VESTNIK UDMURTSKOGO UNIVERSITETA. MATEMATIKA. MEKHANIKA. KOMP'UTERNYE NAUKI
MATHEMATICS 2019. Vol. 29. Issue 4. Pp. 569–582.

MSC2010: 28A33

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CONSTRAINTS OF ASYMPTOTIC NATURE AND ATTAINABILITY PROBLEMS

In control problems, construction and investigation of attainability domains and their analogs are very important. This paper addresses attainability problems in topological spaces. Constraints of asymptotic nature defined in the form of nonempty families of sets are used. The solution of the corresponding attainability problem is defined as an attraction set. Points of this attraction set (attraction elements) are realized in the class of approximate solutions which are nonsequential analogs of the Warga approximate solutions. Some possibilities of applying compactifiers are discussed. Questions of the realization of attraction sets up to a given neighborhood are considered. Some topological properties of attraction sets are investigated. An example with an empty attraction set is considered.

Keywords: attraction set, extension, topological space, compactness.

DOI: 10.20537/vm190408

Introduction

We consider extension of attainability problems with constraints of asymptotic nature. In many control problems, stability of attainability domains under perturbation of standard constraints is absent. In particular, a given stability may be absent with respect to weakening of these constraints. This case is interesting since we obtain some prize under small constraints violation. We note that it is often difficult to give concrete weakening of constraints. Usually, we have a system of weakening constraints (we keep in mind weakening of the initial constraints). In these cases, an asymptotic variant of the setting is natural. Namely, we consider all systems of weakened constraints as a unit. Of course, the Warga approach [1, ch. III] can be used as such a variant. But more general variants are possible. Namely, we can omit from consideration any initial nonperturbed constraints. It is possible to introduce constraints of asymptotic nature immediately. Such constraints can be introduced as a nonempty family of sets; see [2–4] and references therein. In addition, this construction is more informative in the case when the corresponding family is directed. But, in the following, we consider the family of the general form. This family is interpreted as a variant of constraints of asymptotic nature. We obtain very general setting.

In connection with the above-mentioned approach to definition of constraints of asymptotic nature, we introduce asymptotic solutions similar to the approximate solutions of Warga (see [1, ch. III]). In our approach, we use two variants of the corresponding definition: nets and filters. For both variants, our solutions are realized (as objects) with respect to “asymptotic” constraints considered as a whole. But, in the given investigation, we use only nets. In addition, we associate some results to any “asymptotic” solution. The totality of such results form an attraction set (AS). The construction of this AS is our basic aim. For realization of this aim, we use the special instrument; namely, we use compactifiers (see [2]). In this connection, we present the necessary and sufficient conditions for existence of the corresponding compactifier.

Later, we discuss some properties of AS. In particular, we show conditions guaranteeing non-emptyness of AS. Moreover, we consider questions concerning the realization of AS up to arbitrary neighborhood (of this AS). We give some generalization of statements about representation of the basic AS in terms of a continuous image for auxiliary AS.

§ 1. General notions and designations

We use standard set-theoretical symbolics (quantifiers, propositional connectives, and so on); $\triangleq$ is equality by definition. We take an axiom of choice. We call a family any set for which all elements are sets also. For every objects α and β, by $\{\alpha;\beta\}$ we denote the set containing α and β as own elements and not containing no other elements. Then, for every object x, in the form of $\{x\} \triangleq \{x;x\}$, we obtain a singleton containing only x. For any objects u and v, we suppose that $(u,v) \triangleq \{\{u\};\{u;v\}\}$ (see [5, ch. II, § 2, (1)]) is the ordered pair with the first element u and the second element v. If z is an ordered pair, then by $\mathrm{pr}_1(z)$ and $\mathrm{pr}_2(z)$ we denote the first and the second elements of z, respectively; $z = (\mathrm{pr}_1(z), \mathrm{pr}_2(z))$.

If H is a set, then by $\mathcal{P}(H)$ we denote the family of all subsets of H and suppose $\mathcal{P}'(H) \triangleq \mathcal{P}(H) \setminus \{\emptyset\}$ (the family of all nonempty subsets of H); moreover, by $\mathrm{Fin}(H)$ we denote the family of all finite sets of $\mathcal{P}'(H)$ (so, $\mathrm{Fin}(H)$ is the family of all nonempty finite subsets of H). Of course, a family can be used as H.

If $\mathbb{H}$ is a set and $\mathcal{H} \in \mathcal{P}'(\mathcal{P}(\mathbb{H}))$, then

$$\mathbf{C}_{\mathbb{H}}[\mathcal{H}] \triangleq \{\mathbb{H} \setminus H : \ H \in \mathcal{H}\} \in \mathcal{P}'(\mathcal{P}(\mathbb{H}))$$

is the family dual with respect to $\mathcal{H}$. In particular, a topology can be used as $\mathcal{H}$. If $\mathcal{A}$ is a nonempty family and B is a set, then

$$(\mathcal{A}|_B \triangleq \{A \cap B : \ A \in \mathcal{A}\} \in \mathcal{P}'(\mathcal{P}(B)))\,\&\,([\mathcal{A}](B) \triangleq \{A \in \mathcal{A} | B \subset A\} \in \mathcal{P}(\mathcal{A})).$$

For every sets A and B, by B^A we denote the set of all mappings from A into B (for $f \in B^A$ and $a \in A$, in the form of $f(a)$, we obtain the value of the mapping f at the point a, $f(a) \in B$); under $g \in B^A$ and $C \in \mathcal{P}(A)$, we suppose that $g^1(C) \triangleq \{g(x) : \ x \in C\}$, $g^1(C) \neq \emptyset$ for $C \neq \emptyset$. So, we introduce the image of a subset of domain of the mapping g. Let $\mathbb{R}$ be the real line, $\mathbb{N} \triangleq \{1;2;\ldots\}$, and $\overline{1,s} \triangleq \{k \in \mathbb{N} \,|\, k \leq s\}$ $\forall s \in \mathbb{N}$.

If $\mathcal{X}$ is a nonempty family, then

$$\{\cap\}_\sharp(\mathcal{X}) \triangleq \{ \bigcap_{X \in \mathcal{K}} X : \ \mathcal{K} \in \mathrm{Fin}(\mathcal{X})\} \tag{1.1}$$

((1.1) is the family of all finite intersections of sets of $\mathcal{X}$; of course, (1.1) is the family of subsets of the union of all sets of $\mathcal{X}$). If $\mathcal{X}$ is a nonempty family, then

$$(\mathrm{Cen})[\mathcal{X}] \triangleq \{\mathcal{Y} \in \mathcal{P}'(\mathcal{X}) | \bigcap_{Y \in \mathcal{K}} Y \neq \emptyset \ \ \forall \mathcal{K} \in \mathrm{Fin}(\mathcal{Y})\} \tag{1.2}$$

(we introduce the family of all nonempty centered subfamilies of $\mathcal{X}$). In the following, we use nets. Therefore, we introduce some designations connected with nets.

If $\mathbb{H}$ is a nonempty set and $\preceq \in \mathcal{P}(\mathbb{H} \times \mathbb{H})$, then, for every $h_1 \in \mathbb{H}$ and $h_2 \in \mathbb{H}$, we suppose that

$$(h_1 \preceq h_2) \overset{\mathrm{def}}{\Longleftrightarrow} ((h_1, h_2) \in \preceq).$$

In these terms, we introduce preorders in $\mathbb{H}$. Namely,

$$(\mathrm{Ord})[\mathbb{H}] \triangleq \{\preceq \in \mathcal{P}(\mathbb{H} \times \mathbb{H}) | (x \preceq x \ \forall x \in \mathbb{H})\,\&\,(\forall x_1 \in \mathbb{H}\ \forall x_2 \in \mathbb{H}\ \forall x_3 \in \mathbb{H}$$
$$((x_1 \preceq x_2)\,\&\,(x_2 \preceq x_3)) \Longrightarrow (x_1 \preceq x_3))\}$$

is the set of all preorders in $\mathbb{H}$. Then

$$(\mathrm{DIR})[\mathbb{H}] \triangleq \{\preceq \in (\mathrm{Ord})[\mathbb{H}] | \forall x \in \mathbb{H}\ \forall y \in \mathbb{H}\ \exists z \in \mathbb{H}: \ (x \preceq z) \,\&\, (y \preceq z)\}$$

is the set of all directions in $\mathbb{H}$. If $\mathbb{H}$ is a nonempty set and $\sqsubseteq \in (\mathrm{DIR})[\mathbb{H}]$, then $(\mathbb{H}, \sqsubseteq)$ is a (nonempty) directed set; moreover, if Y is a set and $f \in Y^{\mathbb{H}}$, then we consider the triplet $(\mathbb{H}, \sqsubseteq, f)$ as a net in Y. In the following, the expression

$$\exists_M S[M \neq \emptyset] \tag{1.3}$$

replaces the phrase "there exists a nonempty set M". Therefore, for every set T, the expression

$$\exists_{\mathbb{D}} S[\mathbb{D} \neq \emptyset]\ \exists \sqsubseteq\ \in (\mathrm{DIR})[\mathbb{D}]\ \exists f \in T^{\mathbb{D}} : \ldots \tag{1.4}$$

replaces (in fact) the phrase "there exists a net $(\mathbb{D}, \sqsubseteq, f)$ in the set T such that ..."; of course, in (1.3) and (1.4), instead of $M, \mathbb{D}, \sqsubseteq$, and f, arbitrary symbols can be used. These expressions be will used under description AS in a topological space (TS). If $\mathbb{L}$ is a set, then, in the form of

$$\beta[\mathbb{L}] \triangleq \{\mathcal{L} \in \mathcal{P}'(\mathcal{P}(\mathbb{L})) | \forall L_1 \in \mathcal{L}\ \forall L_2 \in \mathcal{L}\ \exists L_3 \in \mathcal{L}: \ L_3 \subset L_1 \cap L_2\},$$

we obtain the family of all nonempty directed subfamilies of $\mathcal{P}(\mathbb{L})$.

Elements of topology. By $(\mathrm{top})[H]$ we denote the family of all topologies [6, ch. 1] on a set H. If $\tau \in (\mathrm{top})[H]$, then (H, τ) is a TS. We use standard notions of T_1-space and T_2-space (for example, see [6, ch. 1]). We note the compactness property [6, ch. 3]; for this, we suppose that

$$(\mathbf{c} - \mathrm{top})[H] \triangleq \{\tau \in (\mathrm{top})[H] \,|\, \forall \zeta \in \mathcal{P}'(\tau)\ (H = \bigcup_{G \in \zeta} G) \Longrightarrow (\exists \mathcal{K} \in \mathrm{Fin}(\zeta): \ H = \bigcup_{G \in \mathcal{K}} G)\}$$

(the family of all topologies of $(\mathrm{top})[H]$ converting the set H in a compact TS). As usual, for every TS (X, τ) (so, X is a set and $\tau \in (\mathrm{top})[X]$) and $Y \in \mathcal{P}(X)$, we find that $\tau|_Y \in (\mathrm{top})[Y]$ and $(Y, \tau|_Y)$ is a subspace of TS (X, τ). In the form of

$$(\tau - \mathrm{comp})[X] \triangleq \{K \in \mathcal{P}(X) \,|\, \tau|_K \in (\mathbf{c} - \mathrm{top})[K]\},$$

we obtain the family of all compact (in TS (X, τ)) subsets of X; moreover,

$$(\tau - \mathrm{comp})^0[X] \triangleq \{H \in \mathcal{P}(X) | \exists K \in (\tau - \mathrm{comp})[X]: \ H \subset K\}$$

is the family of all subsets of X precompact in TS (X, τ). For every TS (X, τ), in the form of $\mathbf{C}_X[\tau]$, we obtain the family of all subsets of X closed in the sense of (X, τ).

If (X, τ) is a TS and $M \in \mathcal{P}(X)$, then

$$N_\tau^0[M] \triangleq \{G \in \tau | M \subset G\}$$

is the family of all open neighborhoods of M in TS (X, τ). For $x \in X$, suppose that $N_\tau^0(x) \triangleq N_\tau^0[\{x\}] = \{G \in \tau | x \in G\}$; then, $N_\tau(x) \triangleq \{Y \in \mathcal{P}(X) | \exists G \in N_\tau^0(x): \ G \subset Y\}$ is the family of all neighborhoods of x in TS (X, τ). If (X, τ) is a TS and $M \in \mathcal{P}(X)$, then

$$\mathrm{cl}(M, \tau) \triangleq \{x \in X | G \cap M \neq \emptyset\ \forall G \in N_\tau^0(x)\} = \{x \in X | H \cap M \neq \emptyset\ \forall H \in N_\tau(x)\}$$

is the closure of M in (X, τ).

Convergence of nets. At first, we take the following statement: if $\mathbf{D}$ and $\mathbf{T}$ are nonempty sets, $\sqsubseteq \in (\mathrm{DIR})[\mathbf{D}]$, and $f \in \mathbf{T}^{\mathbf{D}}$, then

$$\begin{gathered}(\mathbf{T}-\mathrm{ass})[\mathbf{D};\sqsubseteq;f] \triangleq \{M \in \mathcal{P}(\mathbf{T}) | \exists d_1 \in \mathbf{D}\ \forall d_2 \in \mathbf{D} \\ (d_1 \sqsubseteq d_2) \Longrightarrow (f(d_2) \in M)\} \in \mathcal{P}'(\mathcal{P}'(\mathbf{T}))\end{gathered} \tag{1.5}$$

is the filter [6, Section 1.6] (in $\mathbf{T}$) associated with the net $(\mathbf{D}, \sqsubseteq, f)$. Then, we introduce the standard Moore–Smith convergence: if $(\mathbf{T}, \tau)$ is a TS, $(\mathbf{D}, \sqsubseteq, f)$ is a net in the set $\mathbf{T}$ (namely, $\mathbf{D}$ is a nonempty set, $\sqsubseteq \in (\mathrm{DIR})[\mathbf{D}]$, and $f \in \mathbf{T}^{\mathbf{D}}$), and $t \in \mathbf{T}$, then

$$((\mathbf{D}, \sqsubseteq, f) \xrightarrow{\tau} t) \overset{\mathrm{def}}{\Longleftrightarrow} (N_\tau(t) \subset (\mathbf{T}-\mathrm{ass})[\mathbf{D};\sqsubseteq;f]). \tag{1.6}$$

So, by (1.5) and (1.6) the Moore–Smith convergence is defined.

If $(X, \tau_1),\ X \neq \emptyset$, and $(Y, \tau_2),\ Y \neq \emptyset$, are two TS, then

$$C(X, \tau_1, Y, \tau_2) \triangleq \{f \in Y^X | f^{-1}(G) \in \tau_1\ \ \forall G \in \tau_2\}$$

(the set of all (τ_1, τ_2)-continuous mappings from Y^X); moreover, we suppose that (see [7, (2.8.1)])

$$C_{\mathrm{cl}}(X, \tau_1, Y, \tau_2) \triangleq \{f \in C(X, \tau_1, Y, \tau_2) | f^1(F) \in \mathbf{C}_Y[\tau_2]\ \ \forall F \in \mathbf{C}_X[\tau_1]\}$$

obtaining the set of all closed (and continuous) mappings from (X, τ_1) into (Y, τ_2).

§ 2. Attraction sets: general representations

In this section, we discuss general notions connected with AS in the fixed TS $(\mathbf{X}, \tau)$. So, $\mathbf{X}$ is a nonempty set and $\tau \in (\mathrm{top})[\mathbf{X}]$. Moreover, in this section, we fix a nonempty set E; elements of E are considered as usual solutions. Finally, in this section, we fix a mapping $\mathbf{f} \in \mathbf{X}^E$. If $\mathcal{E} \in \mathcal{P}'(\mathcal{P}(E))$, then

$$\begin{gathered}(\mathrm{as})[E;\mathbf{X};\tau;\mathbf{f};\mathcal{E}] \triangleq \{\mathbf{x} \in \mathbf{X} | \exists_{\mathbf{D}} S[\mathbf{D} \neq \emptyset]\ \exists \sqsubseteq \in (\mathrm{DIR})[\mathbf{D}]\ \exists g \in E^{\mathbf{D}} : \\ (\mathcal{E} \subset (E-\mathrm{ass})[\mathbf{D};\sqsubseteq;g])\ \&\ ((\mathbf{D}, \sqsubseteq, \mathbf{f} \circ g) \xrightarrow{\tau} \mathbf{x})\}\end{gathered} \tag{2.1}$$

(as usually, the symbol $\circ$ is used for composition of mappings). We consider the set (2.1) as AS in $(\mathbf{X}, \tau)$ corresponding to the aim operator $\mathbf{f}$ and constraints of asymptotic nature defined by $\mathcal{E}$. Of course, in (2.1), we can use the variant $\mathcal{E} \in \beta[E]$. In addition (see [7, (3.2.8)]), under $\mathcal{E} \in \beta[E]$

$$(\mathrm{as})[E;\mathbf{X};\tau;\mathbf{f};\mathcal{E}] = \bigcap_{\Sigma \in \mathcal{E}} \mathrm{cl}(\mathbf{f}^1(\Sigma), \tau). \tag{2.2}$$

This case is basic since, for every family $\mathfrak{E} \in \mathcal{P}'(\mathcal{P}(E))$, we have the property $\{\cap\}_\sharp(\mathfrak{E}) \in \beta[E]$. Moreover,

$$(\mathrm{as})[E;\mathbf{X};\tau;\mathbf{f};\mathcal{E}] = (\mathrm{as})[E;\mathbf{X};\tau;\mathbf{f};\{\cap\}_\sharp(\mathcal{E})]\ \forall \mathcal{E} \in \mathcal{P}'(\mathcal{P}(E)). \tag{2.3}$$

So, by (2.2) and (2.3) we obtain the following representation of AS in the general case:

$$(\mathrm{as})[E;\mathbf{X};\tau;\mathbf{f};\mathcal{E}] = \bigcap_{\Sigma \in \{\cap\}_\sharp(\mathcal{E})} \mathrm{cl}(\mathbf{f}^1(\Sigma), \tau)\ \forall \mathcal{E} \in \mathcal{P}'(\mathcal{P}(E)). \tag{2.4}$$

In this article, we investigate properties of AS (2.4). We note that employment of nets in (2.1) (or filters, as in [4]) is essential; in this connection, see [8]. Conditions for exhausting sequential realization of AS are presented in [9, Proposition 3.3.1]. For the AS constructing, compactifiers [2] can be used. A more general variant is indicated in [7, Proposition 3.3.1], where an almost perfect operator for extension of $\mathbf{f}$ (see [6, Section 3.7]) was used. In this investigation, we do not consider this more general scheme, but focus on compactifier application.

From (2.4), the obvious property follows:

$$(\text{as})[E;\mathbf{X};\tau;\mathbf{f};\mathcal{E}] \in \mathbf{C}_{\mathbf{X}}[\tau] \quad \forall \mathcal{E} \in \mathcal{P}'(\mathcal{P}(E)). \tag{2.5}$$

We recall that $\mathbf{f}^1(E) = \{\mathbf{f}(x) : \ x \in E\} \in \mathcal{P}'(\mathbf{X})$ (image of E under operation $\mathbf{f}$). From (2.5), we find that

$$(\tau \in (\mathbf{c}-\text{top})[\mathbf{X}]) \Longrightarrow ((\text{as})[E;\mathbf{X};\tau;\mathbf{f};\mathcal{E}] \in (\tau-\text{comp})[\mathbf{X}] \quad \forall \mathcal{E} \in \mathcal{P}'(\mathcal{P}(E))).$$

Proposition 1. *The following equivalence property takes place:*

$$\begin{gathered}(\exists_K S[K \neq \emptyset]\ \exists \vartheta \in (\mathbf{c}-\text{top})[K]\ \exists m \in K^E\ \exists g \in C(K,\vartheta,\mathbf{X},\tau) : \\ \mathbf{f} = g \circ m) \Longleftrightarrow (\mathbf{f}^1(E) \in (\tau-\text{comp})^0[\mathbf{X}]).\end{gathered} \tag{2.6}$$

P r o o f. Let the expression on the left-hand side of (2.6) be true. Fix a nonempty set $\mathbb{K}$, $\bar\vartheta \in (\mathbf{c}-\text{top})[\mathbb{K}]$, $\bar m \in \mathbb{K}^E$, and $\mathbf{g} \in C(\mathbb{K},\bar\vartheta,\mathbf{X},\tau)$ with the property $\mathbf{f} = \mathbf{g} \circ \bar m$. Then $\mathbf{g}^1(\mathbb{K}) \in (\tau-\text{comp})[\mathbf{X}]$ (see [6, Section 3.1]). In addition,

$$\mathbf{f}^1(E) = \mathbf{g}^1(\bar m^1(E)) \subset \mathbf{g}^1(\mathbb{K}).$$

So, $\mathbf{f}^1(E) \in (\tau-\text{comp})^0[\mathbf{X}]$. We obtain the following implication:

$$\begin{gathered}(\exists_K S[K \neq \emptyset]\ \exists \vartheta \in (\mathbf{c}-\text{top})[K]\ \exists m \in K^E\ \exists g \in C(K,\vartheta,\mathbf{X},\tau) : \\ \mathbf{f} = g \circ m) \Longrightarrow (\mathbf{f}^1(E) \in (\tau-\text{comp})^0[\mathbf{X}]).\end{gathered} \tag{2.7}$$

Let $\mathbf{f}^1(E) \in (\tau-\text{comp})^0[\mathbf{X}]$. Then, for a (compact) set $\mathbf{K} \in (\tau-\text{comp})[\mathbf{X}]$, the inclusion $\mathbf{f}^1(E) \subset \mathbf{K}$ is realized. Of course, $\mathbf{K} \neq \emptyset$ and $\tau|_{\mathbf{K}} \in (\mathbf{c}-\text{top})[\mathbf{K}]$. So, $(\mathbf{K},\tau|_{\mathbf{K}})$ is a nonempty compact space for which $\mathbf{f} \in \mathbf{K}^E$. Suppose that $\eta \in \mathbf{X}^{\mathbf{K}}$ is defined by conditions $\eta(\mathbf{x}) \triangleq \mathbf{x}\ \forall \mathbf{x} \in \mathbf{K}$ (we use the inclusion map into $\mathbf{X}$). Of course, $\mathbf{f} = \eta \circ \mathbf{f}$, where η is the continuous mapping from compact space $(\mathbf{K},\tau|_{\mathbf{K}})$ into $(\mathbf{X},\tau)$:

$$\eta \in C(\mathbf{K},\tau|_{\mathbf{K}},\mathbf{X},\tau)$$

(namely, under $G \in \tau$, we find that $\eta^{-1}(G) = \mathbf{K} \cap G \in \tau|_{\mathbf{K}}$). So, we find that $(\mathbf{K},\tau|_{\mathbf{K}},\mathbf{f},\eta)$ is a collection with required properties. Therefore, the implication

$$\begin{gathered}(\mathbf{f}^1(E) \in (\tau-\text{comp})^0[\mathbf{X}]) \\ \Longrightarrow (\exists_K S[K \neq \emptyset]\ \exists \vartheta \in (\mathbf{c}-\text{top})[K]\ \exists m \in K^E\ \exists g \in C(K,\vartheta,\mathbf{X},\tau) :\ \mathbf{f} = g \circ m)\end{gathered}$$

is established. Using (2.7), we obtain (2.6). □

In [2], the following notion was introduced: the collection (K,ϑ,m,g) for which K is a nonempty set, $\vartheta \in (\mathbf{c}-\text{top})[K]$, $m \in K^E$, $g \in C(K,\vartheta,\mathbf{X},\tau)$, and $\mathbf{f} = g \circ m$, was called (in [2]) a compactifier. So, in Proposition 1, we obtain the necessary and sufficient conditions for the compactifier existence. In this connection, for a nonempty set S, we introduce

$$\mathbf{F}^0_{\mathbf{c}}[S;\mathbf{X};\tau] \triangleq \{f \in \mathbf{X}^S | f^1(S) \in (\tau-\text{comp})^0[\mathbf{X}]\} \in \mathcal{P}'(\mathbf{X}^S) \tag{2.8}$$

(all constant functions are elements of the set (2.8)). We note the obvious property: for every nonempty set S, $f \in S^E$, and $g \in \mathbf{F}^0_{\mathbf{c}}[S; \mathbf{X}; \tau]$

$$g \circ f \in \mathbf{F}^0_{\mathbf{c}}[E; \mathbf{X}; \tau].$$

Of course, from (2.8) and Proposition 1, we obtain

$$(\exists_K S[K \neq \emptyset]\ \exists \vartheta \in (\mathbf{c} - \mathrm{top})[K]\ \exists m \in K^E\ \exists g \in C(K, \vartheta, \mathbf{X}, \tau) : \\ \mathbf{f} = g \circ m) \Longleftrightarrow (\mathbf{f} \in \mathbf{F}^0_{\mathbf{c}}[E; \mathbf{X}; \tau]). \tag{2.9}$$

We note that $\forall S \in \mathcal{P}(\mathbf{X})\ \forall F \in [\mathbf{C}_X[\tau]](S)\ \forall A \in \mathcal{P}(S)$

$$\mathrm{cl}(A, \tau|_F) = \mathrm{cl}(A, \tau); \tag{2.10}$$

(2.10) is a simple corollary of definitions. We use (2.10) in the next section; in this section, we will change $(\mathbf{X}, \tau)$ and $\mathbf{f}$.

§ 3. Transformation of attraction sets

In this section, we fix only a nonempty set E. We consider E as the set of usual solutions. Recall that for every TS (Y, τ), $Y \neq \emptyset$, and $f \in Y^E$

$$[\mathbf{C}_Y[\tau]](f^1(E)) = \{F \in \mathbf{C}_Y[\tau] | f^1(E) \subset F\} \in \mathcal{P}'(\mathbf{C}_Y[\tau]);$$

in addition, $\mathbb{F} \neq \emptyset\ \forall \mathbb{F} \in [\mathbf{C}_Y[\tau]](f^1(E))$.

Proposition 2. *If (Y, τ) is a TS with $Y \neq \emptyset$, $f \in Y^E$, $\mathcal{E} \in \mathcal{P}'(\mathcal{P}(E))$, and $F \in [\mathbf{C}_Y[\tau]](f^1(E))$, then*

$$(\mathrm{as})[E; Y; \tau; f; \mathcal{E}] = (\mathrm{as})[E; F; \tau|_F; f; \mathcal{E}] \in \mathbf{C}_F[\tau|_F].$$

The corresponding proof is carried out by obvious combination of (2.3), (2.4), and (2.10). So, we can turn to a closed subspace without changing AS. In this connection, we recall that, for a T_2-space (H, τ), $H \neq \emptyset$, and a set $S \in \mathcal{P}(H)$, the inclusion $[(\tau - \mathrm{comp})[H]](S) \subset [\mathbf{C}_H[\tau]](S)$ holds. As a corollary, for a T_2-space (Y, τ), $Y \neq \emptyset$, $f \in Y^E$, and $\mathbf{K} \in [(\tau - \mathrm{comp})[Y]](f^1(E))$, in the form of $(\mathbf{K}, \tau|_{\mathbf{K}})$, we obtain a nonempty compactum: $\mathbf{K} \neq \emptyset$, $\tau|_{\mathbf{K}} \in (\mathbf{c} - \mathrm{top})[\mathbf{K}]$, and $(\mathbf{K}, \tau|_{\mathbf{K}})$ is a T_2-space. On the other hand, by Proposition 2, for a nonempty T_2-space (Y, τ), $f \in Y^E$, $\mathcal{E} \in \mathcal{P}'(\mathcal{P}(E))$, and $K \in [(\tau - \mathrm{comp})[Y]](f^1(E))$, we find that

$$(\mathrm{as})[E; Y; \tau; f; \mathcal{E}] \in \mathbf{C}_K[\tau|_K]$$

and, as a corollary, the property

$$(\mathrm{as})[E; Y; \tau; f; \mathcal{E}] \in (\tau|_K - \mathrm{comp})[K] \tag{3.1}$$

takes place. In addition, for a TS (Y, τ), $Y \neq \emptyset$, and $f \in Y^E$

$$([(\tau - \mathrm{comp})[Y]](f^1(E)) \neq \emptyset) \Longleftrightarrow (f \in \mathbf{F}^0_{\mathbf{c}}[E; Y; \tau]). \tag{3.2}$$

As a corollary, for a nonempty T_2-space (Y, τ), $Y \neq \emptyset$, $f \in \mathbf{F}^0_{\mathbf{c}}[E; Y; \tau]$, $\mathcal{E} \in \mathcal{P}'(\mathcal{P}(E))$, and $K \in [(\tau - \mathrm{comp})[Y]](f^1(E))$, the inclusion (3.1) is obtained.

Proposition 3. *If (Y, τ) is a nonempty T_2-space, $f \in \mathbf{F}^0_{\mathbf{c}}[E; Y; \tau]$, and $\mathcal{E} \in \mathcal{P}'(\mathcal{P}(E))$, then*

$$((\mathrm{as})[E; Y; \tau; f; \mathcal{E}] \neq \emptyset) \Longleftrightarrow (\mathcal{E} \in (\mathrm{Cen})[\mathcal{P}(E)]).$$

P r o o f. By (2.3) and (2.4) we find that

$$(\mathrm{as})[E;Y;\tau;f;\mathcal{E}] = (\mathrm{as})[E;Y;\tau;f;\{\cap\}_\sharp(\mathcal{E})] = \bigcap_{\Sigma\in\{\cap\}_\sharp(\mathcal{E})} \mathrm{cl}(f^1(\Sigma),\tau). \tag{3.3}$$

Let $\mathcal{E} \in (\mathrm{Cen})[\mathcal{P}(E)]$. Then, $\Xi \neq \emptyset\ \forall \Xi \in \{\cap\}_\sharp(\mathcal{E})$. Using (3.2), we choose and fix a set

$$\mathbb{K} \in [(\tau - \mathrm{comp})[Y]](f^1(E)).$$

Then, $\tau|_\mathbb{K} \in (\mathbf{c} - \mathrm{top})[\mathbb{K}]$ (and, what is more, $(\mathbb{K}, \tau|_\mathbb{K})$ is a nonempty compactum). Since (Y,τ) is a T_2-space, the property $\mathbb{K} \in [\mathbf{C}_Y[\tau]](f^1(E))$ takes place. By Proposition 2,

$$(\mathrm{as})[E;Y;\tau;f;\mathcal{E}] = (\mathrm{as})[E;\mathbb{K};\tau|_\mathbb{K};f;\mathcal{E}] = \bigcap_{\Sigma\in\{\cap\}_\sharp(\mathcal{E})} \mathrm{cl}(f^1(\Sigma),\tau|_\mathbb{K}) \tag{3.4}$$

(we use a representation similar to (3.3)). In addition,

$$\mathcal{F} \triangleq \{\mathrm{cl}(f^1(\Sigma),\tau|_\mathbb{K}) :\ \Sigma \in \{\cap\}_\sharp(\mathcal{E})\} \in \mathcal{P}'(\mathbf{C}_\mathbb{K}[\tau|_\mathbb{K}]).$$

Then, $\{\cap\}_\sharp(\mathcal{E}) \in \beta[E]$. As a corollary, we find that $\mathcal{F} \in (\mathrm{Cen})[\mathbf{C}_\mathbb{K}[\tau|_\mathbb{K}]]$ and therefore

$$\bigcap_{\Sigma\in\{\cap\}_\sharp(\mathcal{E})} \mathrm{cl}(f^1(\Sigma),\tau|_\mathbb{K}) = \bigcap_{F\in\mathcal{F}} F \neq \emptyset \tag{3.5}$$

(we use compactness of TS $(\mathbb{K},\tau|_\mathbb{K})$). From (3.4) and (3.5), we find that $(\mathrm{as})[E;Y;\tau;f;\mathcal{E}] \neq \emptyset$. So,

$$(\mathcal{E} \in (\mathrm{Cen})[\mathcal{P}(E)]) \Longrightarrow ((\mathrm{as})[E;Y;\tau;f;\mathcal{E}] \neq \emptyset). \tag{3.6}$$

Let $(\mathrm{as})[E;Y;\tau;f;\mathcal{E}] \neq \emptyset$. Then, by (2.4),

$$\mathrm{cl}(f^1(\Sigma),\tau) \neq \emptyset\ \ \forall \Sigma \in \{\cap\}_\sharp(\mathcal{E}).$$

As a corollary, $\Sigma \neq \emptyset\ \forall \Sigma \in \{\cap\}_\sharp(\mathcal{E})$. Therefore, $\mathcal{E} \in (\mathrm{Cen})[\mathcal{P}(E)]$ (see (1.1), (1.2)). So, implication

$$((\mathrm{as})[E;Y;\tau;f;\mathcal{E}] \neq \emptyset) \Longrightarrow (\mathcal{E} \in (\mathrm{Cen})[\mathcal{P}(E)]) \tag{3.7}$$

is established. From (3.6) and (3.7) the required statement follows. □

Now, we consider the question of realization of AS up to given neighborhood. These statements are similar to those considered in [7, Section 3.6].

Proposition 4. *If* (Y,τ) *is a nonempty* T_2*-space,* $f \in \mathbf{F}^0_\mathbf{c}[E;Y;\tau]$, *and* $\mathcal{E} \in \beta[E]$, *then*

$$N^0_\tau[(\mathrm{as})[E;Y;\tau;f;\mathcal{E}]] = \bigcup_{\Sigma\in\mathcal{E}} N^0_\tau[\mathrm{cl}(f^1(\Sigma),\tau)].$$

P r o o f. By [7, Proposition 3.6.1] we obtain the inclusion

$$N^0_\tau[(\mathrm{as})[E;Y;\tau;f;\mathcal{E}]] \subset \bigcup_{\Sigma\in\mathcal{E}} N^0_\tau[\mathrm{cl}(f^1(\Sigma),\tau)]. \tag{3.8}$$

On the other hand, for $\Sigma \in \mathcal{E}$ and $G \in N^0_\tau[\mathrm{cl}(f^1(\Sigma),\tau)]$, the following chain of inclusions holds:

$$(\mathrm{as})[E;Y;\tau;f;\mathcal{E}] \subset \mathrm{cl}(f^1(\Sigma),\tau) \subset G;$$

see (2.2). As a corollary, $G \in N^0_\tau[(\mathrm{as})[E;Y;\tau;f;\mathcal{E}]]$. Since the choice of Σ and G was arbitrary, the inclusion opposite with respect to (3.8) is established. □

From Proposition 4, we find that, for every nonempty T_2-space (Y,τ), $f \in \mathbf{F}^0_\mathbf{c}[E;Y;\tau]$, $\mathcal{E} \in \beta[E]$, and $G \in N^0_\tau[(\mathrm{as})[E;Y;\tau;f;\mathcal{E}]]$

$$\exists \Sigma \in \mathcal{E} : (\mathrm{as})[E;Y;\tau;f;\mathcal{E}] \subset \mathrm{cl}(f^1(\Sigma),\tau) \subset G. \tag{3.9}$$

Proposition 5. *If* (Y, τ) *is a nonempty* T_2*-space,* $f \in \mathbf{F}^0_{\mathbf{c}}[E; Y; \tau]$, $\mathcal{E} \in \mathcal{P}'(\mathcal{P}(E))$, *and* $G \in N^0_\tau[(\mathrm{as})[E; Y; \tau; f; \mathcal{E}]]$, *then*

$$\exists \Sigma \in \{\cap\}_\sharp(\mathcal{E}) : \ (\mathrm{as})[E; Y; \tau; f; \mathcal{E}] \subset \mathrm{cl}(f^1(\Sigma), \tau) \subset G.$$

The corresponding proof is an obvious combination of (2.4) and (3.9). Moreover, we obtain the obvious

Corollary 1. *If* (Y, τ), f, $\mathcal{E}$, *and* G *correspond to Proposition 5, then* $\exists \Xi \in \{\cap\}_\sharp(\mathcal{E})$ $\forall \Sigma \in \{\cap\}_\sharp(\mathcal{E})$

$$(\Sigma \subset \Xi) \Longrightarrow ((\mathrm{as})[E; Y; \tau; f; \mathcal{E}] \subset \mathrm{cl}(f^1(\Sigma), \tau) \subset G)).$$

Proposition 6. *If* (Y, τ) *is a nonempty* T_2*-space,* $\mathcal{E} \in (\mathrm{Cen})[\mathcal{P}(E)]$, *and* $f \in \mathbf{F}^0_{\mathbf{c}}[E; Y; \tau]$, *then*

$$(\mathrm{as})[E; Y; \tau; f; \mathcal{E}] \in (\tau - \mathrm{comp})[Y] \setminus \{\emptyset\}.$$

The proof is carried out by the immediate combination of (3.1) and Proposition 3.

Proposition 7. *If* (Y, τ) *is a nonempty* T_2*-space,* $f \in Y^E$, *and* $\mathcal{E} \in \mathcal{P}'(\mathcal{P}(E))$, *then*

$$\mathrm{cl}(f^1(\bigcap_{\Sigma \in \mathcal{E}} \Sigma), \tau) \subset (\mathrm{as})[E; Y; \tau; f; \mathcal{E}].$$

The proof follows from definitions of Section 3.

Corollary 2. *If* (Y, τ) *is a nonempty TS,* $f \in Y^E$, *and* $\mathcal{K} \in \mathrm{Fin}(\mathcal{P}(E))$, *then*

$$(\mathrm{as})[E; Y; \tau; f; \mathcal{K}] = \mathrm{cl}(f^1(\bigcap_{\Sigma \in \mathcal{K}} \Sigma), \tau).$$

P r o o f. Recall that $\bigcap\limits_{\Sigma \in \mathcal{K}} \Sigma \in \{\cap\}_\sharp(\mathcal{K})$; therefore, by (2.4)

$$(\mathrm{as})[E; Y; \tau; f; \mathcal{K}] = (\mathrm{as})[E; Y; \tau; f; \{\cap\}_\sharp(\mathcal{K})] \subset \mathrm{cl}(f^1(\bigcap_{\Sigma \in \mathcal{K}} \Sigma), \tau).$$

Using Proposition 7, we obtain the required equality. □

Proposition 8. *If* (Y, τ_1) *and* (Z, τ_2) *are two nonempty TS,* $\mathcal{E} \in \mathcal{P}'(\mathcal{P}(E))$, $f \in Y^E$, *and* $g \in C(Y, \tau_1, Z, \tau_2)$, *then*

$$g^1((\mathrm{as})[E; Y; \tau_1; f; \mathcal{E}]) \subset (\mathrm{as})[E; Z; \tau_2; g \circ f; \mathcal{E}]. \tag{3.10}$$

P r o o f. Recall [7, Proposition 3.3.1]. Now, we give a small generalization. For this, we note that by (2.4)

$$(\mathrm{as})[E; Y; \tau_1; f; \mathcal{E}] = \bigcap_{\Sigma \in \{\cap\}_\sharp(\mathcal{E})} \mathrm{cl}(f^1(\Sigma), \tau_1).$$

Therefore, using continuity of g, we obtain that

$$g^1((\mathrm{as})[E; Y; \tau_1; f; \mathcal{E}]) \subset \bigcap_{\Sigma \in \{\cap\}_\sharp(\mathcal{E})} g^1(\mathrm{cl}(f^1(\Sigma), \tau_1)) \subset \bigcap_{\Sigma \in \{\cap\}_\sharp(\mathcal{E})} \mathrm{cl}(g^1(f^1(\Sigma)), \tau_2) =$$
$$= \bigcap_{\Sigma \in \{\cap\}_\sharp(\mathcal{E})} \mathrm{cl}((g \circ f)^1(\Sigma), \tau_2) = (\mathrm{as})[E; Z; \tau_2; g \circ f; \mathcal{E}].$$

□

Proposition 9. *If* (Y, τ_1) *is a nonempty compact TS,* (Z, τ_2) *is a nonempty* T_1*-space,* $f \in Y^E$, $g \in C_{\rm cl}(Y, \tau_1, Z, \tau_2)$, *and* $\mathcal{E} \in \beta[E]$, *then*

$$(\mathrm{as})[E; Z; \tau_2; g \circ f; \mathcal{E}] = g^1((\mathrm{as})[E; Y; \tau_1; f; \mathcal{E}]). \tag{3.11}$$

P r o o f. We use Proposition 8; namely, we have the inclusion (3.10). Moreover, we use the property similar to (2.2). In this connection, we note the following obvious statement: for every $m \in \mathbb{N}$ and mapping

$$(\Sigma_i)_{i \in \overline{1,m}} : \ \overline{1,m} \to \mathcal{E},$$

there exists $\Sigma \in \mathcal{E}$ such that

$$\Sigma \subset \bigcap_{i=1}^{m} \Sigma_i \tag{3.12}$$

((3.12) is established by induction on m). So, by (2.2)

$$(\mathrm{as})[E; Z; \tau_2; g \circ f; \mathcal{E}] = \bigcap_{\Sigma \in \mathcal{E}} \mathrm{cl}((g \circ f)^1(\Sigma), \tau_2). \tag{3.13}$$

Let $z_* \in (\mathrm{as})[E; Z; \tau_2; g \circ f; \mathcal{E}]$. Then, under $\Sigma \in \mathcal{E}$, by (3.13) we obtain

$$z_* \in \mathrm{cl}((g \circ f)^1(\Sigma), \tau_2),$$

where $\mathrm{cl}((g \circ f)^1(\Sigma), \tau_2) = \mathrm{cl}(g^1(f^1(\Sigma)), \tau_2) = g^1(\mathrm{cl}(f^1(\Sigma), \tau_1))$ by the choice of g; therefore

$$\mathrm{cl}(f^1(\Sigma), \tau_1) \cap g^{-1}(\{z_*\}) \in \mathbf{C}_Y[\tau_1] \setminus \{\emptyset\}. \tag{3.14}$$

We introduce the following (nonempty) family

$$\mathcal{C} \triangleq \{\mathrm{cl}(f^1(\Sigma), \tau_1) \cap g^{-1}(\{z_*\}) : \ \Sigma \in \mathcal{E}\} \in \mathcal{P}'(\mathbf{C}_Y[\tau_1] \setminus \{\emptyset\}).$$

Using (3.12), we find that $\mathcal{C} \in (\mathrm{Cen})[\mathbf{C}_Y[\tau_1]]$. As a corollary, the intersection of all sets of the family $\mathcal{C}$ is a nonempty subset of Y (we use compactness of TS (Y, τ_1)). Let y_* be an element of the above-mentioned intersection; then

$$y_* \in \bigcap_{\Sigma \in \mathcal{E}} (\mathrm{cl}(f^1(\Sigma), \tau_1) \cap g^{-1}(\{z_*\})).$$

As a corollary (see (2.2)), $y_* \in (\mathrm{as})[E; Y; \tau_1; f; \mathcal{E}]$ and $z_* = g(y_*)$. Then

$$z_* \in g^1((\mathrm{as})[E; Y; \tau_1; f; \mathcal{E}]).$$

Since the choice of z_* was arbitrary, the inclusion

$$(\mathrm{as})[E; Z; \tau_2; g \circ f; \mathcal{E}] \subset g^1((\mathrm{as})[E; Y; \tau_1; f; \mathcal{E}])$$

is established. Recall that opposite inclusion follows from Proposition 8. □

Theorem 1. *For every nonempty compact TS* (Y, τ_1), *nonempty* T_1*-space* (Z, τ_2), $f \in Y^E$, $g \in C_{\rm cl}(Y, \tau_1, Z, \tau_2)$, *and* $\mathcal{E} \in \mathcal{P}'(\mathcal{P}(E))$, *the equality* (3.11) *is true.*

The corresponding proof is carried out by the immediate combination of (2.3), (2.4), and Proposition 9.

In connection with Theorem 1, it is useful to note that, for every nonempty sets Y and Z, $\tau_1 \in (\mathrm{top})[Y]$, $\tau_2 \in (\mathrm{top})[Z]$, $f \in Y^E$, $g \in C_{\mathrm{cl}}(Y, \tau_1, Z, \tau_2)$, and $A \in \mathcal{P}(E)$

$$\mathrm{cl}((g \circ f)^1(A), \tau_2) = g^1(\mathrm{cl}(f^1(A), \tau_1)).$$

Now, we recall the property noted in [9, Proposition 5.2.1]. Moreover, we can consider this property as a simple corollary of Theorem 1. Namely, if (Y, τ_1) is a nonempty compact TS, (Z, τ_2) is a nonempty T_2-space, $f \in Y^E$, and $g \in C(Y, \tau_1, Z, \tau_2)$, then (3.11) is true. Indeed, in the case at hand, $g \in C_{\mathrm{cl}}(Y, \tau_1, Z, \tau_2)$ automatically (see [6, 3.1.12]). Of course, in the above-mentioned case, (Y, τ, f, g) is a compactifier. It is useful to compare Theorem 1 and Proposition 8.

Proposition 10. *Let (Y, τ_1) and (Z, τ_2) be two nonempty T_2-space, $g \in \mathbf{F}^0_{\mathbf{c}}[E; Y; \tau_1]$, $h \in C(Y, \tau_1, Z, \tau_2)$ and $\mathcal{E} \in \mathcal{P}'(\mathcal{P}(E))$. Then*

$$(\mathrm{as})[E; Z; \tau_2; h \circ g; \mathcal{E}] = h^1((\mathrm{as})[E; Y; \tau_1; g; \mathcal{E}]). \tag{3.15}$$

P r o o f. We use (2.9). Let $(M, \theta, \mu, \lambda)$ be a compactifier: (M, θ) is a compact TS, $M \neq \emptyset$, $\mu \in M^E$, $\lambda \in C(M, \theta, Y, \tau_1)$, and $g = \lambda \circ \mu$. Then, from the above-mentioned corollary of Theorem 1, we obtain that

$$(\mathrm{as})[E; Y; \tau_1; g; \mathcal{E}] = \lambda^1((\mathrm{as})[E; M; \theta; \mu; \mathcal{E}]). \tag{3.16}$$

In addition, $h \circ \lambda \in C(M, \theta, Z, \tau_2)$ and

$$(h \circ \lambda) \circ \mu = h \circ (\lambda \circ \mu) = h \circ g = h \circ \lambda \circ \mu.$$

Then $(M, \theta, \mu, h \circ \lambda)$ is a compactifier with respect to E, (Z, τ_2), and $h \circ g : (M, \theta)$ is a nonempty compact space, $\mu \in M^E$, $h \circ \lambda \in C(M, \theta, Z, \tau_2)$, and $h \circ g = (h \circ \lambda) \circ \mu$. We use the above-mentioned corollary of Theorem 1 again:

$$(\mathrm{as})[E; Z; \tau_2; h \circ g; \mathcal{E}] = (h \circ \lambda)^1((\mathrm{as})[E; M; \theta; \mu; \mathcal{E}]) = h^1(\lambda^1((\mathrm{as})[E; M; \theta; \mu; \mathcal{E}])).$$

Using (3.16), we obtain the required equality (3.15). □

§ 4. Example of the problem with an empty attraction set

In this brief section, we consider a variant of our problem (in asymptotic setting) for which AS is the empty set. Namely, we consider the following simplest controlled differential equation:

$$\dot{x}(t) = u(t), \quad x(0) = 0, \quad 0 \leq t \leq 1. \tag{4.1}$$

In (4.1), $u = u(\cdot)$ is a nonnegative real-valued function on $[0, 1]$; moreover, for simplicity, we suppose that u is piecewise constant, continuous to the right on $[0, 1[$ and continuous to the left at the point 1. By $\mathbb{U}$ we denote the set of all functions of the above-mentioned type. So, elements of $\mathbb{U}$ are nonnegative relay functions on $[0, 1]$ and only they. Under $u \in \mathbb{U}$, by $\mathbf{x}_u$ we denote the trajectory of the system (4.1): $\mathbf{x}_u : [0, 1] \longrightarrow \mathbb{R}$ and

$$\mathbf{x}_u(t) \triangleq \int_0^t u(\xi)\, d\xi \quad \forall t \in [0, 1]. \tag{4.2}$$

Introduce the following phase constraints: $u \in \mathbb{U}$ must satisfy the conditions

$$\mathbf{x}_u(t) \geq \frac{1}{t} \;\; \forall t \in]0, 1]. \tag{4.3}$$

Of course, controls $u \in \mathbb{U}$ with the validity of above-mentioned phase constraints do not exist. But, for $\theta \in]0, 1]$, there exists $u_\theta \in \mathbb{U}$ for which

$$\mathbf{x}_u(t) \geq \frac{1}{t} \quad \forall t \in [\theta, 1]. \tag{4.4}$$

We form constraints of asymptotic nature by conditions (4.4) under enumeration of $\theta \in]0, 1]$. In addition, in this example, we suppose that $E = \mathbb{U}$. Moreover, suppose that

$$\mathbb{U}_\theta \triangleq \{u \in \mathbb{U} \,|\, \mathbf{x}_u(t) \geq \frac{1}{t} \quad \forall t \in [\theta, 1]\} \;\; \forall \theta \in]0, 1].$$

Then, in this section, we define $\mathcal{E} \triangleq \{\mathbb{U}_\theta : \; \theta \in]0, 1]\}$. We obtain constraints of asymptotic nature.

We consider the setting for which realization of trajectories and limits of trajectories is important. For brevity, suppose that $I \triangleq [0, 1]$. So, the space of results is $\mathbb{R}^I$ equipped with the topology $\otimes^I(\tau_\mathbb{R})$ of pointwise convergence, where $\tau_\mathbb{R}$ is the usual $|\cdot|$-topology of the real line $\mathbb{R}$. Of course, TS $(\mathbb{R}^I, \otimes^I(\tau_\mathbb{R}))$ is Tychonoff power of $(\mathbb{R}, \tau_\mathbb{R})$ with index set I. In addition, $\mathbf{x}_u \in \mathbb{R}^I$ for $u \in E$. We suppose that the mapping

$$\mathbf{f} : \; E \longrightarrow \mathbb{R}^I$$

is defined by the rule: $\mathbf{f}(u) \triangleq \mathbf{x}_u \; \forall u \in E$. We consider the set $(\mathrm{as})[E; \mathbb{R}^I; \otimes^I(\tau_\mathbb{R}); \mathbf{f}; \mathcal{E}]$ which is a subset of $\mathbb{R}^I$.

We show that $(\mathrm{as})[E; \mathbb{R}^I; \otimes^I(\tau_\mathbb{R}); \mathbf{f}; \mathcal{E}] = \emptyset$. Indeed, let $(\mathrm{as})[E; \mathbb{R}^I; \otimes^I(\tau_\mathbb{R}); \mathbf{f}; \mathcal{E}] \neq \emptyset$. We choose an element $\mathbf{x}^0 \in (\mathrm{as})[E; \mathbb{R}^I; \otimes^I(\tau_\mathbb{R}); \mathbf{f}; \mathcal{E}]$. So, in particular $\mathbf{x}^0 \in \mathbb{R}^I$. By (2.1), for a net $(\mathbf{D}, \sqsubseteq, g)$ in $E = \mathbb{U}$,

$$(\mathcal{E} \subset (E - \mathrm{ass})[\mathbf{D}; \sqsubseteq; g]) \,\&\, ((\mathbf{D}, \sqsubseteq, \mathbf{f} \circ g) \overset{\otimes^I(\tau_\mathbb{R})}{\longrightarrow} \mathbf{x}^0).$$

In addition, $\mathbf{D}$ is a nonempty set, $\sqsubseteq \in (\mathrm{DIR})[\mathbf{D}]$, and $g \in E^\mathbf{D}$. We note that $\mathbf{x}_u(t) \geq 0 \; \forall u \in E \; \forall t \in I$. Therefore, $\mathbf{f}(u)(t) \geq 0 \; \forall u \in E \; \forall t \in I$. As a corollary, $\mathbf{x}^0(t) \geq 0 \; \forall t \in I$. Now, we fix $t^* \in]0, 1]$. Then,

$$t_* \triangleq \frac{t^*}{2 + t^* \mathbf{x}^0(t^*)} \in]0, 1].$$

Recall that $\mathbb{U}_{t_*} \in \mathcal{E}$ is defined. As a corollary, $\mathbb{U}_{t_*} \in (E - \mathrm{ass})[\mathbf{D}; \sqsubseteq; g]$. Using (1.5), we choose $\mathbf{d} \in \mathbf{D}$ for which $\forall \delta \in \mathbf{D}$

$$(\mathbf{d} \sqsubseteq \delta) \Longrightarrow (g(\delta) \in \mathbb{U}_{t_*}).$$

Let $\delta_* \in \mathbf{D}$ and $\mathbf{d} \sqsubseteq \delta_*$. Then, $g(\delta_*) \in \mathbb{U}_{t_*}$. In particular, $g(\delta_*) \in \mathbb{U}$. In addition, $\mathbf{x}_{g(\delta_*)}(t) \geq \frac{1}{t}$ $\forall t \in [t_*, 1]$. In particular, $\mathbf{x}_{g(\delta_*)}(t_*) \geq \frac{1}{t_*}$. By definition of t_* we obtain

$$\mathbf{x}_{g(\delta_*)}(t_*) \geq \frac{2}{t^*} + \mathbf{x}^0(t^*).$$

By (4.2) $\mathbf{x}_{g(\delta_*)}(t_*) \leq \mathbf{x}_{g(\delta_*)}(t^*)$ (indeed, $g(\delta_*)$ is a nonnegative function). Therefore,

$$\mathbf{x}_{g(\delta_*)}(t^*) \geq \frac{2}{t^*} + \mathbf{x}^0(t^*).$$

Since the choice of δ_* was arbitrary, it is established that $\forall\delta \in \mathbf{D}$

$$(\mathbf{d} \sqsubseteq \delta) \Longrightarrow (\mathbf{x}_{g(\delta)}(t^*) \geq \frac{2}{t^*} + \mathbf{x}^0(t^*)).$$

But by the choice of $(\mathbf{D}, \sqsubseteq, g)$ we find (in particular) that, for some $\hat{d} \in \mathbf{D}$

$$|(\mathbf{f} \circ g)(\delta)(t^*) - \mathbf{x}^0(t^*)| < \frac{2}{t^*}$$

under $\delta \in \mathbf{D}$ with the property $\hat{d} \sqsubseteq \delta$ (indeed, $\frac{2}{t^*} \in]0, \infty[$ and the set

$$\{h \in \mathbb{R}^I |\ |h(t^*) - \mathbf{x}^0(t^*)| < \frac{2}{t^*}\}$$

is an open neighborhood of $\mathbf{x}^0$). Using the definition of directed set, we choose $\hat{\delta} \in \mathbf{D}$ such that $\mathbf{d} \sqsubseteq \hat{\delta}$ and $\hat{d} \sqsubseteq \hat{\delta}$. Then,

$$\mathbf{x}_{g(\hat{\delta})}(t^*) \geq \frac{2}{t^*} + \mathbf{x}^0(t^*).$$

On the other hand, the following inequality holds:

$$(\mathbf{f} \circ g)(\hat{\delta})(t^*) - \mathbf{x}^0(t^*) < \frac{2}{t^*}.$$

Since $(\mathbf{f} \circ g)(\hat{\delta})(t^*) = \mathbf{f}(g(\hat{\delta}))(t^*) = \mathbf{x}_{g(\hat{\delta})}(t^*)$, we obtain the obvious contradiction. As a result, the required property $(\mathrm{as})[E; \mathbb{R}^I; \otimes^I(\tau_{\mathbb{R}}); \mathbf{f}; \mathcal{E}] = \emptyset$ is established. So, under compatibility of all weakened constraints, we obtain the case of empty attraction set.

§ 5. Conclusion

In this article, some properties of AS have been considered. In particular, these AS can arise as a result of the weakening of standard constraints in control problems. But it is possible that the corresponding AS is the empty set (in the previous section, such example was considered) although, for every concrete weakening of constraints, the compatibility condition takes place. In this connection, we have investigated conditions for which AS is a nonempty set. Of course, these conditions are connected with the compactness property for solution space of the corresponding generalized problem. In this connection, the question about the compactifier existence is important. We note Rfs. [10–12] in which some concrete versions of attainability problems with constraint of asymptotic nature were considered.

Funding. Research was funded by the Russian Foundation for Basic Research, project number 19–01–00573.

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Received 28.08.2019

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Citation: A. G. Chentsov, E. G. Pytkeev. Constraints of asymptotic nature and attainability problems, *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki*, 2019, vol. 29, issue 4, pp. 569–582.

ВЕСТНИК УДМУРТСКОГО УНИВЕРСИТЕТА. МАТЕМАТИКА. МЕХАНИКА. КОМПЬЮТЕРНЫЕ НАУКИ
МАТЕМАТИКА 2019. Т. 29. Вып. 4. С. 569–582.

А. Г. Ченцов, Е. Г. Пыткеев

Ограничения асимптотического характера и задачи достижимости

Ключевые слова: множество притяжения, расширение, топологическое пространство, компактность.

УДК 517.9

DOI: 10.20537/vm190408

В задачах управления построение и исследование областей достижимости и их аналогов очень важно. Эта статья адресована задачам о достижимости в топологических пространствах. Используются ограничения асимптотической природы, определяемые в виде непустых семейств множеств. Решение соответствующей задачи о достижимости определяется как множество притяжения. Точки этого множества притяжения (элементы притяжения) реализуются в классе приближенных решений, которые являются несеквенциальными аналогами приближенных решений Варги. Обсуждаются некоторые возможности применяемых компактификаторов. Рассматриваются вопросы реализации множеств притяжения с точностью до заданной окрестности. Исследуются некоторые топологические свойства множеств притяжения. Рассмотрен пример с пустым множеством притяжения.

Финансирование. Исследования выполнены при финансовой поддержке РФФИ в рамках научного проекта 19–01–00573.

Поступила в редакцию 28.08.2019

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Цитирование: А. Г. Ченцов, Е. Г. Пыткеев. Ограничения асимптотического характера и задачи достижимости // Вестник Удмуртского университета. Математика. Механика. Компьютерные науки. 2019. Т. 29. Вып. 4. С. 569–582.