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© *K. G. Malyutin, M. V. Kabanko***THE MEROMORPHIC FUNCTIONS OF COMPLETELY REGULAR GROWTH ON THE UPPER HALF-PLANE**

A strictly positive continuous unbounded increasing function  $\gamma(r)$  on the half-axis  $[0, +\infty)$  is called growth function. Let the growth function  $\gamma(r)$  satisfies the condition  $\gamma(2r) \leq M\gamma(r)$  for some  $M > 0$  and for all  $r > 0$ . In the paper, the class  $JM(\gamma(r))^o$  of meromorphic functions of completely regular growth on the upper half-plane with respect to the growth function  $\gamma$  is considered. The criterion for the meromorphic function  $f$  to belong to the space  $JM(\gamma(r))^o$  is obtained. The definition of the indicator of function from the space  $JM(\gamma(r))^o$  is introduced. It is proved that the indicator belongs to the space  $L^p[0, \pi]$  for all  $p > 1$ .

*Keywords:* just meromorphic function, complete measure, function of growth, function of completely regular growth, Fourier coefficients, conjugate series, indicator.

DOI: [10.35634/vm200304](https://doi.org/10.35634/vm200304)**Introduction**

The theory of entire functions of completely regular growth (c. r. g) with respect to the function  $\gamma(r)$  close to a power function was created in late 30's of the XX century independently of each other by B. Ya. Levin [13] and A. Pflüger [18, 19]. This theory has many applications in various areas of the modern complex analysis. A full exposition of this theory as well as its applications can be found in [14]. In [6], A. F. Grishin generalized the Levin–Pflüger theory to subharmonic functions on a complex plane. Using the Fourier series method, developed by L. A. Rubel and B. A. Taylor [22], A. A. Kondratyuk [10–12] generalized the Levin–Pflüger theory of entire functions of c. r. g. to meromorphic functions on a complex plane. In the Kondratyuk theory the growth of a function is measured with respect to an arbitrary non-decreasing continuous function  $\gamma(r)$  satisfying the condition

$$\gamma(2r) \leq M\gamma(r) \quad (0.1)$$

for some  $M > 0$  and all  $r > 0$ . This generalization made it possible to describe asymptotic behaviour of entire functions of completely regular growth in  $L_p$ -metrics. A. A. Kondratyuk also introduced the spaces of meromorphic functions of completely regular growth. Note that the Levin–Pflüger theory is included into the Kondratyuk theory as a special case when  $\gamma(r) = r^{\rho(r)}$ , where  $\rho(r)$  is a proximate order [14],  $\lim_{r \rightarrow \infty} \rho(r) = \rho > 0$ . In particular, the theory of Fourier coefficients allowed Kondratyuk to introduce the concept of an indicator of a meromorphic function of a c. r. g., which in the case of entire functions and  $\gamma(r) = r^{\rho(r)}$  coincides with the classical definition of the indicator in the sense of Phragmén and Lindelöf.

At the same time, it has been developed the theory of functions of c.r.g. on the upper half-plane  $\mathbb{C}_+ = \{z: \Im z > 0\}$  of a complex variable. In the 60's of the XX century, A. F. Grishin [7, 8] and N. Govorov [4, 5] independently of each other developed the Levin–Pflüger theory of functions of a finite order on the half-plane. Meanwhile, if the Govorov theory is related to analytic functions of a c.r.g. with respect to the function  $\gamma(r) = r^\rho$  ( $\rho > 0$  is a fixed number), the Grishin

theory covers subharmonic functions of a c.r.g. on the half-plane with respect to  $\gamma(r) = r^{\rho(r)}$ , where  $\rho \geq 0$ , including the classes of subharmonic functions of the zero order.

Using the theory of Fourier coefficients of delta-subharmonic functions on a half-plane developed at the beginning of this century by the first author of the present article [15], K. G. Malyutin and N. Sadik introduced the concept of delta-subharmonic functions of a c.r.g. on a half-plane [16]. In this work, as well as in works by A. A. Kondratyuk, the growth of the function was measured with respect to an arbitrary non-decreasing continuous function  $\gamma(r)$  satisfying the condition (0.1). Our paper specifies and supplements some results of the work [16]. In particular, we obtain another criterion for the meromorphic function to have completely regular growth different from the announced criterion in [16].

## § 1. Function spaces in the upper half-plane

In this paper we use terminology from [9] and [15, 17]. Besides, following Titchmarsh, we will use the following names and designations. If in some reasoning there is a number which does not depend on the basic variables it is called as a constant. For a designation of absolute positive constants, not necessarily the same, we use letters  $A, M, K$ . One can meet the statement like “ $|v(z)| < M\gamma(2r)$  hence  $|v(z)| < M\gamma(r)$ ” which should not cause misunderstanding.

By  $\mathbb{N}$  we denote the set of natural numbers. By  $\mathbb{C}$  and  $\mathbb{R}$  we denote the set of complex and real numbers respectively. Let  $\mathbb{C}_+ = \{z: \Im z > 0\}$  be the upper half-plane of the complex variable  $z$ . We denote by  $C(a, r)$  the open disk of radius  $r$  with the centre at  $a$ , and by  $\Omega_+$  the intersection of a set  $\Omega$  with the half-plane  $\mathbb{C}_+$ :  $\Omega_+ = \Omega \cap \mathbb{C}_+$ ;  $\overline{G}$  will be the closure of the set  $G$ . If  $0 < r_1 < r_2$ , then  $D_+(r_1, r_2) = \overline{C_+(0, r_2)} \setminus C_+(0, r_1)$  is the closed half-annulus.

A function  $f$  is called a just analytic function [3] in  $\mathbb{C}_+$  if  $f$  is an analytic function in  $\mathbb{C}_+$  and  $\limsup_{z \rightarrow t, z \in \mathbb{C}_+} \log |f(z)| \leq 0$  for any  $t \in (-\infty, \infty)$ . By  $JA$  we denote the space of just analytic functions in  $\mathbb{C}_+$ .

Let  $AK$  [3] be the space of analytic functions in  $\mathbb{C}_+$  such that  $\log |f(z)|$  has a positive harmonic majorant in any bounded domain in  $\mathbb{C}_+$ . The functions  $f$  of the space  $AK$  have the following properties:

- a)  $\log |f(z)|$  has the non-tangential limit  $\log |f(t)|$  almost everywhere on the real axis such that  $\log |f(t)| \in L^1_{loc}(-\infty, \infty)$ ;
- b) for any  $f \in AK$  there exists  $\lim_{y \rightarrow +0} \int_a^b \log |f(t + iy)| dt = \nu([a, b])$ , where  $\nu$  is a signed measure on the real axis,  $a, b \in (-\infty, +\infty) \setminus E_f$ ,  $E_f$  is a countable set. The measure  $\nu$  is called the boundary measure of the function  $f$ ;
- c)  $d\nu(t) = \log |f(t)| dt + d\sigma(t)$ , where the measure  $\sigma$  is singular with respect to the Lebesgue measure.

For the function  $f \in AK$ , following [3], we define the complete measure  $\lambda$  of  $f$  as

$$\lambda(K) := \lambda_f(K) = 2\pi \int_{\mathbb{C}_+ \cap K} \Im \zeta d\mu_f(\zeta) - \nu(K \cap \mathbb{R}),$$

where  $\mu_f$  is the Riesz measure of the function  $\log |f(z)|$ ,  $K \subset \mathbb{C}$  is an arbitrary set. The measure  $\lambda$  has the following properties:

- 1)  $\lambda$  is a finite measure on each compact subset  $K$  of  $\mathbb{C}$ ;
- 2)  $\lambda$  is a positive measure outside  $\mathbb{R}$ ;

3)  $\lambda$  is a zero measure on the half-plane  $\mathbb{C}_- = \{z: \Im z < 0\}$ .

Conversely, if a measure  $\lambda$  with properties 1)–3) is given, then there exists a function  $f \in AK$  such that  $\lambda$  is the complete measure of  $f$ .

The complete measure of a function  $f \in JA$  is a positive measure, which explains our term “just analytic”.

A function  $f$  is called a *just meromorphic function in  $\mathbb{C}_+$*  if  $f$  can be represented as a quotient of two just analytic functions. The space of just meromorphic functions in  $\mathbb{C}_+$  is denoted by  $JM$ . So  $JM = JA/JA$ .

Let  $f_1, f_2 \in JA$ . We will say that functions  $f_1$  and  $f_2$  have no common zeros in  $\overline{\mathbb{C}}_+$  if the complete measure of  $f_1$  and  $f_2$  are mutually singular. Point out that if  $f_1$  and  $f_2$  have no common zeros in  $\mathbb{C}_+$  the same can fail to be valid for  $af_1$  and  $af_2$  if  $|a| \neq 1$ .

The following assertions is true [3]:  $JA \subset AK$  and  $JM = AK/AK$ . For  $f \in JM$  the identity  $f = \frac{f_1}{f_2}$  holds, where  $f_1, f_2$  are just analytic functions without common zeros in  $\overline{\mathbb{C}}_+$ . They are defined uniquely up to the multiplier of the form  $e^{ig(z)}$ , where  $g(z)$  is a real entire functions. In this case we define the complete measure  $\lambda_f$  of  $f$  as  $\lambda_f = \lambda_{f_1} - \lambda_{f_2}$ . Additionally, if  $f_1, f_2 \in JA$ , then  $f_1 f_2 \in JA$ , but it can happen that  $f_1 + f_2$  is out of  $JA$ . However, if  $\alpha$  and  $\beta$  satisfy  $|\alpha| + |\beta| \leq 1$ , then  $\alpha f_1 + \beta f_2 \in JA$ . If  $f = (f_0, f_1, \dots, f_n)$ ,  $f_k \in JA$  ( $k = 0, 1, \dots, n$ ),  $w = (w_0, w_1, \dots, w_n)$ ,  $p(w) = \sum_{\alpha} c_{\alpha} w^{\alpha}$  ( $\alpha$  is multi-index),  $\frac{1}{\beta} = \max_{\max |w_k| \leq 1} |p(w)|$ , then  $\beta p \circ f \in JA$ . The set  $JM$  is a complex linear space and a field, but not a differential field under ordinary differentiation. If  $f$  is meromorphic in  $\overline{\mathbb{C}}_+$  (this case was studied by R. Nevanlinna), then  $f \in JM$ .

We observe that  $JM$  is the largest class of meromorphic functions in  $\mathbb{C}_+$  to which one can ascribe the Nevanlinna characteristic.

**§ 2. Nevanlinna characteristic functions for the complex half-plane**

For a fixed measure  $\lambda$ , let

$$d\lambda_m(\zeta) = \frac{\sin m\varphi}{\sin \varphi} \tau^{m-1} d\lambda(\zeta) \left( \zeta = \tau e^{i\varphi} \right), \quad \lambda_m(r) = \lambda_m(\overline{C(0, r)}),$$

where  $\frac{\sin m\varphi}{\sin \varphi}$  is defined for  $\varphi = 0, \pi$  by continuity.

Let  $f \in JM$  and let  $\lambda = \lambda_f$  be the corresponding complete measure of  $f$ . The next relation is the Carleman’s formula in the Grishin notations [3]:

$$\frac{1}{r^k} \int_0^{\pi} \log |f(re^{i\varphi})| \sin k\varphi d\varphi = \int_{r_0}^r \frac{\lambda_k(t)}{t^{2k+1}} dt + \frac{1}{r_0^k} \int_0^{\pi} \log |f(r_0 e^{i\varphi})| \sin k\varphi d\varphi, \quad k \in \mathbb{N} \quad (2.1)$$

where  $r_0 > 0$  is an arbitrary (usually fixed) positive number (we can set  $r_0 = 1$ ).

In particular, for  $k = 1$  we have

$$\frac{1}{r} \int_0^{\pi} \log |f(re^{i\varphi})| \sin \varphi d\varphi = \int_{r_0}^r \frac{\lambda(t)}{t^3} dt + \frac{1}{r_0} \int_0^{\pi} \log |f(r_0 e^{i\varphi})| \sin \varphi d\varphi \quad (2.2)$$

for all  $r > r_0$ .

For  $f \in JM$ , let  $\lambda_f := \lambda = \lambda_+ - \lambda_-$  be the Jordan decomposition of  $\lambda_f$ . We set up the following notations and terminology

$$m(r, f) := \frac{1}{r} \int_0^{\pi} \log^+ |f(re^{i\varphi})| \sin \varphi d\varphi, \quad N(r, f, r_0) := \int_{r_0}^r \frac{\lambda_-(t)}{t^3} dt,$$

$$T(r, f, r_0) := m(r, f) + N(r, f, r_0) + m\left(r_0, \frac{1}{f}\right), \quad r > r_0,$$

where  $r_0$  is an arbitrary fixed positive number, which we shall drop in the notations (provided that this cannot lead to confusion; for instance, we shall write  $T(r, f)$  in place of  $T(r, f, r_0)$  and so on).

In this notation the Carleman's formula (2.2) can be written as follows:

$$T(r, f) = T\left(r, \frac{1}{f}\right). \quad (2.3)$$

Note also the inequality which will be useful further:

$$\begin{aligned} |\lambda_m(r)| &= \left| \frac{\iint_{\overline{C(0,r)}} d\lambda_m(\zeta)}{\overline{C(0,r)}} \right| = \left| \frac{\iint_{\overline{C(0,r)}} \frac{\sin m\varphi}{\sin \varphi} \tau^{m-1} d\lambda(\zeta)}{\overline{C(0,r)}} \right| \leq \\ &\leq m \iint_{\overline{C(0,r)}} \tau^{m-1} d|\lambda|(\zeta) \leq mr^{m-1} |\lambda|(r). \end{aligned} \quad (2.4)$$

We call a strictly positive continuous unbounded increasing function  $\gamma(r)$  on the half-axis  $[0, +\infty)$  a *growth function*. We assume that the growth function  $\gamma$  satisfies the condition:

$$\liminf_{r \rightarrow \infty} \frac{\gamma(r)}{r} > 0. \quad (2.5)$$

Condition (2.5) holds for  $\gamma(r) = r^\rho$  with  $\rho \geq 1$ , but fails for  $r^\rho$  with  $0 < \rho < 1$ . If (2.5) fails, then we set  $N(r, f) := N(r, f, r/2)$  and  $T(r, f) := T(r, f, r/2)$ ; then all our results below persist.

The function  $f \in JM$  is called a *function of finite  $\gamma$ -type* if there exist constants  $A, B > 0$  such that

$$T(r, f) \leq \frac{A}{r} \gamma(Br), \quad r > r_0.$$

We denote the corresponding space of meromorphic functions of finite  $\gamma$ -type by  $JM(\gamma(r))$ . By  $JA(\gamma(r))$ , we denote the space of just analytic functions of finite  $\gamma$ -type.

A positive measure  $\lambda$  has a *finite  $\gamma$ -density* if there exist positive constants  $A$  and  $B$  such that

$$N(r, \lambda) := \int_{r_0}^r \frac{\lambda(t)}{t^3} dt \leq \frac{A}{r} \gamma(Br)$$

for all  $r > r_0$  (if condition (2.5) fails, then we set  $r_0 = r/2$ ).

A positive measure  $\lambda$  in the complex plane is called a *measure of finite  $\gamma$ -type* if there exist positive constants  $A$  and  $B$  such that for all  $r > 0$ ,

$$\lambda(r) \leq Ar\gamma(Br). \quad (2.6)$$

If a measure  $\lambda$  has a finite  $\gamma$ -density then one is the measure of a finite  $\gamma$ -type (this assertion is Lemma 2 from [15]).

The Fourier coefficients of a function  $f \in JM$  are defined as usual [15]:

$$c_k(r, f) = \frac{2}{\pi} \int_0^\pi \log |f(re^{i\theta})| \sin k\theta d\theta, \quad k \in \mathbb{N}.$$

From (2.1), we obtain the following expressions for the Fourier coefficients for  $r > r_0$ :

$$c_k(r, f) = \alpha_k r^k + \frac{2r^k}{\pi} \int_{r_0}^r \frac{\lambda_k(t)}{t^{2k+1}} dt, \quad k \in \mathbb{N}, \quad (2.7)$$

where  $\alpha_k = r_0^{-k} c_k(r_0, f)$ .

We will need the following two theorems [15].

**Theorem 1.** *Let  $\gamma$  be a growth function and let  $f \in JM$ . Then the following two properties are equivalent:*

- (i)  $f \in JM(\gamma(r))$ ;
- (ii) *the measure  $\lambda_+(f)$  (or  $\lambda_-(f)$ ) has a finite  $\gamma$ -density and*

$$|c_k(r, f)| \leq A\gamma(Br), \quad k \in \mathbb{N},$$

for some positive  $A, B$  and all  $r > 0$ .

**Theorem 2.** *Let  $\gamma$  be a growth function and let  $f \in JA$ . Then the following two properties are equivalent:*

- (i)  $f \in JA(\gamma(r))$ ;
- (ii)  $|c_k(r, f)| \leq A\gamma(Br)$ ,  $k \in \mathbb{N}$ , for some positive  $A, B$  and all  $r > 0$ .

**Definition 1.** Let the growth function  $\gamma(r)$  satisfies (0.1) and (2.5). A function  $f \in JM$  is said to be a *function of a completely regular growth with respect to  $\gamma(r)$*  if there exist a limit

$$\lim_{r \rightarrow \infty} \frac{1}{\gamma(r)} \int_{\eta}^{\varphi} \log |f(re^{i\theta})| \sin \theta \, d\theta.$$

for all  $\eta$  and  $\varphi$  from the interval  $[0, \pi]$ .

We denote the space of meromorphic functions of the c.r.g. with respect to  $\gamma(r)$  by  $JM(\gamma(r))^{\circ}$ . The notation  $JA(\gamma(r))^{\circ}$  will be used to denote a class of just analytic functions of the c.r.g. from  $JM(\gamma(r))^{\circ}$ .

Let  $\tilde{L}^{\infty}[0, \pi]$  be the Banach subspace of  $L^{\infty}[0, \pi]$  generating by the family of characteristic functions of all intervals from  $[0, \pi]$ . According to the Cantor theorem on uniform continuity,  $C[0, \pi] \subset \tilde{L}^{\infty}[0, \pi]$ . Denote by  $\mathcal{L}[0, \pi]$  any of the spaces  $C[0, \pi]$ ,  $\tilde{L}^{\infty}[0, \pi]$  or  $L^1[0, \pi]$ .

Our main result is the following theorem.

**Theorem 3.** *Let  $f \in JM$  and let the function  $\gamma(r)$  satisfy (0.1) and (2.5). Then the following properties are equivalent:*

- (i)  $f \in JM(\gamma(r))^{\circ}$ ;
- (ii)  $f \in JM(\gamma(r))$  and for all  $k \in \mathbb{N}$  there exists

$$\lim_{r \rightarrow \infty} \frac{c_k(r, f)}{\gamma(r)} = c_k; \tag{2.8}$$

- (iii) *the measure  $\lambda_-(f)$  has a finite  $\gamma$ -density and for any function  $\psi$  from  $\mathcal{L}[0, \pi]$  there exists a limit*

$$\lim_{r \rightarrow \infty} \frac{1}{\gamma(r)} \int_0^{\pi} \psi(\theta) \log |f(re^{i\theta})| \sin \theta \, d\theta. \tag{2.9}$$

Note that if  $f$  belongs the class  $JA(\gamma(r))^{\circ}$ , the restriction of the measure  $\lambda_-(f)$  in (iii) is lacking ( $\lambda_-(f) \equiv 0$ ) and the following theorem is true.

**Theorem 4.** Let  $f \in JA$  and let the function  $\gamma(r)$  satisfy (0.1) and (2.5). Then the following properties are equivalent:

- (i)  $f \in JA(\gamma(r))^o$ ;
- (ii)  $f \in JA(\gamma(r))$  and for all  $k \in \mathbb{N}$  there exists  $\lim_{r \rightarrow \infty} \frac{c_k(r, f)}{\gamma(r)} = c_k$ ;
- (iii) for any function  $\psi$  from  $\mathcal{L}[0, \pi]$  there exists a limit

$$\lim_{r \rightarrow \infty} \frac{1}{\gamma(r)} \int_0^\pi \psi(\theta) \log |f(re^{i\theta})| \sin \theta \, d\theta.$$

Analogous criterion for meromorphic functions in the complex plane is got by Kondratyuk.

### § 3. Proof of Theorem 3

Let us prove the theorem by scheme (i)  $\Rightarrow$  (iii)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (i). We prove now the implication (i)  $\Rightarrow$  (iii). Let  $f \in JM(\gamma(r))^o$ . Note first that for each  $f$  the measures  $\lambda_+(f)$  and  $\lambda_-(f)$  has a finite  $\gamma$ -density. The measure  $\lambda_-(f)$  has a finite  $\gamma$ -density by the definition of the space  $JM(\gamma(r))^o$ . The fact that  $\lambda_+(f)$  has a finite  $\gamma$ -density is a consequence of (2.3). The same formula yields

$$\int_0^\pi |\log |f(re^{i\theta})|| \sin \theta \, d\theta \leq A\gamma(Br), \quad (3.1)$$

where  $A, B > 0$  are some constants. We consider the family

$$\mathcal{F}_r[\psi] = \frac{1}{\gamma(r)} \int_0^\pi \psi(\theta) \log |f(re^{i\theta})| \sin \theta \, d\theta$$

of linear continuous operators on  $\mathcal{L}[0, \pi]$ . The norms of these operators are

$$\|\mathcal{F}_r\| = \frac{1}{\gamma(r)} \int_0^\pi |\log |f(re^{i\theta})|| \sin \theta \, d\theta. \quad (3.2)$$

It follows from (0.1) that

$$\gamma(Br) \leq M\gamma(r), \quad M > 0. \quad (3.3)$$

From (3.1) and (3.3), it follows that the norms of the operators  $\mathcal{F}_r$  are uniformly bounded with respect to  $r$ . The linear hull of the family of characteristic functions of all intervals from  $[0, \pi]$  is a dense set in  $\mathcal{L}[0, \pi]$ . By Banach–Steinhaus theorem, we obtain (2.9).

We prove the implication (iii)  $\Rightarrow$  (ii). It follows from (2.9), by Banach–Steinhaus theorem, that the family of norms  $\{\|\mathcal{F}_r\|\}$  are uniformly bounded with respect to  $r$ . Then, by (3.3), (3.2), and (3.1), we have

$$|c_k(r, f)| \leq Ak\gamma(r), \quad k \in \mathbb{N}, \quad A > 0. \quad (3.4)$$

Formula (2.1) yields

$$c_k(r, f) = \frac{1}{2^k} c_k(2r, f) - \frac{2r^k}{\pi} \int_r^{2r} \frac{\lambda_k(t)}{t^{2k+1}} dt,$$

which, in view of (3.3), (2.4), (2.6), and (3.4), gives us the inequality

$$|c_k(r, f)| \leq \frac{k}{2^k} A\gamma(2rB) + \frac{2}{\pi} A\gamma(2rB) \leq D\gamma(r).$$

If  $N(r, \lambda_-) = O(\gamma(r))$ ,  $r \rightarrow \infty$ , then, by Theorem 1, we have  $f \in JM(\gamma(r))$ .

Further, as  $\sin k\theta / \sin \theta \in C[0, \pi]$  for all  $k \in \mathbb{N}$ , then (2.8) holds.

We prove now the implication (ii)  $\Rightarrow$  (iii). It follows from (2.8) and the equality  $2 \sin a \cos b = \sin(a + b) + \sin(a - b)$  that for all  $k \in \mathbb{N} \cup \{0\}$  there exists

$$\lim_{r \rightarrow \infty} \frac{1}{\gamma(r)} \int_0^\pi \log |f(re^{i\theta})| \sin \theta \cos k\theta \, d\theta.$$

The family of the functions  $\{\cos k\theta\}_0^\infty$  is dense set in  $\mathcal{L}[0, \pi]$ . By Banach–Steinhaus theorem and (3.1), we obtain (2.9).

The implication (iii)  $\Rightarrow$  (i) follows from the fact that the family of characteristic functions of all intervals from  $[0, \pi]$  belongs to  $\mathcal{L}[0, \pi]$ .

Theorem 3 is proved.  $\square$

Theorem 4 is a corollary of Theorem 3 and Theorem 2.

#### § 4. Preliminaries on conjugate series

For any real number  $p > 0$ , we denote by  $\mathbf{L}^p = \mathbf{L}^p([0, \pi])$  the set of real-valued measurable functions  $f$  with the period  $\pi$  such that

$$\|f\|_p = \left[ \frac{1}{\pi} \int |f(x)|^p \, dx \right]^{1/p}$$

is finite, the integral being extended over any interval of the length  $\pi$ .

If  $k$  is an integer,  $k \geq 0$ ,  $\mathbf{C}^k = \mathbf{C}^k([0, \pi])$  will denote the set of real-valued functions with the period  $\pi$  and with  $k$  continuous derivatives, and  $\mathbf{C}^\infty = \bigcap \{\mathbf{C}^k : k = 1, 2, \dots\}$ . For brevity,  $\mathbf{C}$  is written in place  $\mathbf{C}^0$ . On  $\mathbf{C}^k$  ( $k$  is an integer  $\geq 0$ ) we introduce the norm  $\|f\|_{(k)} = \sup_{0 \leq h \leq k} \max_{x \in [0, \pi]} D^h f(x)$ ; here and throughout  $D$  is a symbol of derivation. On  $\mathbf{C}^\infty$  we introduce the metric  $\|f - g\|_{(\infty)}$ , where

$$\|f\|_{(\infty)} = \sum_{k=0}^{\infty} \frac{2^{-k} \|f\|_{(k)}}{1 + \|f\|_{(k)}}.$$

Despite the notation,  $\|f\|_{(\infty)}$  is not a norm. In the space  $\mathbf{C}^\infty$  the topology is defined. Thus, if  $u \in \mathbf{C}^\infty$  and if  $(u_k)_{k=1}^\infty$  is a sequence extracted from  $\mathbf{C}^\infty$ , we shall write  $\mathbf{C}^\infty - \lim_{k \rightarrow \infty} u_k = u$  (or  $\lim_{k \rightarrow \infty} u_k = u$  in  $\mathbf{C}^\infty$ ) if and only if  $\lim_{k \rightarrow \infty} \|u_k - u\|_{(\infty)} = 0$ .

By a *distribution* is meant a continuous linear functional on  $\mathbf{C}^\infty$ . Henceforth we shall always denote by  $\mathbf{D}$  the set of distributions. Since  $\mathbf{D}$  is the set of continuous linear functionals on a topological linear space; it carries a natural linear space structure: if  $F_1, F_2 \in \mathbf{D}$  and  $\lambda$  is a scalar, then  $F_1 + F_2$  and  $\lambda F_1$  are the functionals defined by  $(F_1 + F_2)(u) = F_1(u) + F_2(u)$ ,  $(\lambda F_1)(u) = \lambda \cdot F_1(u)$  for  $u \in \mathbf{C}^\infty$ .

For a given distribution  $F$ , there exists a least integer  $m \geq 0$  such that  $|F(u)| \leq c \cdot \sup_{0 \leq p \leq m} \|D^p u\|_{(\infty)}$  holds for suitable ( $F$ -dependent) number  $c$  [2, p. 55]. We then say that  $F$  is a distribution of order  $m$ . We shall henceforth denote by  $\mathbf{D}^m$  ( $m = 0, 1, 2, \dots$ ) the set of distributions of order at most  $m$ .  $\mathbf{D}^m$  is a linear subspace of  $\mathbf{D}$  and

$$\mathbf{L}^p \subset \mathbf{L}^q \subset \mathbf{D}^0 \subset \mathbf{D}^1 \subset \dots \subset \mathbf{D}^m \subset \mathbf{D}^{m+1} \subset \dots, \quad \mathbf{D} = \bigcup_{m=0}^{\infty} \mathbf{D}^m, \quad (4.1)$$

where  $\infty > p > q > 0$ .

In this section we give a few results about special type of series, namely,

$$\sum_{n=1}^{\infty} a_n \sin nx. \quad (4.2)$$

We shall assume throughout that the  $a_n$  are real-valued.

The series (4.2) is an example of the so-called conjugate series. These series play an important role in the theory of functions on the half-plane. Recently, there has been a surge in interest in the theory of conjugate series. Note the studies of A. Yu. Popov, A. P. Solodov [21], A. P. Solodov [23], S. A. Telyakovskii [24].

We will need the following important theorem, found simultaneously and independently by A. I. Plessner, on the one hand, and by A. N. Kolmogorov and G. A. Seliverstov on the other hand [1, Chapter V, § 2].

**Theorem 5 (Kolmogorov–Seliverstov–Plessner).** *If  $\sum_{n=1}^{\infty} a_n^2 \ln n < \infty$ , then the series (4.2) converges almost everywhere.*

### § 5. Indicator of the delta-subharmonic function of a completely regular growth

We introduce the following definition.

**Definition 2.** Let  $f \in JM(\gamma(r))^{\circ}$  and  $c_n$  be defined by equalities (2.8). Then the function

$$h(\theta, f) = \sum_{n=1}^{\infty} c_n \sin nx \quad (5.1)$$

is called the indicator of the function  $f$ .

We will need the following lemma on Pólya peaks [20].

**Lemma 1.** *Let us assume that  $\psi_1, \psi_2, \psi$  are positive continuous functions of  $r$  on the ray  $[r_0, +\infty)$  such that the fraction  $\psi_2(r)/\psi_1(r)$  increases and*

$$\limsup_{r \rightarrow \infty} \frac{\psi(r)}{\psi_1(r)} = \infty, \quad \limsup_{r \rightarrow \infty} \frac{\psi(r)}{\psi_2(r)} = 0.$$

*Then, there is a sequence  $\{r_n\}$ ,  $r_n \rightarrow \infty$  ( $n \rightarrow \infty$ ), such that*

$$\frac{\psi(t)}{\psi_1(t)} \leq \frac{\psi(r_n)}{\psi_1(r_n)}, \quad r_0 \leq t \leq r_n, \quad \frac{\psi(t)}{\psi_2(t)} \leq \frac{\psi(r_n)}{\psi_2(r_n)}, \quad r_n \leq t < \infty.$$

It follows from (0.1) that the order  $\beta := p[\gamma] < \infty$ . Let us formulate this proposition in the form of the following lemma.

**Lemma 2.** *Let us assume that a strictly positive, continuous, increasing and unbounded function  $\gamma(r)$ , defined on a semi-axis  $[0, +\infty)$ , satisfies the condition (0.1). Then, there are numbers  $\beta \in \mathbb{N}$  and  $B > 0$  such that  $\gamma(r) \leq Br^\beta$  for all  $r \in [0, +\infty)$ .*

**Proof.** Let  $M$  be from (0.1). Let us choose  $\beta \in \mathbb{N}$  such that  $M \leq 2^\beta$ . First, let us prove that  $\gamma(2^n) \leq M^n \gamma(1)$  for all  $n \in \mathbb{N}$ . Indeed, this follows from the condition (0.1) for  $n = 1$ . Let this inequality hold for a certain  $n \in \mathbb{N}$ . Then  $\gamma(2^{n+1}) \leq M\gamma(2^n) \leq M^{n+1}\gamma(1)$ . Now, let  $r \in [2^n, 2^{n+1}]$ . We have

$$\gamma(r) \leq \gamma(2^{n+1}) \leq M^{n+1}\gamma(1) \leq (2^{n+1})^\beta \gamma(1) = (2^n)^\beta 2^\beta \gamma(1) \leq Br^\beta,$$

where  $B = 2^\beta \gamma(1)$ . □

**Theorem 6.** *Let the function  $f$  belong to the space  $JM(\gamma(r))^\circ$  and  $\gamma(r)$  satisfy condition (0.1). Then the indicator  $h(\theta, f)$  belongs to  $L^p[0, \pi]$  for all  $p > 1$ .*

*P r o o f.* Let  $\gamma(r)$  satisfy condition (0.1). Then  $\lim_{r \rightarrow \infty} \gamma(r)/r^k = 0$  for all  $k > \beta$  where  $\beta$  as in Lemma 2. The inequality  $|c_k(r, f)| \leq A\gamma(r)$  and the formula for Fourier’s coefficients (2.7) provide

$$c_k(r, f) = -\frac{2r^k}{\pi} \int_r^\infty \frac{\lambda_k(t)}{t^{2k+1}} dt, \quad k > \beta. \tag{5.2}$$

In fact, we obtain from (2.7) that

$$\frac{c_k(r, f)}{r^k} = \alpha_k + \frac{2}{\pi} \int_{r_0}^r \frac{\lambda_k(t)}{t^{2k+1}} dt, \quad k \in \mathbb{N}.$$

Passing to the limit as  $r \rightarrow \infty$ , we obtain

$$\alpha_k = -\frac{2}{\pi} \int_{r_0}^\infty \frac{\lambda_k(t)}{t^{2k+1}} dt, \quad k \in \mathbb{N}, \quad k > \beta.$$

Substituting this value for  $\alpha_k$  in (2.7), we get (5.2).

Using the definition of the measure  $\lambda_k(t)$  from § 2 and applying the formula of integration by parts to the integral in (5.2), we obtain

$$c_k(r, f) = -\frac{1}{\pi k r^k} \iint_{\substack{\mathbb{C}_+(0, r) \\ \Im \zeta}} \frac{\sin k\varphi}{\Im \zeta} \tau^k d\lambda(\zeta) - \frac{r^k}{\pi k} \iint_{|\zeta| \geq r} \frac{\sin k\varphi}{\tau^k \Im \zeta} d\lambda(\zeta), \quad \zeta = \tau e^{i\varphi}, \tag{5.3}$$

for all  $k > \beta$ , where the kernel  $\frac{\sin k\varphi}{\tau^k \Im \zeta}$  is extended by continuity to the points on the real axis.

Set  $\tilde{\lambda} = |\lambda|$ ,

$$N_1(r) = N_1(r, f) := \int_{r_0}^r \frac{\tilde{\lambda}(t)}{t^3} dt.$$

It follows from Theorem 1 that the measure  $\tilde{\lambda}$  has a finite  $\gamma$ -density. Really, by the definition of the space  $JM(\gamma(r))$  and the inequality  $N(r) \leq T(r, f)$ , the measure  $\lambda_-$  has a finite  $\gamma$ -density. By Theorem 1, the measure  $\lambda_+$  has a finite  $\gamma$ -density. Then the measure  $\tilde{\lambda} = \lambda_+ + \lambda_-$  has a finite  $\gamma$ -density. Formula (5.3) provides the inequality

$$|c_k(r, f)| \leq \frac{1}{\pi r^k} \int_0^r t^{k-1} d\tilde{\lambda}(t) + \frac{r^k}{\pi} \int_r^\infty \frac{d\tilde{\lambda}(t)}{t^{k+1}}, \quad k > \beta.$$

Applying the integration by parts formula in the right-hand side of the inequality, we obtain

$$\begin{aligned} |c_k(r, f)| &\leq \frac{(k+1)r^k}{\pi} \int_r^\infty \frac{\tilde{\lambda}(t)}{t^{k+2}} dt - \frac{k-1}{r^k \pi} \int_0^r t^{k-2} \tilde{\lambda}(t) dt = \\ &\frac{(k+1)r^k}{\pi} \int_r^\infty \frac{dN_1(t)}{t^{k-1}} - \frac{k-1}{r^k \pi} \int_0^r t^{k+1} dN_1(t) = \\ &\frac{(k^2-1)}{\pi} \left\{ \int_r^\infty \left(\frac{r}{t}\right)^k N_1(t) dt + \int_0^r \left(\frac{t}{r}\right)^k N_1(t) dt \right\} - \frac{2k}{\pi} r N_1(r) \end{aligned} \tag{5.4}$$

for all  $k > \beta$ .

Let  $\beta$  be the order of the function  $N_1(r)$ , i. e.,  $\beta = \limsup_{r \rightarrow \infty} \frac{N_1(r)}{\log r}$ . By Lemma 2,  $\beta < \infty$ . Then, for all  $\varepsilon > 0$ ,  $\limsup_{r \rightarrow \infty} N_1(r)/r^{\beta-\varepsilon} = \infty$ . Applying Lemma 1 to the functions  $\psi(r) = N_1(r)$ ,  $\psi_1(r) = r^{\beta-\varepsilon}$ ,  $\psi_2(r) = r^{\beta+\varepsilon}$ , we obtain the sequence  $\{r_n\}$ ,  $r_n \rightarrow \infty$  as  $n \rightarrow \infty$ , such that

$$N_1(t) \leq \left(\frac{t}{r_n}\right)^{\beta-\varepsilon}, \quad r_0 \leq t \leq r_n; \quad N_1(t) \leq \left(\frac{t}{r_n}\right)^{\beta+\varepsilon}, \quad r_n \leq t < \infty. \quad (5.5)$$

Using (5.5), we obtain from (5.4) that

$$|c_k(r_n, f)| \leq \frac{2k}{\pi} N(r_n) \left\{ \frac{k^2 + \beta - \varepsilon k}{(k - \varepsilon)^2 - \beta^2} - 1 \right\} \leq \frac{Ak}{\pi} \gamma(r_n) \left\{ \frac{k^2 + \beta - \varepsilon k}{(k - \varepsilon)^2 - \beta^2} - 1 \right\}, \quad k > \beta.$$

Since, by Theorem 3, there exists

$$c_k = \lim_{r \rightarrow \infty} \frac{c_k(r, f)}{\gamma(r)},$$

it follows that the latter inequality entails that

$$|c_k| = \lim_{r \rightarrow \infty} \frac{|c_k(r, f)|}{\gamma(r)} = \lim_{n \rightarrow \infty} \frac{|c_k(r_n, f)|}{\gamma(r_n)} \leq \frac{Ak}{\pi} \left\{ \frac{k^2 + \beta - \varepsilon k}{(k - \varepsilon)^2 - \beta^2} - 1 \right\},$$

when  $k > \beta$ . Since  $\varepsilon > 0$  is any sufficiently small number, we have

$$|c_k| \leq \frac{Ak}{\pi} \left\{ \frac{\beta^2 + \beta}{k^2 - \beta^2} \right\}, \quad k > \beta. \quad (5.6)$$

By Theorem 5, the series (5.1) converges almost everywhere; by (5.6),  $h(\theta, f) \in \mathbf{L}^p[0, \pi]$  for all  $p > 1$ . Theorem 6 is proved completely.  $\square$

**Lemma 3.** *Let  $f \in JM(\gamma(r))^o$ . Then, there is a finite limit in  $L_p$ -metrics ( $p > 1$ )*

$$\lim_{r \rightarrow \infty} \frac{1}{\gamma(r)} \int_0^\pi \log |f(re^{i\theta})| \sin k\theta \, d\theta = \int_0^\pi h(\theta, f) \sin k\theta \, d\theta$$

for all  $k \in \mathbb{N}$ .

**P r o o f.** This equality is obtained by means of expanding the integrand in the right-hand side into the Fourier series, its integration term by term, and passing to the limit in the left-hand side of the equality.  $\square$

**Lemma 4.** *Let  $f \in JM(\gamma(r))^o$ . Then  $h(\theta, f) \in \mathbf{D}^k$  for all  $k \in \mathbb{N} \cup \{0\}$ .*

**P r o o f.** This equality is a corollary of Theorem 6 and the relations (4.1).  $\square$

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### Мероморфные функции вполне регулярного роста в верхней полуплоскости

*Ключевые слова:* истинно мероморфная функция, полная мера, функция роста, функция вполне регулярного роста, коэффициенты Фурье, сопряженный ряд, индикатор.

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Строго положительная, непрерывная, неограниченная, возрастающая функция  $\gamma(r)$  на полуоси  $[0, +\infty)$  называется функцией роста. Пусть функция роста  $\gamma(r)$  для некоторого  $M > 0$  и для всех  $r > 0$  удовлетворяет условию  $\gamma(2r) \leq M\gamma(r)$ . В статье рассматривается пространство  $JM(\gamma(r))^o$  мероморфных функций вполне регулярного роста в верхней полуплоскости относительно функции роста  $\gamma$ . Получен критерий принадлежности мероморфной функции  $f$  к пространству  $JM(\gamma(r))^o$ . Введено определение индикатора функции пространства  $JM(\gamma(r))^o$ . Доказано, что индикатор принадлежит пространству  $L^p[0, \pi]$  для всех  $p > 1$ .

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