MATHEMATICS 2022. Vol. 32. Issue 3. Pp. 433–446.

MSC2020: 90C05, 90C90, 90-10

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KEEPING ORDER OF VESSELS IN PROBLEM OF SAFE MERGING AIRCRAFT FLOWS

Nowadays, the problem of creating an optimal safe schedule for arrival of aircraft coming in several flows to a checkpoint, where these flows join into one, is very important for air-traffic management. Safety of the resultant queue is present if there is a safe interval between neighbor arrivals to the merge point. Change of an arrival instant of an aircraft is provided by changing its velocity and/or usage of fragments of the air-routes scheme, which elongate or shorten the aircraft path. Optimality of the resultant queue is considered from the point of some additional demands: minimization of the deviation of the actual aircraft arrival instant from the nominal one, minimization of order changes in the resultant queue in comparison with the original one, minimization of fuel expenditures, etc. The optimality criterion to be minimized, which reflects these demands, is often taken as a sum of penalties for deviations of the assigned arrival instants from the nominal ones. Each individual penalty is considered in almost all papers as either the absolute value of the difference between the assigned and nominal arrival instants or a similar function with asymmetric branches (which punishes delays and accelerations of an aircraft in different ways). The problem can be divided into two subproblems: one is a search for an optimal order of aircraft in the resultant queue, and the other is a search for optimal arrival instants for a given order. The second problem is quite simple since it can be formalized in the framework of linear programming and solved quite efficiently. However, the first one is very difficult and now is solved by various methods. The paper suggests sufficient conditions for the problem, which guarantee that the order of the optimal assigned instants is the same as the order of the nominal ones and, therefore, exclude the first subproblem.

Keywords: aircraft, airway merge point, non-conflict flows merging, nominal arrival instants, assigned arrival instants, joined aircraft queue.

DOI: 10.35634/vm220306

Introduction

At the present time, aircraft motion is performed along airways consisting of height echelons and air corridors in horizontal planes. With that, the airways can split or join. At the joining points, a problem of merging aircraft flows into a joint queue appears. This problem is especially actual near airports and in approach zones where the air traffic is very dense.

The main requirement to the merging process is providing a minimal safe time interval between two neighbor aircraft arrivals to the merge point. There are two main tools for changing the arrival instant of an aircraft. The first is the control of the aircraft speed, which allows one to achieve a relatively small acceleration or delay of the aircraft. To achieve longer accelerations or delays, they use the second tool, namely, path alignment schemes or delay schemes.

Often, the problem is formalized as optimizational. The criterion, usually, to be minimized reflects additional demands to the resultant queue such as minimization of fuel expenditure for aircraft maneuvering, minimization of deviation of appointed arrival instants from the nominal ones, minimization of interactions between air-traffic managers and pilots, etc.

With that, the usual form of the functional is a sum over all aircraft of some penalties for deviations of the assigned arrival instants from their nominal values. Individual penalties are chosen as the absolute value of the difference between the assigned and nominal arrival instants or a similar asymmetric function, which punishes delays and accelerations in different ways. The criterion can be deterministic or stochastic. Sometimes, a multi-criteria problem is considered.

In general, the problem of constructing a safe joint queue can be considered as a discrete-continuous one. The discrete subproblem deals with the search for the optimal order of the aircraft in the resultant queue. The continuous one concerns the search of the optimal arrival instants for some given order of the aircraft. If the individual penalties are taken as an "absolute value"-like function, then the continuous subproblem under a fixed order of the aircraft can be reduced to a linear programming problem (despite non-linearity of the functional) and, therefore, can be solved efficiently.

At the same time, the discrete subproblem is, generally speaking, extremely hard. As the set of aircraft becomes relatively large, the order cannot be chosen by means of an exhaustive search. So, one should involve either some heuristic approaches (see, for example, [7, 12] and references within), branch and bound method of some kind, deterministic or stochastic (see, for example, [6, 19] and references within). Also, some versions of the dynamic programming approach can be used (see, for example, [1, 17] and references within). A very detailed review of results on aircraft scheduling problems up to 2011 is set forth in [3]. More or less exhaustive reviews of works made later can be found in [21–23, 25]. Despite the aircraft scheduling problem is considered since the 1970s, till now it is under great attention of many researchers, both theoreticians and practitioners (see, for example, [4, 5, 8–11, 13–16, 18, 24]). And till now, various numerous methods are applied, namely, to overcome the problem of gigantic enumeration arising during search of the optimal aircraft order. The performance becomes extremely important in real-time systems for support of air-traffic management.

So, if there is a possibility to obtain directly the optimal aircraft order in the resultant queue or sufficiently decrease the enumeration to get it, then it would be very helpful for solving the problem. In this paper, we suggest and prove some sufficient conditions for the problem, which provide that the order of optimal assigned arrival instants coincides with the order of the nominal ones.

One of the conditions is that the individual penalty functions are the same for all aircraft and are convex unimodal piecewise-linear. This condition is not too restrictive, since as it is said earlier, the usual form of the penalties is "absolute value"-like functions. However, the second condition demands equality of all characteristics of all aircraft: at first, safety intervals between each pair of aircraft and, at second, values for maximal delay and maximal acceleration. The most unrealistic demand is the latter one because different aircraft move along different routes, which, obviously, have different capabilities for delay and acceleration. Nevertheless, in some situations, these conditions can be fulfilled more or less exactly. For example, if a group of aircraft is considered in terminal maneuvering area, then all delay/alignment schemes are already passed. In this situation, aircraft velocity almost cannot be varied, and the possible change of the arrival instant can be provided by the final point merging scheme only. This scheme is common for all incoming flows and, therefore, allows the same variation for all aircraft.

The paper is organized as follows. Section 1 gives a general formulation of the aircraft flow merge problem as a constrained finite-dimension optimization problem. Section 2 is devoted to theoretical facts about convex piecewise-linear and unimodal functions. They allow one to state that for considered penalty criterion type, the order of aircraft in the resultant queue coincides with the one of the nominal arrival instants. In small Section 3, the original problem is reformulated as a linear programming one. The paper is finalized by a conclusion (Section 4) and a reference list.

§ 1. Problem statement

At the beginning, we are given with a collection $\mathbf{t}^{\mathrm{nom}} = \{t_i^{\mathrm{nom}}\}_{i=1}^N$ of nominal instants of aircraft arrivals to the joining point. The value N is the total number of aircraft in all flows. It is assumed that the collection is sorted in ascending order. With that, there are no safety conflicts between aircraft coming in one flow, but there can be conflicts between aircraft from different flows.

The objective is to obtain a new collection $\mathbf{t} = \{t_i\}_{i=1}^N$ of arrival instants to the joining point. The new collection might not be sorted because the order of aircraft in this collection can, possibly, change in comparison with the original collection.

The obtained set $\mathbf{t}$ must obey the following conditions. At first, for each $t_i \in \mathbf{t}$, it is true that $t_i \in [t_i^{\mathrm{nom}} - t_i^{\mathrm{acc}}, t_i^{\mathrm{nom}} + t_i^{\mathrm{dec}}]$. Here, the values t_i^{acc} and t_i^{dec} show how long the ith aircraft can be accelerated or decelerated according to the velocity change or possible direct routes and delay schemes allocated along the airways. Of course, different airways and different aircraft can have different configurations.

Also, the new collection should obey the safety demands: for all pairs of indices $1 \le i < j \le N$, one should have $|t_i - t_j| \ge \tau_{i,j}^{\mathrm{safe}}$. Here, $\tau_{i,j}^{\mathrm{safe}}$ is the length of the time interval between the ith and jth aircraft, which provides their safe passage.

Some penalty criterion $F(\mathbf{t}, \mathbf{t}^{\mathrm{nom}})$ should be minimized, which describes the optimality of the obtained schedule from the point of view of air-traffic managers and airport services.

Thus, one gets a constrained optimization problem:

$$F(\mathbf{t}, \mathbf{t}^{\mathrm{nom}}) = \sum_{i=1}^{N} f_i(t_i, t_i^{\mathrm{nom}}) \to \min, \tag{1.1}$$

$$\text{s. t. } t_i \in [t_i^{\mathrm{nom}} - t_i^{\mathrm{acc}}, t_i^{\mathrm{nom}} + t_i^{\mathrm{dec}}], \quad t_i \ge 0, \tag{1.2}$$

$$\forall\, 1 \le i < j \le N\colon |t_i - t_j| \ge \tau_{i,j}^{\mathrm{safe}}. \tag{1.3}$$

Here, the criterion F to be minimized consists of the additive penalties f_i for each aircraft.

An essential issue of this formalization is that inequalities (1.3) make the constraint set disconnected. Namely, the enumeration of its connectivity components corresponds to the enumeration of the aircraft order in the resultant queue.

§ 2. Piecewise-linear unimodal functions

At first, assume that the penalty functions f_i are convex piecewise-linear and the same for all aircraft. The first assumption is usual for the formulations of the problem where this common function f is taken as the absolute value of the deviation.

Piecewise-linearity of the function f prevents the problem to be considered as a linear programming one. But there are approaches allowing one to get rid of this non-linearity and to make the functional linear. One of such approaches is set forth below. So, further in the section, we consider unimodal convex piecewise-linear functions.

Definition 1. A continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is called *unimodal*, if there is a value A such that f monotonically decreases on the semi-axis $(-\infty, A]$ and monotonically increases on the semi-axis $[A, +\infty)$. The increasing and decreasing can be non-strict.

Definition 2. A function $f\colon \mathcal{D} \subseteq \mathbb{R}^N \to \mathbb{R}$ is called *convex*, if the domain $\mathcal{D}$ is a convex set and for all $x, y \in \mathcal{D}$ and $\alpha \in [0, 1]$, the following inequality holds:

$$f\big(\alpha \cdot x + (1 - \alpha) \cdot y\big) \le \alpha \cdot f(x) + (1 - \alpha) \cdot f(y).$$

In other words, a function is convex, if its epigraph is a convex set.

Linearization of Convex Continuous Piecewise-Linear Functional

Let points (x_k, y_k), x_k, $y_k \in \mathbb{R}$, $k = \overline{1, m}$, be given such that $x_k < x_{k+1}$, $k = \overline{1, m-1}$. They define some piecewise-linear function in the following way. Each pair of points (x_k, y_k) and (x_{k+1}, y_{k+1}), $k = \overline{1, m-1}$, is connected by a segment. Outside the interval $[x_1, x_m]$, the function grows as an extension of the linear parts on the segments $[x_1, x_2]$ and $[x_{m-1}, x_m]$, respectively. Due to the definition, the function is continuous. Also, it is assumed that the points are such that the function defined in this way is convex (see Fig. 1).

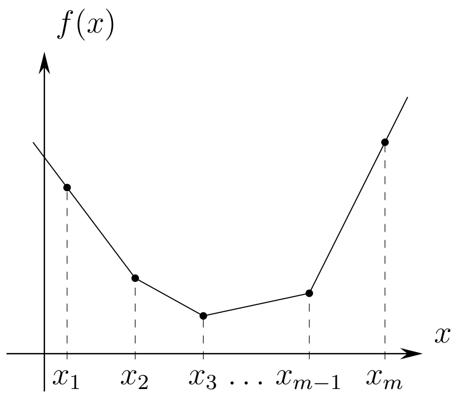

Figure 1: Representation of a convex piecewise-linear function as a set of linear parts

Such a piecewise-linear function can be represented as

$$f(x) = \sum_{k=2}^{m-1} a_k |x - x_k| + Ax + B \tag{2.1}$$

with non-negative coefficients a_k.

The coefficients a_k, A can be found by solving a system of $(m-1)$ equations:

$$\begin{cases} -a_2 - a_3 - \ldots - a_{m-1} + A = K_1, \\ \quad a_2 - a_3 - \ldots - a_{m-1} + A = K_2, \\ a_2 + a_3 - a_4 \ldots - a_{m-1} + A = K_3, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \quad a_2 + a_3 + \ldots + a_{m-1} + A = K_{m-1}. \end{cases}$$

The scalars K_k defining the slope of the corresponding linear parts are found as follows:

$$K_k = \frac{y_{k+1} - y_k}{x_{k+1} - x_k}, \quad k = \overline{1, m-1}.$$

So, each equation of the system ensures the equality of the slope coefficients after opening the absolute values in (2.1) and the required value K_k on each of $(m-1)$ intervals (x_k, x_{k+1}).

The free term B can be found by solving the equation

$$f(x_1) = \sum_{k=2}^{m-1} -a_k(x_1 - x_k) + Ax_1 + B = y_1,$$

which follows from the condition that the entire function passes through the point (x_1, y_1).

If one searches the minimum of function f (2.1), then such a problem can be reformulated as a linear programming one. To do this, additional variables $q_k := |x - x_k|$, $k = \overline{2, m-1}$, are introduced. After the variable change, the following problem is obtained:

$$f(x) = \sum_{k=2}^{m-1} a_k q_k + Ax \to \min \quad \text{s. t.} \; - q_k \le x - x_k \le q_k, \quad k = \overline{2, m-1}.$$

The summand B of the function f can be omitted since it does not affect the minimum point, but only the minimal value of the function.

Another way to formulate a linear programming problem for searching the minimum of a convex piecewise-linear function is presented in [20]. Both our method and the method from [20] work for minimization, not maximization, of a convex piecewise-linear function.

Properties of Convex Unimodal Functions

Lemma 1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a unimodal convex function and have the minimum (possibly, non-strict) at some point M. Then, for all $M \le x < y$ and $\delta > 0$, it is true that*

$$f(x + \delta) - f(x) \le f(y + \delta) - f(y). \tag{2.2}$$

In other words, the growth rate of a unimodal convex function can only increase with moving away from the minimum point.

A similar property is true for decreasing rate of the function on the semi-axis to the left from the minimum point: for all $y < x \le M$ and $\delta > 0$, it is true that

$$f(x - \delta) - f(x) \le f(y - \delta) - f(y). \tag{2.3}$$

P r o o f. Consider the points $A\big(x, f(x)\big)$, $B\big(x+\delta, f(x+\delta)\big)$, $C\big(y, f(y)\big)$, $D\big(y+\delta, f(y+\delta)\big)$ belonging to the graph of f. Denote by l the straight line passing through the points A and B. Denote by l' the straight line passing through the point C parallel to the line l.

There are two cases: $x < y < x + \delta$ and $x < x + \delta \le y$.

Let us prove inequality (2.2) in the first case by contradiction.

The point C lies on the line l or below it due to convexity of the function f. If it is true that $f(y + \delta) - f(y) < f(x + \delta) - f(x)$, then the point D is located below the line l' (the dashed line in Fig. 2). Therefore, the point D lies below the line l, and we have a contradiction to the convexity of the function f: the point B is situated above the segment $[AD]$.

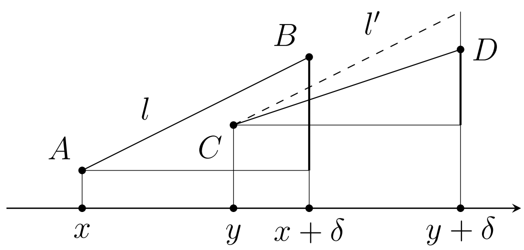

Figure 2: To the proof of Lemma 1, the case $x < y < x + \delta$

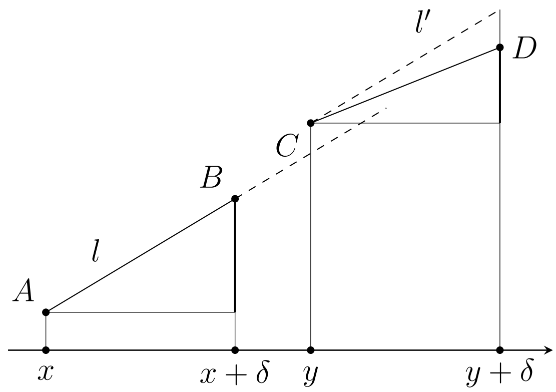

Figure 3: To the proof of Lemma 1, the case $x < x + \delta \le y$

In the second case $x+\delta \leq y$, inequality (2.2) is proved also by contradiction. Due to convexity of the function f, the point C is located not below the line l. If the relation $f(y+\delta) - f(y) < f(x+\delta) - f(x)$ holds, then again we have a contradiction with the convexity of the function f. Namely, the point C is situated above the segment $[AD]$ since the point is located on the line l' and the segment $[AD]$ is below this line.

The relation for the interval of decreasing f can be proved in a similar way. □

Lemma 2. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a unimodal convex function having its minimum at the origin. Take some values $A < B$ and define $f_1(x) = f(x-A)$ and $f_2(x) = f(x-B)$. Then, for any values $a < b$, one has*

$$f_1(a) + f_2(b) \leq f_1(b) + f_2(a). \tag{2.4}$$

P r o o f. There are six cases of relative positions of the points A, B, a, b on the semi-axes of decrease and increase of the functions f_1 and f_2:

$$A \leq a < b \leq B, \quad (2.5) \qquad a \leq A < b \leq B, \quad (2.8)$$
$$a < b \leq A < B, \quad (2.6) \qquad A \leq a < B \leq b, \quad (2.9)$$
$$A < B \leq a < b, \quad (2.7) \qquad a \leq A < B \leq b. \quad (2.10)$$

Consider case (2.5) (see Fig. 4). Inequality (2.4) holds due to the inequalities $f_1(a) \leq f_1(b)$ and $f_2(b) \leq f_2(a)$.

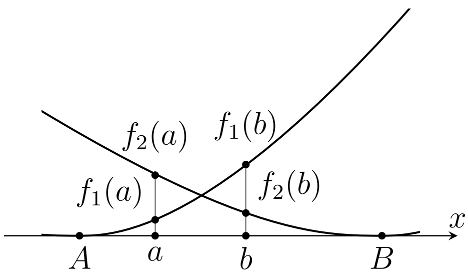

Figure 4: To the proof of Lemma 2, the case $A \leq a < b \leq B$

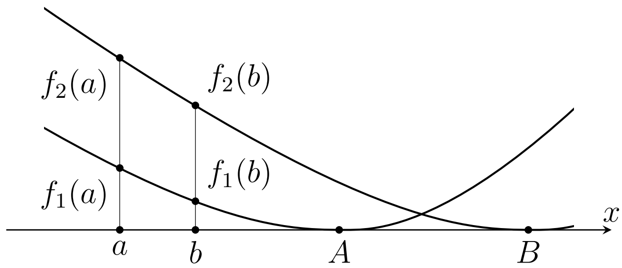

Figure 5: To the proof of Lemma 2, the case $a < b \leq A < B$

To prove the subsequent cases, we transform inequality (2.4) to be proved:

$$f_2(a) - f_1(a) + f_1(b) - f_2(b) \geq 0. \tag{2.11}$$

Let us study case (2.6) (see Fig. 5). In inequality (2.11), we pass to the function f:

$$f_2(a) - f_1(a) + f_1(b) - f_2(b) = f(a-B) - f(a-A) + f(b-A) - f(b-B) \geq 0.$$

This inequality is true due to inequality (2.3) from Lemma 1, where $M = 0$, $x = b - A$, $y = b - B$, and $\delta = b - a$.

Case (2.7) can be proved in a similar way involving inequality (2.2).

Consider case (2.9) (see Fig. 6). To prove this case, one needs to transform the left-hand side of inequality (2.11) by adding and subtracting some terms:

$$\begin{aligned} f_2(a) - f_1(a) + f_1(b) - f_2(b) &= \big[\pm f_1(B) \pm f_2(B)\big] \\ &= \big(f_1(B) - f_1(a)\big) + \big(f_2(a) - f_2(B)\big) + \big(f_1(b) - f_1(B)\big) - \big(f_2(b) - f_2(B)\big). \end{aligned} \tag{2.12}$$

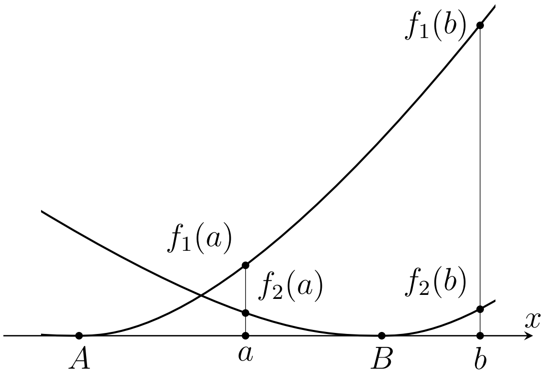

Figure 6: To the proof of Lemma 2, the case $A \le a < B \le b$

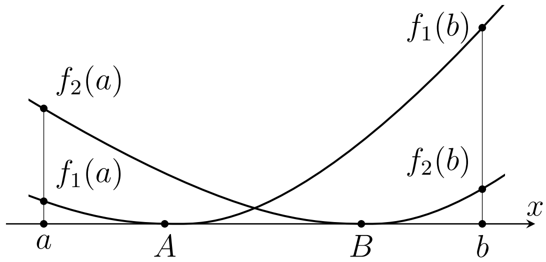

Figure 7: To the proof of Lemma 2, the case $a \le A < B \le b$

The term $\big(f_1(B) - f_1(a)\big)$ is non-negative due to monotonic increase of the function f_1 on the semi-axis $[A, +\infty)$. The term $f_2(a) - f_2(B)$ is non-negative due to monotonic decrease of the function f_2 on the semi-axis $(-\infty, B]$. The difference

$$\big(f_1(b) - f_1(B)\big) - \big(f_2(b) - f_2(B)\big) = \big(f(b - A) - f(B - A)\big) - \big(f(b - B) - f(B - B)\big)$$

is non-negative due to inequality (2.2) of Lemma 1, where $M = 0$, $x = B - B$, $y = B - A$, and $\delta = B - a$. So, entire expression (2.12) is non-negative.

Case (2.8) is proved in a similar way by adding and subtracting the terms $f_1(A)$ and $f_2(A)$ and applying inequality (2.3).

Consider case (2.10) (see Fig. 7). To prove this case, we transform the left-hand side of inequality (2.11) as follows:

$$\begin{aligned} f_2(a) - f_1(a) + f_1(b) - f_2(b) &= \big[\pm f_1(A) \pm f_2(A) \pm f_1(B) \pm f_2(B)\big] \\ &= \big(f_1(B) - f_1(A)\big) + \big(f_2(A) - f_2(B)\big) + \big(f_2(a) - f_2(A)\big) - \big(f_1(a) - f_1(A)\big) \\ &\qquad + \big(f_1(b) - f_1(B)\big) - \big(f_2(b) - f_2(B)\big). \end{aligned} \tag{2.13}$$

The term $\big(f_1(B) - f_1(A)\big)$ is non-negative due to monotonic increase of the function f_1 on the semi-axis $[A, +\infty)$. The term $\big(f_2(A) - f_2(B)\big)$ is non-negative due to monotonic decrease of the function f_2 on the semi-axis $(-\infty, B]$. The difference

$$\big(f_2(a) - f_2(A)\big) - \big(f_1(a) - f_1(A)\big) = \big(f(a - B) - f(A - B)\big) - \big(f(a - A) - f(A - A)\big)$$

is non-negative due to inequality (2.2) from Lemma 1, where $M = 0$, $x = A - A$, $y = A - B$, and $\delta = A - a$. The difference

$$\big(f_1(b) - f_1(B)\big) - \big(f_2(b) - f_2(B)\big) = \big(f(b - A) - f(B - A)\big) - \big(f(b - B) - f(B - B)\big).$$

is non-negative due to inequality (2.3) from Lemma 1, where $M = 0$, $x = B - B$, $y = B - A$, and $\delta = B - b$. Thereby, entire expression (2.13) is proved to be non-negative.

So, in all cases (2.5)–(2.10), inequality (2.11) holds, therefore, inequality (2.4) holds too.

□

Lemma 3. *Let*

1) *some constants* $A_i \in \mathbb{R}$, $i = \overline{1, N}$, *obey inequalities* $A_1 < A_2 < \ldots < A_N$*;*
2) $\mathcal{D}_i = [A_i - \underline{\alpha}, A_i + \overline{\alpha}] \subset \mathbb{R}$, $i = \overline{1, N}$, *for some* $\underline{\alpha}, \overline{\alpha} > 0$*;*

3) *for some* $\tau > 0$, $\widehat{\mathcal{D}} = \big\{(x_1, \ldots, x_N) \mid x_i \in \mathcal{D}_i, \forall\, 1 \le i' < i'' \le N\, |x_{i'} - x_{i''}| \ge \tau\big\} \subset \mathbb{R}^N$; *the constant* τ *be not too large such that* $\widehat{\mathcal{D}} \ne \emptyset$;
4) $f\colon \mathbb{R} \to \mathbb{R}$ *be a convex unimodal function, which has its minimum at the origin,* $f_i(x) = f(x - A_i)$, *and*

$$F(x) = F(x_1, \ldots, x_N) = \sum_{i=1}^{N} f_i(x_i);$$

Then among the minimum points of F *over* $\widehat{\mathcal{D}}$, *there is such a point* $x^* = (x_1^*, x_2^*, \ldots, x_N^*)$ *that* $x_1^* < x_2^* < \ldots < x_N^*$.

P r o o f. Let $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N) \in \widehat{\mathcal{D}}$ be some minimum point of F. And let there be two indices $1 \le i' < i'' \le N$ such that $\bar{x}_{i'} > \bar{x}_{i''}$. (Note that equality of coordinates is impossible due to the second condition in the definition of $\widehat{\mathcal{D}}$ in item 3.)

Let us show that a) the point $\bar{y} = (\bar{x}_1, \ldots, \bar{x}_{i'-1}, \bar{x}_{i''}, \bar{x}_{i'+1}, \ldots, \bar{x}_{i''-1}, \bar{x}_{i'}, \bar{x}_{i''+1}, \ldots, \bar{x}_N)$, that is, the point obtained from $\bar{x}$ by swapping positions of the components $\bar{x}_{i'}$ and $\bar{x}_{i''}$, also belongs to $\widehat{\mathcal{D}}$; and b) $F(\bar{y}) \le F(\bar{x})$.

a) Since $\bar{x}_{i'} \in \mathcal{D}_{i'}$ and $\bar{x}_{i''} \in \mathcal{D}_{i''}$, then

$$A_{i'} - \underline{\alpha} \le \bar{x}_{i'} \le A_{i'} + \overline{\alpha}, \quad A_{i''} - \underline{\alpha} \le \bar{x}_{i''} \le A_{i''} + \overline{\alpha}. \tag{2.14}$$

If to take into account the inequalities $A_{i'} < A_{i''}$ and $\bar{x}_{i'} > \bar{x}_{i''}$, inequalities (2.14) can be rewritten as

$$A_{i'} - \underline{\alpha} < A_{i''} - \underline{\alpha} \le \bar{x}_{i''} < \bar{x}_{i'} \le A_{i'} + \overline{\alpha} < A_{i''} + \overline{\alpha}.$$

Therefore, one can conclude that $\bar{x}_{i''} \in \mathcal{D}_{i'}$ and $\bar{x}_{i'} \in \mathcal{D}_{i''}$. Thus, the first condition in the definition of $\widehat{\mathcal{D}}$ in item 3 is true for the components of $\bar{y}$.

The second part of the definition of $\widehat{\mathcal{D}}$ in item 3 is true for the components of $\bar{y}$ because the inequalities are true for all pairs of components of $\bar{x}$, and $\bar{y}$ has just the same components as $\bar{x}$.

b) The difference of the values $F(\bar{x})$ and $F(\bar{y})$ is in two summands only: $f_{i'}(\bar{x}_{i'}) + f_{i''}(\bar{x}_{i''})$ and $f_{i'}(\bar{x}_{i''}) + f_{i''}(\bar{x}_{i'})$, respectively. Due to Lemma 2, inequality

$$f_{i'}(\bar{x}_{i'}) + f_{i''}(\bar{x}_{i''}) \ge f_{i'}(\bar{x}_{i''}) + f_{i''}(\bar{x}_{i'}) \tag{2.15}$$

holds. Indeed, in the framework of Lemma 2, we have $A = A_{i'} < B = A_{i''}$,

$$f_1(x) = f(x - A) = f_{i'}(x) = f(x - A_{i'}), \quad f_2(x) = f(x - B) = f_{i''}(x) = f(x - A_{i''}),$$

and $a = \bar{x}_{i''} < b = \bar{x}_{i'}$. So, due to inequality (2.15), one has $F(\bar{x}) \ge F(\bar{y})$.

Since $\bar{x}$ is a minimum point of F over $\widehat{\mathcal{D}}$, then $F(\bar{x}) \le F(\bar{y})$. Consequently, $F(\bar{x}) = F(\bar{y})$, and $\bar{y}$ is also a minimum point of F.

Thus, from existence of some minimum point of F over $\widehat{\mathcal{D}}$, it follows that in $\widehat{\mathcal{D}}$ there exists a minimum point of F, which coordinates are situated in ascending order. □

§ 3. Conditions and linear programming formalization

At first, we demand similarity of all aircraft. That is, for all $1 \le i < j \le N$, we suppose that $\tau_{i,j}^{\text{safe}} = \tau^{\text{safe}}$. Also, for all $1 \le i \le N$, it is assumed that $t_i^{\text{acc}} = t^{\text{acc}}$, $t_i^{\text{dec}} = t^{\text{dec}}$.

As it is demanded at the beginning of Section 2, the penalty functions f_i in (1.1) are the same and the common function f is unimodal convex piecewise-linear. As it is explained in Section 2 (the part devoted to linearization of a convex continuous piecewise-linear functional), the functional (1.1) can be linearized by introducing new variables and constraints.

Now, the only obstacle to consider problem (1.1)–(1.3) as a linear programming one is that inequalities (1.3) define a disconnected set, which, of course, is not convex. One of the usual approaches is to consider the problem by introducing additional *binary* variables as a mixed integer linear programming one (see, for example, [2]).

However, due to Lemma 3, one can assert that there is an optimal solution of problem (1.1)–(1.3), for which the appointed arrival instants have the same order as the nominal ones. Indeed, in the framework of Lemma 3, $A_i = t_i^{\text{nom}}$, $\underline{\alpha} = t^{\text{acc}}$, $\overline{\alpha} = t^{\text{dec}}$, $\tau = \tau^{\text{safe}}$. The conclusion of the lemma allows us to declare conservation of the order of aircraft in the resultant queue.

Under the premises made and this conclusion, problem (1.1)–(1.3) can be rewritten as

$$F(\mathbf{t}, \mathbf{t}^{\text{nom}}) = \sum_{i=1}^{N} f(t_i, t_i^{\text{nom}}) \to \min, \tag{3.1}$$

$$\text{s.t. } t_i \in [t_i^{\text{nom}} - t^{\text{acc}}, t_i^{\text{nom}} + t^{\text{dec}}], \quad t_i \geq 0, \tag{3.2}$$

$$\forall\, 1 \leq i < j \leq N : t_j - t_i \geq \tau^{\text{safe}}. \tag{3.3}$$

In comparison with (1.1)–(1.3), the main difference is in inequalities (3.3).

But, of course, this is true only for the situation of aircraft of the same type (having the same safe time interval τ^{safe} between any pair of aircraft) under equality of routes (that is, equality of t^{acc} and t^{dec} for all aircraft) when the same penalty function is applied for all aircraft. As any difference appears, problem (1.1)–(1.3), generally speaking, cannot be considered as a linear programming one.

If the penalty function f in (3.1) is convex piecewise-linear of type (2.1)

$$f(t_i, t_i^{\text{nom}}) = \sum_{k=2}^{m-1} a_k |t_i - \hat{t}_k| + A t_i$$

with some constants $\hat{t}_k$, then by introducing new variables q_{ik}, problem (3.1)–(3.3) is transformed to the new form:

$$F(\mathbf{t}, \mathbf{t}^{\text{nom}}) = \sum_{i=1}^{N} \sum_{k=2}^{m-1} (a_k q_{ik} + A t_i) \to \min, \tag{3.4}$$

$$\text{s.t. } t_i \in [t_i^{\text{nom}} - t^{\text{acc}}, t_i^{\text{nom}} + t^{\text{dec}}], \quad t_i \geq 0, \tag{3.5}$$

$$\forall\, 1 \leq i < j \leq N : t_j - t_i \geq \tau^{\text{safe}}, \tag{3.6}$$

$$\forall\, i = \overline{1, N},\ k = \overline{2, m-1} : -q_{ik} \leq t_i - \hat{t}_k \leq q_{ik}. \tag{3.7}$$

Actually, criterion F (3.4) depends now only on t_i and the new variables q_{ik}. The nominal arrival instants t_i^{nom} now participate in constraints (3.5) and (3.7) only. The obtained problem (3.4)–(3.7) is of linear programming type and can be efficiently solved by, for example, simplex method even for a quite large number of aircraft in the considered set.

§ 4. Conclusion

In the paper, the problem of safe merging aircraft flows with a given nominal schedule of their arrival to the point of airways joining is considered as a problem of global constrained minimization of some optimality criterion. The constraints are connected both to the maximal possible values of acceleration/deceleration of an aircraft and to the conditions of safety of each consequent pair of aircraft at the merge point. The distinctive feature of the problem statement is that the flows of aircraft of the same type with the equal possibilities of maximum acceleration and delay are considered.

Several theoretic statements are proved, which show that under the assumptions of similarity of all aircraft and air-routes the order of the aircraft arrivals in the resultant queue coincides with the order of the nominal arrivals. This allows one to apply methods of linear programming to the problem solving.

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Received 29.03.2022

Accepted 30.07.2022

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Citation: A. A. Spiridonov, S. S. Kumkov. Keeping order of vessels in problem of safe merging aircraft flows, *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki*, 2022, vol. 32, issue 3, pp. 433–446.

2022. Т. 32. Вып. 3. С. 433–446.

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Сохранение порядка самолетов в задаче безопасного слияния потоков воздушных судов

Ключевые слова: воздушные суда, точка слияния воздушных трасс, бесконфликтное слияние потоков, номинальные моменты прибытия, назначенные моменты прибытия, объединенная очередь самолетов.

УДК 519.852.3

DOI: 10.35634/vm220306

В настоящее время в рамках управления воздушным движением крайне важной является задача формирования оптимального безопасного расписания прибытия самолетов в точку слияния воздушных трасс. Безопасность результирующей очереди обеспечивается наличием безопасного временнóго интервала между соседними прибытиями в точку слияния. Изменение момента прибытия может обеспечиваться изменением скорости движения самолета и/или использованием схем, удлиняющих или укорачивающих его траекторию. Оптимальность результирующей очереди рассматривается с точки зрения дополнительных требований: минимизации отклонения назначенных моментов прибытия от номинальных, минимизации количества изменений порядка самолетов в очереди, минимизации расхода топлива и т. д. Минимизируемый критерий оптимальности, отражающий эти требования, часто выбирается как сумма индивидуальных штрафов каждому судну за отклонение назначенного момента прибытия от номинального. Функция индивидуального штрафа почти во всех статьях рассматривается либо как модуль отклонения, либо как функция, похожая на модуль, но с различными наклонами ветвей, что приводит к разному штрафу за задержку и ускорение. В целом, задача может быть разделена на две: одна связана с поиском оптимального порядка прибытия судов, вторая — с выбором оптимальных моментов прибытия при заданном порядке. Последняя подзадача достаточно просто решается, поскольку чаще всего может быть формализована как задача линейного программирования. Однако первая решается значительно сложнее, для ее решения применяются разнообразные методы — от эвристических и генетических процедур до подходов смешанного целочисленного линейного программирования. В статье предлагаются условия на параметры задачи, достаточные для того, чтобы порядок оптимальных моментов прибытия самолетов в точку слияния совпадал с порядком номинальных моментов. Это позволяет исключить первую подзадачу из решения всей задачи.

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Поступила в редакцию 29.03.2022

Принята к публикации 30.07.2022

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Цитирование: А. А. Спиридонов, С. С. Кумков. Сохранение порядка самолетов в задаче безопасного слияния потоков воздушных судов // Вестник Удмуртского университета. Математика. Механика. Компьютерные науки. 2022. Т. 32. Вып. 3. С. 433–446.