$$

and eventually by the property of quasi density,

$$\delta_c\Big(\big\{k \in \mathbb{N} \colon \mathcal{R}_{\|x_k-x_0\|_{\mathcal{G}}}(\vartheta) \geq r_1 + r_2 + \eta\big\}\Big) = 0.$$

Hence, $x_k \xrightarrow{st_q^{(r_1+r_2)}(\mathcal{G})} x_0$.

For the converse part, let us assume that

$$x_k \xrightarrow{st_q^{(r_1+r_2)}(\mathcal{G})} x_0. \tag{2.3}$$

Define $y = (y_k)$ by

$$y_k = \begin{cases} x_0, & \text{if } \mathcal{R}_{\|x_k-x_0\|_{\mathcal{G}}}(\vartheta) \leq r_2, \\ x_k + r_2 \frac{x_0 - x_k}{\mathcal{R}_{\|x_k-x_0\|_{\mathcal{G}}}(\vartheta)}, & \text{otherwise.} \end{cases}$$

Then, it is easy to observe that $\mathcal{R}_{\|x_k-y_k\|_{\mathcal{G}}}(\vartheta) \leq r_2$ for all $k \in \mathbb{N}$.

Moreover,

$$\mathcal{R}_{\|y_k-x_0\|_{\mathcal{G}}}(\vartheta) = \begin{cases} 0, & \text{if } \mathcal{R}_{\|x_k-x_0\|_{\mathcal{G}}}(\vartheta) \leq r_2, \\ \mathcal{R}_{\|x_k-x_0\|_{\mathcal{G}}}(\vartheta) - r_2, & \text{otherwise.} \end{cases}$$

By (2.3), for every $0 < \vartheta \leq 1$ and $\eta > 0$,

$$\delta_c\Big(\big\{k \in \mathbb{N} \colon \mathcal{R}_{\|x_k-x_0\|_{\mathcal{G}}}(\vartheta) \geq r_1 + r_2 + \eta\big\}\Big) = 0.$$

Now, as the inclusion

$$\big\{k \in \mathbb{N} \colon \mathcal{R}_{\|x_k-x_0\|_{\mathcal{G}}}(\vartheta) \geq r_1 + r_2 + \eta\big\} \supseteq \big\{k \in \mathbb{N} \colon \mathcal{R}_{\|y_k-x_0\|_{\mathcal{G}}}(\vartheta) \geq r_1 + \eta\big\}$$

holds, so we must have

$$\delta_c\Big(\big\{k \in \mathbb{N} \colon \mathcal{R}_{\|y_k-x_0\|_{\mathcal{G}}}(\vartheta) \geq r_1 + \eta\big\}\Big) = 0.$$

Hence, $y_k \xrightarrow{st_q^{r_1}(\mathcal{G})} x_0$ and the proof ends. □

Remark 2.4. A sequence $x = (x_k)$ in a GNLS $(X, \|\cdot\|_{\mathcal{G}})$ is $st_q^r(\mathcal{G})$-convergent to $x_0 \in X$ with roughness degree $r \geq 0$ if and only if there exists a sequence $y = (y_k)$ in X such that $x_k \xrightarrow{st_q(\mathcal{G})} x_0$ and $\mathcal{R}_{\|x_k-y_k\|} \leq r$ for all $k \in \mathbb{N}$.

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ВЕСТНИК УДМУРТСКОГО УНИВЕРСИТЕТА. МАТЕМАТИКА. МЕХАНИКА. КОМПЬЮТЕРНЫЕ НАУКИ
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Ч. Чоудхури

Квазистатистически грубая сходимость последовательностей в градуальных нормированных линейных пространствах

Ключевые слова: градуальное число, градуальное нормированное линейное пространство, квазиплотность, $st_q^r(\mathcal{G})$-сходимость, $st_q^r(\mathcal{G})$-предельное множество.

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В настоящей статье мы излагаем новое понятие квазистатистически грубой сходимости в градуальных нормированных линейных пространствах. Мы устанавливаем важные результаты, которые представляют несколько фундаментальных свойств этого нового понятия. Мы также вводим понятие $st_q^r(\mathcal{G})$-предельного множества и доказываем, что оно градуально замкнуто, выпукло и играет важную роль для квазистатистической ограниченности последовательности в градуальном нормированном линейном пространстве.

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