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ON VOLUMES OF MATRIX BALL OF THIRD TYPE AND GENERALIZED LIE BALLS

The third-type matrix ball and the generalized Lie ball that are connected with classical domains play a crucial role in the theory of several complex variable functions. In this paper the volumes of the third type matrix ball and the generalized Lie ball are calculated. The full volumes of these domains are necessary for finding kernels of integral formulas for these domains (kernels of Bergman, Cauchy–Szegö, Poisson etc.). In addition, it is used for the integral representation of a function holomorphic on these domains, in the mean value theorem and other important concepts. The results obtained in this article are the general case of results of Hua Lo-ken and his results in particular cases coincides with our results.

Keywords: classical domains, matrix ball of the first type, matrix balls of the second type, matrix balls of the third type, generalized Lie ball.

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§ 1. Introduction

In 1935 E. Cartan proved that there are only six possible types of irreducibly transitive restricted symmetric domains (see [1–3]) and four of them are defined as follows:

$$\Re_I = \left\{ Z \in \mathbb{C}\,[m \times k] : I^{(m)} - Z\overline{Z}' > 0 \right\},$$

$$\Re_{II} = \left\{ Z \in \mathbb{C}\,[m \times m] : I^{(m)} - Z\overline{Z} > 0, \quad \forall Z' = Z \right\},$$

$$\Re_{III} = \left\{ Z \in \mathbb{C}\,[m \times m] : I^{(m)} + Z\overline{Z} > 0, \quad \forall Z' = -Z \right\},$$

$$\Re_{IV} = \left\{ z \in \mathbb{C}^n : |zz'|^2 + 1 - 2\overline{z}z' > 0, \quad |zz'| < 1 \right\}.$$

Here $I^{(m)}$ is the identity matrix of order m, $\overline{Z}'$ is the complex conjugate of transposed matrix Z' (for an Hermitian matrix H, it is assumed, as usual, that H is positive definite: $H > 0$).

Dimensions of these domains are equal respectively

$$mk,\ m(m+1)/2,\ m(m-1)/2,\ n.$$

Cartan's classical domains $\Re_V$, $\Re_{VI}$ in $\mathbb{C}^{16}$, $\mathbb{C}^{27}$, respectively, has very special cases. The question of an efficient description of these two domains still remains open.

The first, second, and third types of matrix balls were produced according to these classical domains [4].

Let $\mathbb{C}\,[m \times m]$ be a space of $[m \times m]$ matrices with complex elements. The direct product of n instances $\mathbb{C}\,[m \times m]$ is denoted by $\mathbb{C}^n\,[m \times m]$.

The domain

$$B^{(1)}_{m,n} = \{(Z_1, \dots, Z_n) = Z \in \mathbb{C}^n\,[m \times m] : \ I - \langle Z, Z \rangle > 0\},$$

is called the matrix ball (of the first type), where $\langle Z, Z \rangle = Z_1 Z_1^* + \dots + Z_n Z_n^*$ is the «scalar multiplication», I is the unit $[m \times m]$ matrix, $Z_\nu^* = \overline{Z}'_\nu$ is the conjugate matrix transposed to Z_ν,

$\nu = 1, 2, \ldots, n$. Here $I - \langle Z, Z \rangle > 0$ means that the Hermitian matrix is positively definite, i.e., all eigenvalues are positive.

We define the matrix balls of the second $B^{(2)}_{m,n}$ and third types $B^{(3)}_{m,n}$, respectively:

$$B^{(2)}_{m,n} = \{(Z_1, \ldots, Z_n) = Z \in \mathbb{C}^n [m \times m] : I - \langle Z, Z \rangle > 0, \quad \forall Z'_\nu = Z_\nu, \ \nu = 1, 2, \ldots, n\}$$

and

$$B^{(3)}_{m,n} = \{(Z_1, \ldots, Z_n) = Z \in \mathbb{C}^n [m \times m] : I + \langle Z, Z \rangle > 0, \forall Z'_\nu = -Z_\nu, \quad \nu = 1, 2, \ldots, n\}.$$

We denote the skeletons (Shilov boundaries) of the matrix balls $B^{(k)}_{m,n}$ by $X^{(k)}_{m,n}$, $k = 1, 2, 3$, i.e.,

$$X^{(1)}_{m,n} = \{Z \in \mathbb{C}^n [m \times m] : \langle Z, Z \rangle = I\},$$

$$X^{(2)}_{m,n} = \{Z \in \mathbb{C}^n [m \times m] : \langle Z, Z \rangle = I, \quad Z'_\nu = Z_\nu, \ \nu = 1, 2, \ldots, n\},$$

$$X^{(3)}_{m,n} = \left\{Z \in \mathbb{C}^n [m \times m] : I + \langle Z, Z \rangle = 0, \ Z'_\nu = -Z_\nu, \quad \nu = 1, 2, \ldots, n\right\}.$$

If $n = 1$, $m > 1$, then $B^{(k)}_{m,1}$, $k = 1, 2, 3$, are the classical domains of the first, second and third types (according to E. Cartan's classification), the cores $X^{(1)}_{m,1}$, $X^{(2)}_{m,1}$, and $X^{(3)}_{m,1}$ are the unitary, symmetric unitary and skew-symmetric unitary matrices, respectively [1].

The first type of matrix ball was considered by G. Khudayberganov, A. Khalknazarov [5]. In [6], the volume of a matrix ball of the first type and its skeleton was considered.

The properties of the second type matrix ball were studied by G. Khudayberganov and Z. Matyakubov [9]. The third type of matrix ball was studied by G. Khudayberganov, U. Rakhmonov and integral formulas were found [7]. For these integral formulas, the volume of the third-type matrix ball was used as a finite constant. In this work we consider the volume of the third-type matrix ball.

The results on fourth-type classical domains (Lie ball) were obtained by B. A. Shaimkulov [8]. Moreover, in this article

$$\Re^n_{IV}(r) = \left\{z \in \mathbb{C}^n : |zz'|^2 - 2r^2 \overline{z} z' + r^4 > 0, \quad |zz'| < r^2\right\}$$

is calculated as the volume of the generalized Lie ball.

The full volumes of these domains are necessary for finding kernels of integral formulas for these domains (kernels of Bergman, Cauchy–Szegö, Poisson etc.) [10–12]. In addition, they are used for the integral representation of a function holomorphic on these domains, in the mean value theorem and other important concepts.

The volume of the matrix ball of the second type is calculated using the following theorem.

Theorem 1. *Let Z_ν be an $[m \times m]$ symmetric matrix and $m \geq 2$. Set*

$$J(\lambda) = \int\limits_{I - \langle Z, Z \rangle > 0} [\det(I - \langle Z, Z \rangle)]^\lambda \dot{Z},$$

where $\dot{Z} = \prod\limits_{i=1}^{m} \prod\limits_{j=1}^{mn} dx_{ij} dy_{ij}, x_{ij} + i y_{ij} = z_{ij}$. Then

$$J(\lambda) = \frac{\pi^{\frac{m(m+1)}{2} n}}{(\lambda + 1) \cdot \ldots \cdot (\lambda + mn)} \cdot \frac{\Gamma(2\lambda + 3)\Gamma(2\lambda + 5) \cdot \ldots \cdot \Gamma(2\lambda + 2mn - 1)}{\Gamma(2\lambda + mn + 2)\Gamma(2\lambda + mn + 3) \cdot \ldots \cdot \Gamma(2\lambda + 2mn)}.$$

In particular, when $\lambda = 0$, the volume of the matrix ball of the second type is

$$V(B^{(2)}_{m,n}) = \frac{\pi^{\frac{m(m+1)}{2} n}}{m!} \cdot \frac{2!4! \ldots (2mn - 3)!}{(mn + 1)!(mn + 2)! \ldots (2mn - 1)!}. \quad (1)$$

From the result (1), in particular, when $n = 1$, we obtain the well-known formulas for finding the total volume of classical domain of the second type [1].

§ 2. Proofs of the main results

The volume of the matrix ball of the third type is calculated using the following theorem.

Theorem 2. *Let Z_ν be a $[m \times m]$ skew-symmetric matrix and $m \geq 2$. Set*

$$J(\lambda) = \int\limits_{I+\langle Z,Z\rangle>0} [\det(I + \langle Z, Z\rangle)]^\lambda \dot{Z},$$

where $\dot{Z} = \prod_{i=1}^{m}\prod_{j=1}^{mn} dx_{ij}dy_{ij}, x_{ij} + iy_{ij} = z_{ij}$. Then

$$J(\lambda) = \pi^{\frac{m(m-1)}{2}n} \cdot \frac{\Gamma(2\lambda+1)\Gamma(2\lambda+3)\ldots\Gamma(2\lambda+2mn-3)}{\Gamma(2\lambda+mn)\Gamma(2\lambda+mn+1)\ldots\Gamma(2\lambda+2mn-2)}.$$

In particular, when $\lambda = 0$, the volume of the matrix ball of the third type is

$$V(B^{(3)}_{m,n}) = \pi^{\frac{m(m-1)}{2}n} \cdot \frac{2!4!\ldots(2mn-4)!}{(mn-1)!(mn)!\ldots(2mn-3)!}. \tag{2}$$

P r o o f. Let $Z = (Z_1, \ldots, Z_n)$ be the matrix in the form

$$Z = \begin{pmatrix} Z_{m-1,mn-1} & -u' \\ u & 0 \end{pmatrix},$$

where Z_1 is the skew-symmetric $[(m-1)\times(mn-1)]$-order matrix, and u is the $(m-1)$-dimensional vector. Then

$$I + \langle Z, Z\rangle = \begin{pmatrix} I + Z_{m-1,mn-1}\overline{Z}_{m-1,mn-1} - u'\overline{u} & -Z_{m-1,mn-1}u' \\ u\overline{Z}_{m-1,mn-1} & 1 - u\overline{u}' \end{pmatrix}.$$

The condition $I + \langle Z, Z\rangle > 0$ is equivalent to the following two conditions [13, 14]

$$I + Z_{m-1,mn-1}\overline{Z}_{m-1,mn-1} - u'\overline{u} > 0, \tag{3}$$

$$1 - u\overline{u}' + u\overline{Z}_{m-1,mn-1}(I + Z_{m-1,mn-1}\overline{Z}_{m-1,mn-1} - u'\overline{u})^{-1}Z_{m-1,mn-1}\overline{u}' > 0. \tag{4}$$

Moreover,

$$\det(I + \langle Z, Z\rangle) =$$
$$= (1 - u\overline{u}' + u\overline{Z}_{m-1,mn-1}(I + Z_{m-1,mn-1}\overline{Z}_{m-1,mn-1} - u'\overline{u})^{-1}Z_{m-1,mn-1}\overline{u}')\times$$
$$\times \det(I + Z_{m-1,mn-1}\overline{Z}_{m-1,mn-1} - u'\overline{u}).$$

Let the matrix Γ satisfy the relation $I + Z_{m-1,mn-1}\overline{Z}_{m-1,mn-1} = \Gamma\overline{\Gamma}$. Make a substitution $u = \nu\Gamma'$, where ν is the $(m-1)$-dimensional vector. From inequality (3), it follows that $I + Z_{m-1,mn-1}\overline{Z}_{m-1,mn-1} > 0$ and $1 - u\overline{u}' > 0$. The left-hand side of inequality (4) transforms to

$$1 - \nu\Gamma'\overline{\Gamma}\overline{\nu}' + \nu\Gamma'\overline{Z}_{m-1,mn-1}\overline{\Gamma}'^{-1}(I - \nu'\overline{\nu})^{-1}\Gamma^{-1}Z_{m-1,mn-1}\overline{\Gamma}\overline{\nu}' =$$

$$= 1 - \nu\Gamma'\overline{\Gamma}\overline{\nu}' + \nu\Gamma'\overline{Z}_{m-1,mn-1}\overline{\Gamma}'^{-1}\Gamma^{-1}Z_{m-1,mn-1}\overline{\Gamma}\overline{\nu}' - \frac{|\nu\Gamma'\overline{Z}_{m-1,mn-1}\overline{\Gamma}'^{-1}\overline{\nu}'|^2}{1 - \overline{\nu}\nu'} =$$

$$= 1 - \nu\overline{\nu}' - \frac{|\nu\Gamma'\overline{Z}_{m-1,mn-1}\overline{\Gamma}'^{-1}\overline{\nu}'|^2}{1-\overline{\nu}\nu'} = 1 - \nu\overline{\nu}'.$$

Here we have used the fact that the matrix $\Gamma'\overline{Z}_{m-1,mn-1}\overline{\Gamma}'^{-1}$ is skew-symmetric. Consequently,

$$J_m(\lambda) = \int\limits_{I+Z_{m-1,mn-1},\overline{Z}_{m-1,mn-1}>0} [\det(I + Z_{m-1,mn-1}, \overline{Z}_{m-1,mn-1}]^{\lambda+1}\dot{Z}_{m-1,mn-1}\times$$

$$\times \int\limits_{1-\nu'\overline{\nu}>0} (1-\nu'\overline{\nu})^{2\lambda}\dot{\nu}.$$

Using the well-known formula [1]

$$\int\limits_{x_1^2+x_2^2+\ldots+x_{2m}^2<1} (1 - x_1^2 - x_2^2 - \ldots - x_{2m}^2)^{\mu-1}\, dx_1 \ldots dx_{2m} = \pi^m \frac{\Gamma(\mu)}{\Gamma(\mu+m)} \quad (\mu > 0),$$

we get

$$J_m(\lambda) = \pi^{n(m-1)}\frac{\Gamma(2\lambda+1)}{\Gamma(2\lambda+mn)} J_{m-1}(\lambda+1).$$

Continuing this process, we get

$$J_2(\lambda+mn-2) = \iint\limits_{|z|<1}(1-|z|^2)^{2\lambda+2mn-4} = \pi^n\frac{\Gamma(2\lambda+2mn-3)}{\Gamma(2\lambda+2mn-2)}.$$

This implies that

$$V(B^{(3)}_{m,n}) = \pi^{\frac{m(m-1)}{2}n}\frac{2!4!\ldots(2mn-4)!}{(mn-1)!(mn)!\ldots(2mn-3)!}, \quad \text{if} \quad \lambda = 0.$$

It gives us the volume of the matrix ball $B^{(3)}_{m,n}$. The theorem is proved. □

From the result (2), in particular, when $n = 1$, we obtain the well-known formulas for finding the total volume of the classical domain of the third type [1].

§ 3. The volume of the generalization of the Lie ball

Consider an n-dimensional complex $\mathbb{C}^n$ space whose points are ordered sets of n complex numbers $z = (z_1, z_2, \ldots, z_n)$.

The domain $\Re_{IV}$ (the Lie-ball) consists of all n-dimensional complex vectors z satisfying the conditions

$$\Re_{IV} = \Big\{z \in \mathbb{C}^n : |zz'|^2 - 2\overline{z}z' + 1 > 0, \quad |zz'| < 1\Big\}.$$

where z' is the transposed vector z

These regions are the classical domains of the fourth type (according to E. Cartan's classification [1]) or the Lie ball. The skeleton of the domain of $\Re_{IV}$ is defined as follows:

$$\aleph_{IV} = \Big\{z \in \mathbb{C}^n : |zz'|^2 - 2\overline{z}z' + 1 = 0, \quad |zz'| = 1\Big\}.$$

Generalizations of the Lie ball of radius r are defined as an image of $\Re_{IV}$ under homothetic $z \to \frac{1}{r}z$, i.e.,

$$\Re^n_{IV}(r) = \Big\{z \in \mathbb{C}^n : |zz'|^2 - 2r^2\overline{z}z' + r^4 > 0, \quad |zz'| < r^2\Big\}.$$

The latter relation shows that these two inequalities can be replaced by one (see [1]):

$$r^2 - \overline{z}z' > \sqrt{(\overline{z}z') - |zz'|^2}. \tag{5}$$

Theorem 3. *The volume of a Lie ball $\Re_{IV}^n(r)$ is equal to*

$$V(\Re_{IV}^n(r)) = \frac{\pi^n r^{2n}}{2^{n-1} n!}.$$

First we will prove the following lemma.

Lemma 1. For $\alpha > -1$ and $\beta > -(n+\alpha)$ the following relation holds:

$$I_n = \int\limits_{\Re_{IV}^n(r)} \left(r^2 - \overline{z}z' - \sqrt{(\overline{z}z')^2 - |zz'|^2}\right)^\alpha \left(r^2 - \overline{z}z' + \sqrt{(\overline{z}z')^2 - |zz'|^2}\right)^\beta \dot{z} =$$

$$= \frac{\pi^n r^{2(\alpha+\beta+n)}}{2^{n-1}(\alpha+\beta+n)(n-1)!}.$$

P r o o f. For $n = 1$ we set $z = x + iy$, where x and y are real numbers. Then

$$\overline{z}z' = |zz'| = x^2 + y^2$$

and therefore, $\Re_{IV}^n(r)$ is the disc in the complex plane.

Consequently,

$$I_1 = \iint\limits_{x^2+y^2<r^2} \left(1 - x^2 - y^2\right)^{\alpha+\beta} dx\, dy = \frac{\pi r^2}{\alpha+\beta+1}.$$

So, for $n = 1$ the lemma is proved.

For $n \geq 2$ we set $z = x + iy$, where x and y are real vectors. Inequality (5) has the form

$$r^2 - xx' - yy' > 2\sqrt{xx'yy' - (xy')^2}. \tag{6}$$

Hence,

$$I_n = \int\limits_{x,y} \left(r^2 - xx' - yy' - 2\sqrt{xx'yy' - (xy')^2}\right)^\alpha \left(r^2 - xx' - yy' + 2\sqrt{xx'yy' - (xy')^2}\right)^\beta \dot{x}\dot{y},$$

where the integral is taken on the domain described by inequality (6). For every fixed x we can find an orthogonal matrix H with determinant 1 such that

$$xH = \left(\sqrt{xx'}, 0, \dots, 0\right).$$

We set $yH = (\xi, w)$, where ξ is some real number, and w is an $(n-1)$-dimensional vector. Using the last substitution, inequality (6) can be written in the form:

$$r^2 - xx' - \xi^2 - ww' > 2\sqrt{xx'(\xi^2 + ww') - xx'\xi^2} = 2\sqrt{xx'ww'}. \tag{7}$$

Consequently, after the substitution, the integral transforms to

$$I_n = \int\limits_{\xi,w,x} \left(r^2 - \xi^2 - xx' - ww' - 2\sqrt{xx'ww'}\right)^\alpha \left(r^2 - \xi^2 - xx' - ww' + 2\sqrt{xx'ww'}\right)^\beta d\xi \dot{w}\dot{x},$$

where the integral is taken by the domain described by inequality (7). Furthermore, making the substitutions

$$x = u\sqrt{r^2 - \xi^2}, \quad w = v\sqrt{r^2 - \xi^2},$$

we obtain

$$I_n = \int_{-r}^{r} \left(r^2 - \xi^2\right)^{\alpha+\beta+n-\frac{1}{2}} d\xi \times$$

$$\times \int\limits_{1-uu'-vv'>2\sqrt{uu'vv'}} \left(1 - uu' - vv' - 2\sqrt{uu'vv'}\right)^{\alpha} \left(1 - uu' - vv' + 2\sqrt{uu'vv'}\right)^{\beta} \dot{u}\dot{v} =$$

$$= \frac{\pi^{\frac{1}{2}} r^{2(\alpha+\beta+n)} 2^{2n-1} \Gamma\left(\alpha+\beta+n+\frac{1}{2}\right)}{\Gamma\left(\alpha+\beta+n+1\right)} \cdot J, \tag{8}$$

where

$$J = \int\limits_{\substack{1-uu'-vv'>2\sqrt{uu'vv'} \\ u_\mu \geq 0,\, v_\gamma \geq 0}} \left(1 - uu' - vv' - 2\sqrt{uu'vv'}\right)^{\alpha} \left(1 - uu' - vv' + 2\sqrt{uu'vv'}\right)^{\beta} \dot{u}\dot{v}.$$

Now we set $\eta^2 = uu'$, $\zeta^2 = vv'$. Then we obtain

$$J = \iint\limits_{\substack{\eta+\zeta<1 \\ \eta\geq 0,\, \zeta\geq 0}} \left(1 - (\eta+\zeta)^2\right)^{\alpha} \left(1 - (\eta-\zeta)^2\right)^{\beta} d\eta\, d\zeta \times$$

$$\times \int\limits_{\substack{u_2^2+\ldots+u_n^2\leq\eta^2 \\ u_\mu\geq 0}} \cdots \int \frac{\eta\, du_2 \ldots du_n}{\sqrt{\eta^2 - u_2^2 - \ldots - u_n^2}} \times$$

$$\times \int\limits_{\substack{v_2^2+\ldots+v_n^2\leq\zeta^2 \\ v_\nu\geq 0}} \cdots \int \frac{\zeta\, dv_3 \ldots dv_n}{\sqrt{\zeta^2 - v_3^2 - \ldots - v_n^2}}. \tag{9}$$

Let us use the fact that

$$\int\limits_{\substack{u_2^2+\ldots+u_n^2\leq\eta^2 \\ u_\mu\geq 0}} \cdots \int \frac{\eta\, du_2 \ldots du_n}{\sqrt{\eta^2 - u_2^2 - \ldots - u_n^2}} =$$

$$= \eta^{n-1} \int\limits_{\substack{u_2^2+\ldots+u_n^2\leq 1 \\ u_\mu\geq 0}} \cdots \int \frac{du_2 \ldots du_n}{\sqrt{1 - u_2^2 - \ldots - u_n^2}} = \eta^{n-1} \frac{\pi^{\frac{n}{2}}}{2^{n-1}\Gamma\left(\frac{n}{2}\right)}.$$

From the formula (9) it follows that

$$J = \frac{2^{-(2n-3)}\pi^{n-\frac{1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \iint\limits_{\substack{\eta+\zeta\leq 1 \\ \eta\geq 0,\, \zeta\geq 0}} \left(1 - (\eta+\zeta)^2\right)^{\alpha} \left(1 - (\eta-\zeta)^2\right)^{\beta} \eta^{n-1}\zeta^{n-2} d\eta\, d\zeta =$$

$$= \frac{2^{-(2n-3)}\pi^{n-\frac{1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \iint\limits_{\substack{\eta+\zeta\leq 1 \\ 0\leq\eta\leq\zeta}} \left(1 - (\eta+\zeta)^2\right)^{\alpha} \left(1 - (\eta-\zeta)^2\right)^{\beta} \eta^{n-2}\zeta^{n-2} \left(\eta+\zeta\right) d\eta\, d\zeta.$$

By setting $\zeta - \eta = \tau$, $\zeta + \eta = \sigma$,

$$J = \frac{2^{-(2n-3)}\pi^{n-\frac{1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \iint\limits_{0\le\tau\le\sigma\le1} \left(1-\sigma^2\right)^{\alpha}\left(1-\tau^2\right)^{\beta}\left(\frac{\sigma+\tau}{2}\right)^{n-2}\left(\frac{\sigma-\tau}{2}\right)^{n-2}\sigma\frac{d\sigma\, d\tau}{2} =$$

$$= \frac{2^{-(4n-6)}\pi^{n-\frac{1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \int\limits_0^1 \left(1-\tau^2\right)^{\beta} d\tau \int\limits_{\tau}^1 \left(1-\sigma^2\right)^{\alpha}\left(\sigma^2-\tau^2\right)^{n-2}\sigma\, d\sigma.$$

Making the substitution $\omega = \dfrac{\sigma^2-\tau^2}{1-\tau^2}$, in the inside integral, we have

$$J = \frac{2^{-(4n-5)}\pi^{n-\frac{1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \int\limits_0^1 \left(1-\tau^2\right)^{\alpha+\beta+n-1} d\tau \int\limits_0^1 \left(1-\omega\right)^{\alpha}\omega^{n-2}\, d\omega =$$

$$= \frac{\pi^{n-\frac{1}{2}}2^{-(4n-4)}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \cdot \frac{\pi^{\frac{1}{2}}\Gamma\left(\alpha+\beta+n\right)}{\Gamma\left(\alpha+\beta+n+\frac{1}{2}\right)} \cdot \frac{\Gamma\left(\alpha+1\right)\Gamma\left(n-1\right)}{\Gamma\left(\alpha+n\right)}.$$

Substituting this formula into (8), we obtain

$$I_n = r^{2(\alpha+\beta+n)} \cdot \frac{\pi^{n+\frac{1}{2}}2^{-(4n-3)}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \cdot \frac{\Gamma\left(\alpha+\beta+n\right)}{\Gamma\left(\alpha+\beta+n+1\right)} \cdot \frac{\Gamma\left(\alpha+1\right)\Gamma\left(n-1\right)}{\Gamma\left(\alpha+n\right)} =$$

$$= \frac{\pi^n}{2^{n-1}\left(\alpha+\beta+n\right)\left(n-1\right)!} \cdot r^{2(\alpha+\beta+n)}.$$

The lemma is proved. □

By applying this lemma for $\alpha = \beta = 0$, we obtain the formula for the volume of a Lie ball $\Re_{IV}^n\left(r\right)$

$$V\left(\Re_{IV}^n\left(r\right)\right) = \frac{\pi^n r^{2n}}{2^{n-1}n!}.$$

This proves Theorem 3.

Notice, that for $r = 1$, Theorem 3 completely coincides with Theorem 2.5.1 from [1].

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Rakhmonov Uktam Sodikovich, Senior Researcher, Senior Lecturer, Tashkent State Technical University, ul. Universitetskaya, 2, Tashkent, 100095, Uzbekistan.
E-mail: uktam_rakhmonov@mail.ru

Abdullayev Jonibek Shokirovich, PhD Student, National University of Uzbekistan named after M. Ulugbek, ul. Universitetskaya, 4, Tashkent, 100174, Uzbekistan.
E-mail: jonibek-abdullayev@mail.ru

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У. С. Рахмонов, Ж. Ш. Абдуллаев

Об объемах матричного шара третьего типа и обобщенных шаров Ли

Ключевые слова: классические области, матричный шар первого типа, матричный шар второго типа, матричный шар третьего типа, обобщенный шар Ли.

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Матричный шар третьего типа и обобщенный шар Ли, связанные с классическими областями, играют важную роль в теории функций многих комплексных переменных. В данной работе вычислены объемы матричного шара третьего типа и обобщенного шара Ли. Полные объемы этих областей необходимы для нахождения ядер интегральных формул для этих областей (ядра Бергмана, Коши–Сегё, Пуассона и т. д.). Кроме того, он используется для интегрального представления функции, голоморфной на этих областях, в теореме о среднем значении и других важных понятиях. Результаты, полученные в этой статье, являются общим случаем результатов Хуа Ло-кена, и его результаты в частных случаях совпадают с нашими результатами.

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Рахмонов Уктам Содикович, старший преподаватель, старший научный сотрудник, Ташкентский государственный технический университет, 100095, Узбекистан, г. Ташкент, ул. Университетская, 2.
E-mail: uktam_rakhmonov@mail.ru

Абдуллаев Жонибек Шокирович, докторант (PhD), Национальный университет Узбекистана, 100174, Узбекистан, г. Ташкент, ул. Университетская, 4.
E-mail: jonibek-abdullayev@mail.ru

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