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ON THE PROBLEM OF CONTROLLING A SECOND-ORDER NONLINEAR SYSTEM BY MEANS OF DISCRETE CONTROL UNDER DISTURBANCE

The problem of bringing a trajectory to a neighborhood of zero under disturbance is considered in terms of a differential pursuit game. The dynamics are described by a nonlinear autonomous system of second-order differential equations. The set of values of the pursuer's controls is finite, and that of the evader (disturbance) is compact. The goal of the control, that is, the goal of the pursuer, is to bring, within a finite time, the trajectory to any predetermined neighborhood of zero, regardless of the actions of the disturbance. To construct the control, the pursuer knows only the phase coordinates and the value of the velocity at some discrete moments of time and the choice of the disturbance control is unknown. Conditions are obtained for the existence of a set of initial positions, from each point of which a capture occurs in the specified sense. Moreover, this set contains a certain neighborhood of zero. The winning control is constructed constructively and has an additional property specified in the theorem. In addition, an estimate of the time required to bring the speed from one given point to the neighborhood of another given point under disturbance conditions was obtained.

Keywords: differential game, nonlinear dynamic systems, control, disturbance.

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Introduction

Differential game theory is a branch of mathematical control theory, which studies the control of objects in conflict situations described by differential equations. The works of R. Isaacs [1] originally considered differential games of pursuit–evasion and laid the foundations of the theory. At present, differential games are a substantial theory and have a wide field of research [2–7]. Methods for solving different classes of game problems have been developed: Isaacs method, based on the analysis of a certain partial derivative equation and its characteristics, Krasovsky's method of extreme aiming, Pontryagin's method and others. N. N. Krasovsky and representatives of his scientific school have created the theory of positional games, which is based on the concept of the maximal statistical bridge and the rule of extreme aiming. However, the effective construction of such bridges for the study of real conflict-controlled processes, primarily nonlinear differential games, is very difficult or even impossible. It is more convenient to construct bridges that are not maximal, but possess the property of stability and giving effectively realizable control procedures for certain classes of games having additional properties. The construction of approximations of stable bridges in nonlinear differential games, including numerically, is considered, in particular, in the works [8, 9].

Sufficient conditions for solvability of the pursuit problem in the nonlinear example of L. S. Pontryagin are obtained in [10]. In [11], sufficient conditions for the solvability of the pursuit problem in a nonlinear differential game under some additional conditions on the set of values of the right-hand side of the system of differential equations are presented. In [12], sufficient conditions for capture in a nonlinear game described by a stationary nonlinear system are obtained, and the optimality of the capture time for some case in the plane is investigated. In [12], the pursuer uses a counterstrategy. In [13] it is considered a nonlinear control problem with disturbance using the formalization of a differential game. Sufficient conditions for the existence of a winning strategy are obtained. In [14] a nonlinear two-person differential game with an integral quality criterion is considered. Players use piecewise program controls of a specific form, and the time interval is divided in two parts. Necessary and sufficient conditions for the existence of a saddle point for the game under consideration are obtained. In [15] the differential pursuit game on the plane, the dynamics of which is described by a nonlinear system of differential equations of a specific form, is considered. The target set is the origin of coordinates. The conditions for realizing capture by means of a positional counterstrategy and the characteristics of the game in explicit form are obtained, and some examples are given.

In [16] the concept of positive basis was introduced, which was effectively used in the works of [17–19] to study the controllability property of nonlinear systems described by differential equations in a finite-dimensional Euclidean space. The conditions of controllability to zero of a nonlinear autonomous system by means of discrete control with a finite set of control values are obtained.

In papers [19] and [20] the problem of capture in a nonlinear differential game analogous to the differential game of the presented paper was considered. In these papers, sufficient conditions on the parameters of the game were obtained for the existence of a neighborhood of zero from which catching occurs. Among these conditions, the key one was that some set of vectors forms a positive basis. In [20], additional properties of the winning strategy are obtained.

The presented work is a continuation of studies of [19, 20] with substantially more general dynamics. We consider the problem of bringing the trajectory to a neighborhood of zero under the influence of a disturbance in terms of a differential pursuit game. The control objective, i. e., the goal of the pursuer, is to bring the trajectory, within finite time, to any predetermined neighborhood of zero regardless of the disturbance action. In this paper, we obtain conditions for the existence of a set of initial positions from each point of which capture in the above sense occurs. In addition, an estimate of the time of velocity conversion from one given point to the neighborhood of another given point under the disturbance action is obtained.

§ 1. Statement of the problem

In the space $\mathbb{R}^k$, $k \geqslant 2$, we consider a differential game $\Gamma(x_0, \dot{x}_0)$ of two persons — a pursuer P and an evader E. The dynamics of the game are described by the system of differential equations with a discontinuous right-hand side

$$\ddot{x} = f(x, \dot{x}, u, v), \quad u \in U, \quad v \in V, \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0,$$

where x, $\dot{x} \in \mathbb{R}^k$, x is the phase vector, $\dot{x}$ is the velocity vector, u and v are controls, $U = \{u_1, \ldots, u_m\}$, $u_i \in \mathbb{R}^l$, $i = 1, \ldots, m$. The set $V \subset \mathbb{R}^s$ is a compact set. For each $u \in U$ the function $f\colon \mathbb{R}^k \times \mathbb{R}^k \times U \times V \to \mathbb{R}^k$ is continuous in the set of variables x, $\dot{x}$, v and satisfies the Lipschitz condition in x, $\dot{x}$ with a constant L independent of v. That is, the following inequality holds:

$$\big\|f(x_1, \dot{x}_1, u_i, v) - f(x_2, \dot{x}_2, u_i, v)\big\| \leqslant L\big(\|x_1 - x_2\| + \|\dot{x}_1 - \dot{x}_2\|\big),$$
$$x_1, x_2, \dot{x}_1, \dot{x}_2 \in \mathbb{R}^k, \quad v \in V, \quad i = 1, \ldots, m.$$

Here and everywhere below, we consider the Euclidean norm. By a partition σ of the interval $[0, T]$ we mean a finite partition $\{\tau_q\}_{q=0}^{\eta}$, where $0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_\eta = T$.

Definition 1.1. A *piecewise constant strategy* W of pursuer P on the interval $[0, T]$ is a pair (σ, W_σ), where σ is a partition of the interval $[0, T]$ and W_σ is a family of mappings d_r, $r = 0, 1, \ldots, \eta - 1$, that take the values $(\tau_r, x(\tau_r), \dot{x}(\tau_r))$ to the constant control $\overline{u}_r(t) \equiv \overline{u}_r \in U$, $t \in [\tau_r, \tau_{r+1})$.

Admissible control of the evader is an arbitrary measurable function $v\colon [0,\infty)\to V$. To construct control for evader E (disturbance) at the initial moment of time, the initial position x_0, $\dot x_0$ and the chosen strategy W of pursuer P are known. In addition, the players know the dynamics of the system, i. e., the function f and the sets U and V.

Definition 1.2. An *ε-capture* occurs in a game $\Gamma(x_0,\dot x_0)$ if there exists $T>0$ such that for any $\hat\varepsilon>0$ there exists a piecewise constant strategy W of pursuer P on the interval $[0,T]$ such that for any admissible evader control $v(\cdot)$ one has the inequality $\|x(\tau)\|<\hat\varepsilon$ for some $\tau\in[0,T]$.

The goal of the pursuer is to perform the ε-capture. The goal of the evader is to prevent this.

Note that for any $u\in U$ and any measurable function $v\colon[0,\infty)\to V$, $t\geqslant 0$, $x,\dot x\in\mathbb{R}^k$, the inequality holds n

$$\big\|f\big(x,\dot x,u,v(t)\big)\big\|=\big\|f\big(x,\dot x,u,v(t)\big)-f\big(0,0,u,v(t)\big)+f\big(0,0,u,v(t)\big)\big\|\leqslant \\ \leqslant L\big(\|x\|+\|\dot x\|\big)+\big\|f\big(0,0,u,v(t)\big)\big\|\leqslant L\big(\|x\|+\|\dot x\|\big)+B,$$

for some positive B. Such a B exists by due to the continuity of the function f with respect to v, the compactness of V and the finiteness of U.

Consequently, for any $T>0$ and any partition σ of the interval $[0,T]$, on each interval of the partition, the Caratheodory conditions of existence, uniqueness and right extendability of a solution of the Cauchy problem for a first-order system obtained from a given second-order system by standard substitution are satisfied. Thus, the problem statement is correct.

Let us introduce the following notation: $\operatorname{Int}A$ is the interior of the set A; $\operatorname{co}A$ is a convex hull of the set A; $D_\varepsilon(x)$ is a closed ball of radius ε centered at point x; $O_\varepsilon(x)$ is an open ball of radius ε centered at point x; $\langle a,b\rangle$ is the inner product of vectors a,b.

§ 2. ε-capture theorem

Theorem 2.1. *Let*

$$\min_{p\in\mathbb{R}^k,\|p\|=1}\ \max_{u\in U}\min_{v\in V}\langle f(0,0,u,v),p\rangle>0.$$

Then there exist $\varepsilon_0>0$, $\theta>0$ and $T>0$ such that an ε-capture occurs in time T for any x_0, $\dot x_0$ such that $\|x_0\|+\theta\|\dot x_0\|<\varepsilon_0$. Moreover, the pursuer can use a fixed-step partitioning of the time interval to construct a strategy.

P r o o f. $\mathbf{1^0}$. In this item, the value of ε_0 from the theorem condition, a number of parameters to be applied to construct the pursuer's strategy, and the value of the time interval partitioning step will be obtained.

Since the function f, for each fixed u, is continuous with respect to the set of variables x, $\dot x$, v, there exists $\varepsilon_0>0$ such that for any $\overline{x},\dot{\overline{x}}\in D_{\varepsilon_0}(0)$ the following inequality holds:

$$\min_{p\in\mathbb{R}^k,\|p\|=1}\ \max_{u\in U}\min_{v\in V}\big\langle f(\overline{x},\dot{\overline{x}},u,v),p\big\rangle>0.$$

Therefore,

$$\overline{\alpha}=\min_{\overline{x},\dot{\overline{x}}\in D_{\varepsilon_0}(0)}\ \min_{p\in\mathbb{R}^k,\|p\|=1}\ \max_{u\in U}\min_{v\in V}\big\langle f(\overline{x},\dot{\overline{x}},u,v),p\big\rangle>0. \tag{2.1}$$

Then, we show that the given ε_0 is the required one from the condition of the theorem, $x_0,\dot x_0\in O_{\varepsilon_0}(0)$, $x_0\neq 0$.

We take

$$D=\max_{\overline{x},\dot{\overline{x}}\in D_{\varepsilon_0}(0),\,u\in U,v\in V}\|f(\overline{x},\dot{\overline{x}},u,v)\|.$$

Let us define a number h of the form

$$h = \frac{\mu \overline{\alpha}}{2L},$$

where $\mu \in (0,1)$ is arbitrary fixed number. Let $\overline{x}, \dot{\overline{x}} \in D_{\varepsilon_0}(0)$, $p \in \mathbb{R}^k$, $\|p\| = 1$, $x \in D_h(\overline{x})$, $\dot{x} \in D_h(\dot{\overline{x}})$, $v \in V$ and

$$\max_{u\in U} \min_{v\in V} \big\langle f(\overline{x}, \dot{\overline{x}}, u, v), p \big\rangle = \big\langle f(\overline{x}, \dot{\overline{x}}, \overline{u}, \overline{v}), p \big\rangle .$$

Note that, by (2.1), for any $v \in V$,

$$\big\langle f(\overline{x}, \dot{\overline{x}}, \overline{u}, v), p \big\rangle \geqslant \overline{\alpha}.$$

We obtain the following estimate:

$$\begin{aligned} \langle f(x, \dot{x}, \overline{u}, v), p\rangle &= \big\langle f(\overline{x}, \dot{\overline{x}}, \overline{u}, v) - \big(f(\overline{x}, \dot{\overline{x}}, \overline{u}, v) - f(x, \dot{x}, \overline{u}, v)\big), p \big\rangle \geqslant \\ &\geqslant \overline{\alpha} - L\big(\|x - \overline{x}\| + \|\dot{x} - \dot{\overline{x}}\|\big) \geqslant \overline{\alpha} - 2Lh. \end{aligned}$$

It follows that, for any $\overline{x}, \dot{\overline{x}} \in D_{\varepsilon_0}(0)$, $p \in \mathbb{R}^k$, $\|p\| = 1$, $x \in D_h(\overline{x})$, $\dot{x} \in D_h(\dot{\overline{x}})$, $v \in V$, the following inequality holds

$$\langle f(x, \dot{x}, \overline{u}, v), p\rangle \geqslant \overline{\alpha}(1-\mu) = \alpha.$$

Note that $\alpha < D$.

Let us fix a number $0 < \delta < \varepsilon_0$. Choose a partitioning step

$$\Delta = \min\left\{\frac{\alpha\delta}{D^2 N}, \frac{h}{D}, \frac{h}{\varepsilon_0}\right\},$$

where $N \geqslant 1$ is an arbitrary number. Thus $\tau_1 = \Delta, \ldots, \tau_\eta = \eta\Delta$. Note that if $x(t), \dot{x}(t) \in D_{\varepsilon_0}(0)$ for all $t \in [\tau_i, \tau_{i+1}]$, $i \in \{0, \ldots, \eta - 1\}$, then $\big\|x(t) - x(\tau_i)\big\| \leqslant \varepsilon_0(t - \tau_i) \leqslant \varepsilon_0\Delta \leqslant h$ and $\big\|\dot{x}(t) - \dot{x}(\tau_i)\big\| \leqslant D(t - \tau_i) \leqslant D\Delta \leqslant h$, $t \in [\tau_i, \tau_{i+1}]$.

$\mathbf{2^0}$. In the given item, we estimate the distance between $\dot{x}(t)$ and some target point $\xi \in O_{\varepsilon_0}(0)$ for all $t \in [\tau_i, \tau_{i+1}]$, $i \in \{0, \ldots, \eta - 1\}$, when the evader's control $v(\cdot)$ is arbitrary admissible, the pursuer's control $u(\cdot)$ is constant at each partitioning interval. The values of the pursuer's control at each partitioning interval are defined below.

Assume that $x(t) \in O_{\varepsilon_0}(0)$, $t \in [\tau_i, \tau_{i+1})$. Next, in $\mathbf{4^0}$ the conditions guaranteeing the fulfillment of this inclusion will be obtained.

Let $D_{\|\dot{x}(\tau_i)-\xi\|}(\xi) \subset D_{\varepsilon_0}(0)$ be true for the target point ξ. Then, if $\big\|\dot{x}(t) - \xi\big\| < \big\|\dot{x}(\tau_i) - \xi\big\|$, $t \in (\tau_i, \tau_{i+1})$, then $\dot{x}(t) \in O_{\varepsilon_0}(0)$, $t \in [\tau_i, \tau_{i+1})$.

We will choose the value of the pursuer's control as follows. If $\xi - \dot{x}(\tau_i) = 0$, then $\overline{u}_0 \in U$ is arbitrary. Then, the inequality $\big\|\xi - \dot{x}(t)\big\| \leqslant (t - \tau_i)D \leqslant \Delta D \leqslant \alpha\delta/D < \delta$ for all $t \in [\tau_i, \tau_{i+1}]$ is true. If $\xi - \dot{x}(\tau_i) \neq 0$, then denote $p_i = \big(\xi - \dot{x}(\tau_i)\big)/\big\|\xi - \dot{x}(\tau_i)\big\|$ and $\overline{u}_i \in U$ is chosen from the following maximum:

$$\max_{u\in U} \min_{v\in V} \big\langle f\big(x(\tau_i), \dot{x}(\tau_i), u, v\big), p_i \big\rangle = \big\langle f\big(x(\tau_i), \dot{x}(\tau_i), \overline{u}_i, \overline{v}\big), p_i \big\rangle .$$

Thus, due to $\mathbf{1^0}$, for any $t \in [\tau_i, \tau_{i+1})$, $v \in V$, the following inequality holds:

$$\big\langle f\big(x(t), \dot{x}(t), \overline{u}_i, v\big), p_i \big\rangle \geqslant \alpha.$$

Let $t \in [\tau_i, \tau_{i+1})$. Let's estimate the square of the norm $\xi - \dot{x}(t)$.

$$\big\|\xi - \dot{x}(t)\big\|^2 = \Big\|\xi - \dot{x}(\tau_i) - \int_{\tau_i}^{t} f\big(x(s), \dot{x}(s), \overline{u}_i, v(s)\big)\, ds\Big\|^2 = \big\|\xi - \dot{x}(\tau_i)\big\|^2 +$$
$$+ \Big\|\int_{\tau_i}^{t} f\big(x(s), \dot{x}(s), \overline{u}_i, v(s)\big)\, ds\Big\|^2 - 2\int_{\tau_i}^{t} \big\langle f\big(x(s), \dot{x}(s), \overline{u}_i, v(s)\big), \xi - \dot{x}(\tau_i)\big\rangle\, ds \leqslant$$
$$\leqslant \big\|\xi - \dot{x}(\tau_i)\big\|^2 + D^2(t-\tau_i)^2 - 2(t-\tau_i)\alpha\big\|\xi - \dot{x}(\tau_i)\big\|.$$

Let's estimate the last polynomial $A = \big\|\xi - \dot{x}(\tau_i)\big\|^2 + D^2(t-\tau_i)^2 - 2(t-\tau_i)\alpha\big\|\xi - \dot{x}(\tau_i)\big\|$.

If $\big\|\xi - \dot{x}(\tau_i)\big\| \leqslant \delta$, then polynomial A reaches its maximum at $\big\|\xi - \dot{x}(\tau_i)\big\| = 0$ or at $\big\|\xi - \dot{x}(\tau_i)\big\| = \delta$. Then, if $\big\|\xi - \dot{x}(\tau_i)\big\| = 0$, then

$$\Delta \leqslant \frac{\alpha\delta}{D^2 N} \Rightarrow A = D^2(t-\tau_i)^2 \leqslant D^2\Delta^2 \leqslant \frac{\alpha^2\delta^2}{D^2N^2} < \delta^2.$$

If $\big\|\xi - \dot{x}(\tau_i)\big\| = \delta$, then

$$A = \delta^2 + D^2(t-\tau_i)^2 - 2(t-\tau_i)\alpha\delta \leqslant \delta^2 + D^2(t-\tau_i)\Delta - 2(t-\tau_i)\alpha\delta \leqslant$$
$$\leqslant \delta^2 + \frac{D^2(t-\tau_i)\alpha\delta}{D^2N} - 2(t-\tau_i)\alpha\delta \leqslant \delta^2 - (t-\tau_i)\alpha\delta < \delta^2.$$

If $\big\|\xi - \dot{x}(\tau_i)\big\| \geqslant \delta$ and $t \in (\tau_i, \tau_{i+1})$, then

$$A \leqslant \big\|\xi - \dot{x}(\tau_i)\big\|^2 + \frac{D^2(t-\tau_i)\alpha\delta}{D^2N} - 2(t-\tau_i)\alpha\big\|\xi - \dot{x}(\tau_i)\big\| \leqslant$$
$$\leqslant \big\|\xi - \dot{x}(\tau_i)\big\|^2 + (t-\tau_i)\alpha\delta - 2(t-\tau_i)\alpha\big\|\xi - \dot{x}(\tau_i)\big\| \leqslant$$
$$\leqslant \big\|\xi - \dot{x}(\tau_i)\big\|^2 - (t-\tau_i)\alpha\big\|\xi - \dot{x}(\tau_i)\big\| < \big\|\xi - \dot{x}(\tau_i)\big\|^2.$$

If $\big\|\xi - \dot{x}(\tau_i)\big\| \geqslant \delta$ and $t = \tau_{i+1}$, then

$$A \leqslant \big\|\xi - \dot{x}(\tau_i)\big\|^2 + \frac{D^2\Delta\alpha\delta}{D^2N} - 2\Delta\alpha\big\|\xi - \dot{x}(\tau_i)\big\| =$$
$$= \big\|\xi - \dot{x}(\tau_i)\big\|^2 - \big\|\xi - \dot{x}(\tau_i)\big\|\left(2\Delta\alpha - \frac{\Delta\alpha}{N}\right) - \frac{\Delta\alpha\big\|\xi - \dot{x}(\tau_i)\big\|}{N} + \frac{\Delta\alpha\delta}{N} <$$
$$< \big\|\xi - \dot{x}(\tau_i)\big\|^2 - \left(2\Delta\alpha - \frac{\Delta\alpha}{N}\right)\big\|\xi - \dot{x}(\tau_i)\big\| < \left(\big\|\xi - \dot{x}(\tau_i)\big\| - \Delta\alpha\left(1 - \frac{1}{2N}\right)\right)^2.$$

Thus, if $\big\|\xi - \dot{x}(\tau_i)\big\| \geqslant \delta$, then $\big\|\xi - \dot{x}(t)\big\| < \big\|\xi - \dot{x}(\tau_i)\big\|$ for all $t \in (\tau_i, \tau_{i+1}]$ and

$$\big\|\xi - \dot{x}(\tau_{i+1})\big\| < \big\|\xi - \dot{x}(\tau_i)\big\| - \Delta\alpha\left(1 - \frac{1}{2N}\right).$$

If $\big\|\xi - \dot{x}(\tau_i)\big\| < \delta$, then $\big\|\xi - \dot{x}(t)\big\| < \delta$ for all $t \in (\tau_i, \tau_{i+1}]$.

$\mathbf{3^0}$. In this item, we estimate the time required for $\dot{x}(\cdot)$ to reach $O_\delta(\xi)$.

Without loss of generality, we consider the velocity function from time $\tau_0 = 0$. If $\dot{x}_0 \in O_\delta(\xi)$, then the time to reach $O_\delta(\xi)$ equals zero. Let us consider the case $\dot{x}_0 \notin O_\delta(\xi)$. At each interval $[\tau_i, \tau_{i+1})$, we will choose the pursuer's control according to $\mathbf{2^0}$. Due to the estimates of $\mathbf{2^0}$, if $\big\|\xi - \dot{x}(\tau_i)\big\| > \delta$, then

$$\big\|\xi - \dot{x}(\tau_{i+1})\big\| < \big\|\xi - \dot{x}(\tau_i)\big\| - \Delta\alpha\left(1 - \frac{1}{2N}\right).$$

Therefore, $\|\xi - \dot{x}(\tau_q)\| < \delta$ for some q that satisfies the following inequality:

$$q \leqslant \left[\frac{\|\xi - \dot{x}_0\| - \delta}{\Delta\alpha\big(1 - 1/(2N)\big)} \right] + 1 = \hat{q}.$$

In fact, if $\|\xi - \dot{x}(\tau_{\hat{q}})\| \geqslant \delta$, then, by the estimates of item $\mathbf{2^0}$, $\|\xi - \dot{x}(\tau_i)\| > \delta$ for all $i = 0, \ldots, \hat{q}-1$. Then

$$\|\xi - \dot{x}(\tau_{\hat{q}})\| < \|\xi - \dot{x}_0\| - \hat{q}\Delta\alpha\left(1 - \frac{1}{2N}\right) < 0.$$

It's a contradiction.

Let us estimate τ_q:

$$\begin{aligned}\tau_q = q\Delta &\leqslant \frac{\big(\|\xi - \dot{x}_0\| - \delta\big)\Delta}{\Delta\alpha\big(1 - 1/(2N)\big)} + \Delta = \frac{\|\xi - \dot{x}_0\| - \delta}{\alpha\big(1 - 1/(2N)\big)} + \Delta \leqslant \\ &\leqslant \frac{\|\xi - \dot{x}_0\| - \delta}{\alpha\big(1 - 1/(2N)\big)} + \frac{\alpha\delta}{D^2 N} < \frac{\|\xi - \dot{x}_0\| - \delta}{\alpha\big(1 - 1/(2N)\big)} + \frac{\delta}{\alpha N} = \\ &= \frac{\|\xi - \dot{x}_0\| - \delta\big(1 - 1/N + 1/(2N^2)\big)}{\alpha\big(1 - 1/(2N)\big)} \doteq \hat{T}(\dot{x}_0, \xi, \delta).\end{aligned}$$

$\mathbf{4^0}$. Here we construct the pursuer's strategy.

First, we choose $\xi = 0$ as the target point. Take δ such that $\delta \leqslant \varepsilon_0/3$ and $\varepsilon_0/(3\delta) = l$ for some $l \in \mathbb{N}$. We denote by $\hat{t}_0$ the moment of partition of the time interval for which $\dot{x}(\hat{t}_0) \in O_\delta(0)$. According to the item $\mathbf{3^0}$, $\hat{t}_0 \leqslant \hat{T}(\dot{x}_0, 0, \delta)$. Wherein

$$\hat{T}(\dot{x}_0, 0, \delta) < \frac{\|\dot{x}_0\|}{\alpha\big(1 - 1/(2N)\big)}. \tag{2.2}$$

Due to the estimates of $\mathbf{2^0}$, $\dot{x}(t) \in O_{\varepsilon_0}(0)$ for all $t \in [0, \hat{t}_0]$ if $x(t) \in O_{\varepsilon_0}(0)$ for all $t \in [0, \hat{t}_0]$. Hence, if

$$\|x_0\| + \hat{t}_0\|\dot{x}_0\| < \varepsilon_0,$$

then $x(t) \in O_{\varepsilon_0}(0)$ for all $t \in [0, \hat{t}_0]$.

Further, if $x(\hat{t}_0) = 0$, then the game is complete. Otherwise, we denote $\zeta = -x(\hat{t}_0)/\|x(\hat{t}_0)\|$ and choose $\delta\zeta$ as the target point. By the choice of δ, this target point satisfies the constraints of item $\mathbf{2^0}$. Let us denote by $\hat{t}_1$ the moment of time interval partitioning for which $\dot{x}(\hat{t}_1) \in O_\delta(\delta\zeta)$. According to $\mathbf{3^0}$, $\hat{t}_1 - \hat{t}_0 \leqslant \hat{T}\big(\dot{x}(\hat{t}_0), \delta\zeta, \delta\big)$. And in this case

$$\hat{T}\big(\dot{x}(\hat{t}_0), \delta\zeta, \delta\big) < \frac{2\delta - \delta\big(1 - 1/N + 1/(2N^2)\big)}{\alpha\big(1 - 1/(2N)\big)} = \frac{\delta + \delta\big(1/N - 1/(2N^2)\big)}{\alpha\big(1 - 1/(2N)\big)} \doteq \hat{\Delta}. \tag{2.3}$$

Next, we choose $2\delta\zeta$ as the target point and denote by $\hat{t}_2$ the moment of time interval partitioning for which $\dot{x}(\hat{t}_2) \in O_\delta(2\delta\zeta)$. According to $\mathbf{3^0}$, $\hat{t}_2 - \hat{t}_1 \leqslant \hat{T}\big(\dot{x}(\hat{t}_1), 2\delta\zeta, \delta\big)$. In this case, $T\big(\dot{x}(\hat{t}_1), 2\delta\zeta, \delta\big)$ is estimated similarly to (2.3).

And so on, until the trajectory $\dot{x}(\cdot)$ reaches the set $O_\delta(l\delta\zeta)$ by the moment $\hat{t}_l$. After that and until the end of the game, we choose $l\delta\zeta = \zeta\varepsilon_0/3$ as the target point. According to the estimates of item $\mathbf{2^0}$, the trajectory $\dot{x}(\cdot)$ will continue to stay in $O_\delta(l\delta\zeta)$.

Let us introduce the following notation: $\varphi(t) = \dot{x}(t) - i\delta\zeta$, when $t \in [\hat{t}_{i-1}, \hat{t}_i)$, $i = 1, \ldots, l$; $\varphi(t) = \dot{x}(t) - \zeta\varepsilon_0/3$, when $t \geqslant \hat{t}_l$; $\psi(t) = i\delta\zeta$, when $t \in [\hat{t}_{i-1}, \hat{t}_i)$, $i = 1, \ldots, l$; $\psi(t) = \zeta\varepsilon_0/3$, when $t \geqslant \hat{t}_l$.

Thus, the form of $\dot{x}(t) = \psi(t)+\varphi(t)$, $t \geqslant \hat{t}_0$, is valid. Note that $\|\varphi(t)\| \leqslant 2\delta$, when $t \in [\hat{t}_0, \hat{t}_l)$, and $\|\varphi(t)\| \leqslant \delta$, when $t \geqslant \hat{t}_l$.

Estimate the norm of $x(t)$, when $t \in [\hat{t}_0, \hat{t}_1)$:

$$\|x(t)\| = \Big\|x(\hat{t}_0) + (t-\hat{t}_0)\delta\zeta + \int_{\hat{t}_0}^{t} \varphi(s)\,ds\Big\| \leqslant \big|\|x(\hat{t}_0)\| - (t-\hat{t}_0)\delta\big| + (t-\hat{t}_0)2\delta. \tag{2.4}$$

If for some $t \in [\hat{t}_0, \hat{t}_1)$ the equality $\|x(\hat{t}_0)\| - (t-\hat{t}_0)\delta = 0$ is true, then we denote this moment by $\hat{T}$ and assume that the game is over. Otherwise, we have the following estimate:

$$\|x(\hat{t}_1)\| \leqslant \|x(\hat{t}_0)\| + (\hat{t}_1 - \hat{t}_0)\delta < \|x(\hat{t}_0)\| + \hat{\Delta}\delta.$$

Estimate the norm of $x(t)$ when $t \in [\hat{t}_1, \hat{t}_2)$. Similarly to (2.4), we obtain the following estimation:

$$\|x(t)\| \leqslant \Big|\|x(\hat{t}_0)\| - (\hat{t}_1 - \hat{t}_0)\delta - (t-\hat{t}_1)2\delta\Big| + (t-\hat{t}_0)2\delta.$$

If for some $t \in [\hat{t}_1, \hat{t}_2)$ the equality $\|x(\hat{t}_0)\| - (\hat{t}_1-\hat{t}_0)\delta - (t-\hat{t}_1)2\delta = 0$ holds, then we denote this moment by $\hat{T}$ and assume the game to be over. Otherwise, we have the following estimate

$$\|x(\hat{t}_2)\| \leqslant \|x(\hat{t}_0)\| + (\hat{t}_1-\hat{t}_0)\delta - (\hat{t}_2-\hat{t}_1)2\delta + (\hat{t}_2-\hat{t}_1)2\delta < \|x(\hat{t}_0)\| + \hat{\Delta}\delta.$$

Next, we similarly estimate the norm of $x(\hat{t}_3)$:

$$\|x(\hat{t}_2)\| < \|x(\hat{t}_0)\| + \hat{\Delta}\delta - (\hat{t}_3 - \hat{t}_2)\delta.$$

And so on, until the moment $\hat{t}_l$. Note that the following inequality is true

$$\|x(t)\| < \|x(\hat{t}_0)\| + \hat{\Delta}\delta, \quad t \geqslant \hat{t}_0. \tag{2.5}$$

If $\hat{T}$ is not determined to the moment $\hat{t}_l$, we determine it from the following equality:

$$\|x(\hat{t}_0)\| - (\hat{t}_1-\hat{t}_0)\delta - \ldots - (\hat{t}_l - \hat{t}_{l-1})l\delta - (\hat{T}-\hat{t}_l)l\delta = 0.$$

Since $\|x(\hat{t}_0)\| < \varepsilon_0$ and $l = \varepsilon_0/(3\delta)$, then $\hat{T} - \hat{t}_l < 3$.

Thus, we have the following estimate of the norm of $x(\hat{T})$:

$$\|x(\hat{T})\| \leqslant \Big\|\int_{\hat{t}_0}^{\hat{T}} \varphi(s)\,ds\Big\| \leqslant (\hat{t}_l - \hat{t}_0)2\delta + (\hat{T}-\hat{t}_l)\delta < l\hat{\Delta}2\delta + 3\delta.$$

The last expression tends to 0 when $\delta \to 0$.

Since $\hat{t}_0 \leqslant \hat{T}(\dot{x}_0, 0, \delta)$, then, by (2.2), we can determine θ from the condition of the theorem as follows:

$$\theta = \max\left\{1, \frac{\varepsilon_0}{\alpha\big(1 - 1/(2N)\big)}\right\}.$$

Then, if $\|x_0\| + \theta\|\dot{x}_0\| < \varepsilon_0$, then $x_0, \dot{x}_0 \in O_{\varepsilon_0}(0)$ and $x(t), \dot{x}(t) \in O_{\varepsilon_0}(0)$ for all $t \in [0, \hat{t}_0]$. Then, by (2.5), choosing δ with an additional constraint of the form $\|x_0\| + \theta\|\dot{x}_0\| + \hat{\Delta}\delta \leqslant \varepsilon_0$, ensure that the inequality $\|x(t)\| < \varepsilon_0$, $t \in [\hat{t}_0, \hat{T}]$ holds.

Let us estimate $\hat{T}$ using (2.2) and (2.3):

$$\begin{aligned}\hat{T} &= \hat{t}_0 + (\hat{t}_l - \hat{t}_0) + (\hat{T} - \hat{t}_l) < \hat{T}(\dot{x}_0, 0, \delta) + l\hat{\Delta} + 3 < \\ &< \frac{\|\dot{x}_0\|}{\alpha\big(1 - 1/(2N)\big)} + \frac{l\delta + l\delta\big(1/N - 1/(2N^2)\big)}{\alpha\big(1 - 1/(2N)\big)} + 3 < \\ &< \frac{\varepsilon_0}{\alpha\big(1 - 1/(2N)\big)} + \frac{\frac{\varepsilon_0}{3} + \frac{\varepsilon_0}{3}\big(1/N - 1/(2N^2)\big)}{\alpha\big(1 - 1/(2N)\big)} + 3 \doteq T.\end{aligned} \tag{2.6}$$

The obtained T does not depend on δ, so it is required.

The theorem is proved. □

Remark 2.1. Since, obtained in the proof of Theorem 2.1, the pursuer's winning strategy guarantees the fulfillment of the inclusion $x(t), \dot{x}(t) \in O_{\varepsilon_0}(0)$, $t \in [0, T]$, it is sufficient that the function f satisfies on x, $\dot{x}$ the local Lipschitz condition with a constant independent of v.

Corollary 2.1. *Let the conditions of Theorem 2.1 are satisfied, $x_0, \dot{x}_0, \xi \in O_{\varepsilon_0}(0)$ and*

$$\frac{\|\xi - \dot{x}_0\|}{\alpha(\varepsilon_0)} < \frac{\varepsilon_0 - \|x_0\|}{\varepsilon_0},$$

where

$$\alpha(r) = \min_{x, \dot{x} \in D_r(0)} \min_{p \in \mathbb{R}^k, \|p\|=1} \max_{u \in U} \min_{v \in V} \langle f(x, \dot{x}, u, v), p \rangle.$$

Then for any $\delta > 0$ there exists a piecewise constant strategy W of the pursuer P on the interval $[0, T_\xi]$ such that for any admissible control of the evader $v(\cdot)$ the following inequality holds $\big\|\xi - \dot{x}(\tau)\big\| < \delta$ for some $\tau \in [0, T_\xi]$, where $T_\xi = \|\xi - \dot{x}_0\|/\alpha(\varepsilon_0)$.

P r o o f. In item $\mathbf{4^0}$ of the proof of Theorem 2.1, we performed the procedure of bringing the trajectory $\dot{x}(\cdot)$ from $O_\delta(0)$ to $O_\delta(\hat{\xi})$, where $\hat{\xi} = -\dfrac{x(\hat{t}_0)\varepsilon_0}{3\big\|x(\hat{t}_0)\big\|}$. In this case, the trajectory $\dot{x}(\cdot)$ does not leave the set $\{y \in \mathbb{R}^k \mid y = \lambda\hat{\xi},\ \lambda \in [0, 1]\} + O_{2\delta}(0)$. An estimate of the time of this translation is obtained in (2.6) (second summand). Similarly, we will bring the trajectory $\dot{x}(\cdot)$ from $\dot{x}_0$ to $O_\delta(\xi)$.

Thus, if $x(t) \in O_{\varepsilon_0}(0)$ for all $t \in \big[0, T_\xi(\mu, N)\big]$, where

$$T_\xi(\mu, N) = \frac{\|\xi - \dot{x}_0\| + \|\xi - \dot{x}_0\|\big(1/N - 1/(2N^2)\big)}{\alpha\big(1 - 1/(2N)\big)} = \frac{\|\xi - \dot{x}_0\|\big(1 + 1/N - 1/(2N^2)\big)}{\overline{\alpha}(1 - \mu)\big(1 - 1/(2N)\big)},$$

then for any $\delta > 0$ the trajectory $\dot{x}(\cdot)$ can be brought from $\dot{x}_0$ to $O_\delta(\xi)$ in time $T_\xi(\mu, N)$. Note that this happens for any $\mu \in (0, 1)$, $N \geqslant 1$. Furthermore, to fulfill the inclusion $\dot{x}(t) \in O_{\varepsilon_0}(0)$ for all $t \in \big[0, T_\xi(\mu, N)\big]$, it suffices to fulfill the inequality $2\delta < \min\big\{\varepsilon_0 - \|\xi\|, \varepsilon_0 - \|\dot{x}_0\|\big\}$.

Note that

$$\inf\big\{T_\xi(\mu, N) \mid \mu \in (0, 1),\ N \geqslant 1\big\} = \frac{\|\xi - \dot{x}_0\|}{\overline{\alpha}} \doteq T_\xi,$$

and for any $\overline{T} > T_\xi$ there exist $\mu \in (0, 1)$, $N \geqslant 0$ such that $T_\xi(\mu, N) < \overline{T}$. Therefore, for any $\overline{T} > T_\xi$ и $\delta > 0$, the trajectory $\dot{x}(\cdot)$ can be brought from $\dot{x}_0$ в $O_\delta(\xi)$ in time $\overline{T}$. Since the trajectories $x(\cdot)$, $\dot{x}(\cdot)$ do not leave the set $O_{\varepsilon_0}(0)$, then the function f is bounded by the value D from clause $\mathbf{1^0}$ of the proof of Theorem 2.1. Let $\overline{T} < T_\xi + \delta/(2D)$. The trajectory $\dot{x}(\cdot)$ can be brought to $O_{\delta/2}(\xi)$ in time $\overline{T}$, that is, at some moment $\tau \in [0, \overline{T}]$, the inequality $\big\|\xi - \dot{x}(\tau)\big\| < \delta/2$

is true. Therefore, $\|\dot{x}(\tau - \delta/(2D)) - \xi\| < \delta$, that is, the trajectory $\dot{x}(\cdot)$ is brought to $O_\delta(\xi)$ in time T_ξ.

Since $\overline{\alpha} = \alpha(\varepsilon_0)$, the conditions of Corollary 2.1 guarantee that the trajectory $\dot{x}(\cdot)$ is brought to $O_\delta(\xi)$ in time T_ξ for any pregiven $\delta > 0$.

The corollary is proved. □

Example 2.1. Consider the dynamics in $\mathbb{R}^2$ of the following form:

$$\ddot{x} = A(x, \dot{x}, v)u,$$
$$U = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}, \quad V = [-\beta, \beta],$$

where $0 < \beta < \frac{\pi}{4}$. Here $A(\cdot,\cdot,\cdot)$ is a square matrix defined over the entire space $\mathbb{R}^2 \times \mathbb{R}^2 \times V$, the elements of the matrix are Lipschitz functions over the set of arguments x, $\dot{x}$, v. Moreover, $A(0, 0, v)$ is a rotation matrix over the angle v.

Let us find the minimum from the condition of the Theorem 2.1. It has the following form

$$\min_{p \in \mathbb{R}^k, \||p\||=1} \max_{u \in U} \min_{v \in V} \langle A(0, 0, v)u, p\rangle = \gamma.$$

It is not difficult to check geometrically that $\gamma = \cos(\pi/4 + \beta)$. Note that

$$\min_{p \in \mathbb{R}^k, \|p\|=1} \min_{v \in V} \max_{u \in U} \langle A(0, v)u, p\rangle = \frac{\sqrt{2}}{2} \neq \gamma.$$

However, in the previous work with dynamics of the form $\ddot{x} = f(x, \dot{x}, u) + g(x, \dot{x}, v)$ the following equality is true for all $x, \dot{x}, p \in \mathbb{R}^k$:

$$\max_{u \in U} \min_{v \in V} \langle f(x, \dot{x}, u) + g(x, \dot{x}, v), p\rangle = \min_{v \in V} \max_{u \in U} \langle f(x, \dot{x}, u) + g(x, \dot{x}, v), p\rangle.$$

§ 3. Comparison with results for less general dynamics

In [19] and [20], we considered a problem similar to the problem of the present paper with less general dynamics of the form

$$\ddot{x} = f(x, \dot{x}, u) + g(x, \dot{x}, v), \quad u \in U, \quad v \in V, \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0.$$

Here, the sets U, V are defined similarly. The function $f\colon \mathbb{R}^k \times \mathbb{R}^k \times U \to \mathbb{R}^k$ for every $u \in U$ is Lipschitz continuous with respect to the variables x and $\dot{x}$. The function $g\colon \mathbb{R}^k \times \mathbb{R}^k \times V \to \mathbb{R}^k$ is Lipschitz continuous with respect to the set of variables. That is, there exist positive numbers L_1, L_2 such that

$$\|f(x_1, \dot{x}_1, u_i) - f(x_2, \dot{x}_2, u_i)\| \leqslant L_1(\|x_1 - x_2\| + \|\dot{x}_1 - \dot{x}_2\|),$$
$$x_1, x_2, \dot{x}_1, \dot{x}_2 \in \mathbb{R}^k, \quad i = 1, \dots, m,$$
$$\|g(x_1, \dot{x}_1, v_1) - g(x_2, \dot{x}_2, v_2)\| \leqslant L_2(\|x_1 - x_2\| + \|\dot{x}_1 - \dot{x}_2\| + \|v_1 - v_2\|),$$
$$x_1, x_2, \dot{x}_1, \dot{x}_2 \in \mathbb{R}^k, \quad v_1, v_2 \in V.$$

In [20] the theorem on ε-capture is proved, for the formulation of which it is necessary to introduce an additional definition.

Definition 3.1 (see [16]). The set of vectors $a_1, \dots, a_n \in \mathbb{R}^k$ is called *positive basis* if for any point $\xi \in \mathbb{R}^k$ there exist real numbers $\mu_1, \dots, \mu_n \geqslant 0$ such that $\xi = \sum_{i=1}^{n} \mu_i a_i$.

Theorem 3.1 (see [20]). *Let $f(0,u_1),\ldots,f(0,u_m)$ set up a positive basis and*

$$-g(0,0,V) \subset \mathrm{Int}\left(\mathrm{co}\left\{f(0,0,u_1),\ldots,f(0,0,u_m)\right\}\right).$$

Then there exist $\varepsilon_0>0$, $\theta>0$ and $T>0$ such that an ε-capture occurs in time T for any $x_0,\dot{x}_0$ such that $\|x_0\|+\theta\|\dot{x}_0\|<\varepsilon_0$. Moreover, the pursuer only needs to use a fixed-step partitioning of the time interval to construct a strategy.

We use the following property of a positive basis: vectors $a_1,\ldots,a_n\in\mathbb{R}^k$ form a positive basis if and only if for any vector $p\in\mathbb{R}^k$, $\|p\|=1$ there exists an index $j\in\{1,\ldots,n\}$ such that $\langle a_j,p\rangle>0$.

Due to the conditions of Theorem 3.1 and the properties of the positive basis (see [16]), for each $v\in V$ the vectors $f(0,0,u_1)+g(0,0,v),\ldots,f(0,0,u_m)+g(0,0,v)$ form a positive basis. Hence, for any $v\in V$, $p\in\mathbb{R}^k$, $\|p\|=1$, there exists an index $j\in\{1,\ldots,n\}$ such that $\langle f(0,0,u_j)+g(0,0,v),p\rangle>0$. Thus, due to the Lipschitz properties of the functions f and g, the following inequality is true

$$\min_{p\in\mathbb{R}^k,\|p\|=1}\max_{u\in U}\min_{v\in V}\langle f(0,0,u)+g(0,0,v),p\rangle>0.$$

That is, the condition of Theorem 2.1 is satisfied.

On the other hand, if the last inequality is true, i. e., if the condition of Theorem 2.1 holds, then for any $v\in V$, $p\in\mathbb{R}^k$, $\|p\|=1$, there exists an index $j\in\{1,\ldots,n\}$ such that $\langle f(0,0,u_j)+g(0,0,v),p\rangle>0$. Hence, by virtue of the properties of the positive basis (see [16]), there exists a vector $\xi\in\mathbb{R}^k$ such that the functions $\hat{f}(x,\dot{x},u)=f(x,\dot{x},u)-\xi$ and $\hat{g}(x,\dot{x},v)=g(x,\dot{x},v)+\xi$ satisfies the conditions of Theorem 3.1. Thus, Theorem 2.1 and Theorem 3.1 are equivalent for dynamics that are admissible for both theorems.

Conclusion

The problem of bringing the trajectory to the neighborhood of zero under the influence of a disturbance is considered in terms of a differential pursuit game in which the dynamics is described by a nonlinear autonomous system of differential equations of the second order. The control takes place on a finite interval, during which the pursuer needs to ensure that the system can be brought to any predetermined neighborhood of zero, regardless of the disturbance. To construct the control the pursuer knows only phase coordinates and velocity value at some discrete moments of time and the choice of the disturbance control is unknown. Conditions for the existence of a set of initial positions from each point of which a capture in the specified sense occurs are obtained. Moreover, this set contains some neighborhood of zero. The winning control is built constructively and can be constructed with a fixed time interval partitioning step. In addition, an estimate of the time to bring the velocity from one given point to the neighborhood of another given point under disturbance conditions is obtained. In this case, the time estimation does not depend on the radius of the selected neighborhood. Additionally, a comparison with previous results with less general dynamics is performed, which shows the identity of the results for the same dynamics.

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О задаче управления нелинейной системой второго порядка посредством дискретного управления в условиях воздействия помехи

Ключевые слова: дифференциальная игра, нелинейные динамические системы, управление, помеха.

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Рассматривается задача приведения траектории в окрестность нуля в условиях воздействия помехи в терминах дифференциальной игры преследования. Динамика описывается нелинейной автономной системой дифференциальных уравнений второго порядка. Множество значений управлений преследователя является конечным, убегающего (помехи) — компакт. Целью управления, то есть целью преследователя, является приведение, в рамках конечного времени, траектории в любую наперед заданную окрестность нуля вне зависимости от действий помехи. Для построения управления преследователю известны только фазовые координаты и значение скорости в некоторые дискретные моменты времени и неизвестен выбор управления помехи. Получены условия существования множества начальных положений, из каждой точки которого происходит поимка в указанном смысле. Причем это множество содержит некоторую окрестность нуля. Выигрышное управление строится конструктивно и имеет дополнительное свойство, указанное в теореме. Кроме того, получена оценка времени приведения скорости из одной заданной точки в окрестность другой заданной точки в условиях воздействия помехи.

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