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ON THE CONSTRUCTION OF PARTIALLY NON-ANTICIPATIVE MULTISELECTOR AND ITS APPLICATION TO DYNAMIC OPTIMIZATION PROBLEMS

Let sets of functions Z and Ω on the time interval T be given, let there also be a multifunction (m/f) α acting from Ω to Z and a finite set Δ of moments from T. The work deals with the following questions: the first one is the connection between the possibility of stepwise construction (specified by Δ) of a selector z of $\alpha(\omega)$ for an unknown step-by-step implemented argument $\omega \in \Omega$ and the existence of a multiselector (m/s) β of the m/f α with a non-anticipatory property of special kind (we call it partially or Δ-non-anticipated); the second question is when and how non-anticipated m/s could be expressed by means of partially non-anticipated one; and the last question is how to build the above Δ-non-anticipated m/s β for a given pair (α, Δ).

The consideration of these questions is motivated by the presence of such step-by-step procedures in the differential game theory, for example, in the alternating integral method, in pursuit–evasion problems posed with use of counter-strategies, and in the method of guide control.

It is shown that the step-by-step construction of the value $z \in \alpha(\omega)$ can be carried out for any steps-implemented argument ω if and only if the above m/s β is non-empty-valued. The key point of the work is the description of finite-step procedure for calculation of this Δ-non-anticipated m/s β. Conditions are given that guarantee the m/s β be a non-anticipative one. Illustrative examples are considered that include, in particular, control problems with disturbance.

Keywords: non-anticipative multi-selectors, set-valued strategies, optimization of guarantee.

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Introduction

Let there be non-empty sets of functions Z and Ω defined on the interval T. Let there also be a m/f S defined on Ω with values in Z and a set $N \subset \mathrm{Z}$. We will interpret these objects as an abstract control problem under conditions of uncertainty (namely, an abstract game problem of retention): the uncontrolled factor ω from the set Ω, acting on the dynamics S of the system, determines the bundle of possible trajectories $S(\omega) \subset \mathrm{Z}$; the task of the control side is to select the motion $z \in S(\omega)$ satisfying the phase constraints N: $z \in N$.

Without additional informational requirements, the solution of the problem is built explicitly: we put $\alpha(\omega) \triangleq N \cap S(\omega)$, $\omega \in \Omega$; then the criterion for the solvability of the problem is the non-emptiness of the m/f α values, and the solution is any of its selectors. So, if we knew the current disturbance $\bar{\omega} \in \Omega$, the retention problem would be solved by calling any trajectory z of $\alpha(\bar{\omega})$.

At the same time, in most control problems, information about the acting disturbance is not available at all. In the rest cases, the best that the control side can count on by the time $\tau \in \mathrm{T}$ is the knowledge of the disturbance on the interval $[t_0, \tau + \delta]$ with a small $\delta > 0$ (t_0 is the initial moment of the process). Such a prediction of disturbance behavior is admissible in certain control problems, as well as in auxiliary control structures, due to properties of the system dynamics (the mapping S) and restrictions describing the set of admissible disturbances Ω.

Thus, we suppose that the control side has the m/f α of global responses and the possibility of a small time-ahead disturbance prediction. Under these conditions, the following step-by-step scheme of the desired trajectory constructing naturally arises.

Let $\mathrm{T} \triangleq [t_0, \vartheta]$ and Δ, $\Delta \subset \mathrm{T}$, be a finite set of instants that splits T into a finite set of half-intervals (control steps) of length shorter then the above value δ: $\Delta \triangleq \{t_0 = \tau_0 < \tau_1 < \ldots$ $\ldots < \tau_{n_\Delta} = \vartheta\}$. We call Δ *the partition* of T and the pair (α, Δ) will be called *step-by-step conditions*. The procedure of step-by-step construction of the trajectory that meets the conditions (α, Δ) and the unknown disturbance $\bar{\omega} \in \Omega$ works as follows:

— at the instant τ_0, by the found disturbance $\omega_1 \in \Omega$ such that $(\omega_1|[\tau_0, \tau_1)) = (\bar{\omega}|[\tau_0, \tau_1))$, the control side chooses the trajectory $h_1 \in \mathrm{Z}$ corresponding to the disturbance ω_1, i. e., such that $h_1 \in \alpha(\omega_1)$; here, $(f|C)$ is the restriction of a function f to the set C;

— at any instant $\tau_i \in \Delta$, $i \in 1..(n_\Delta - 1)$, the control side finds a disturbance $\omega_{i+1} \in \Omega$ reconstructing the unknown $\bar{\omega}$ up to the moment τ_{i+1}:

$$(\omega_{i+1}|[\tau_0, \tau_{i+1})) = (\bar{\omega}|[\tau_0, \tau_{i+1}));$$

and, so, ω_{i+1} coincides with ω_i up to the moment τ_i

$$(\omega_{i+1}|[\tau_0, \tau_i)) = (\omega_i|[\tau_0, \tau_i)).$$

At the previous steps of partition Δ, we have already supplied the desired trajectory h_i for the disturbance ω_i, i. e., $h_i \in \alpha(\omega_i)$; taking this into account, the control side looks for the trajectory $h_{i+1} \in \alpha(\omega_{i+1})$ corresponding to the disturbance ω_{i+1}, that also coincides with the choice at the previous steps:

$$(h_{i+1}|[\tau_0, \tau_i)) = (h_i|[\tau_0, \tau_i)).$$

The procedure is repeated for all moments of the partition Δ except the last one, τ_{n_Δ}. As a result, we get the desired trajectory h_{n_Δ} that corresponds to the unknown in advance disturbance $\omega_{n_\Delta} = \bar{\omega}$:

$$h_{n_\Delta} \in \alpha(\omega_{n_\Delta}) = \alpha(\bar{\omega}).$$

The possibility of realization the above step-by-step procedure in response to any admissible disturbance $\omega \in \Omega$ will be referred to as *the feasibility of the conditions* (α, Δ).

The work deals with two questions. The first one is the connection between the feasibility of conditions (α, Δ) and the existence of some special multi-selector of the m/f α. The second question is the construction of this m/s for given conditions (α, Δ).

The motivation for the consideration was the above step-by-step scheme and similar ones, which starting from the convergence problem [1, Sect. III] in theory of differential games arise, for example, in the method of alternating integral [2], in pursuit–evasion problems using counter-strategies [3–5], or in controlling with a guide under functional constraints on a disturbance (see [6,7] and references). Besides game-theoretical problems, the above informational conditions for control side can be found in the field of robotics: suppose a robot-manipulator extracts from the container and submits for the further processing some parts poured into it. In this case the disturbance/uncertainty is the arrangement of the parts in the container. It changes (unpredictably as a rule) after the extraction of a part and remains practically constant/unchanged during inactivity time. So, the disturbance can be effectively predicted.

It is clear that the non-anticipatory and non-emptiness of values of α or of its m/s implies feasibility of (α, Δ) for any Δ. Thus, the conditions for the existence of non-anticipative m/s (see, for example, [8–10]) are sufficient conditions for the feasibility property. Note that the existence of a non-anticipative selector [11–13] is also closely related to the existence of a non-anticipative m/s.

In this paper, we show that the feasibility of the conditions (α, Δ) is equivalent to the existence of a *partially non-anticipative* and non-empty-valued m/s of the m/f α: the above step-by-step process can be implemented by means of this partially non-anticipative m/s for any disturbance.

Here, the property of partial non-anticipatory is understood as the classical non-anticipatory property that is satisfied at moments from Δ only. This property is certainly weaker than the classical one when it should be satisfied for all moments from T. Moreover, even the feasibility of the conditions (α, Δ) for any Δ does not in general ensure the existence of a non-anticipative non-empty-valued m/s of α.

So, to implement the above step-by-step procedure under conditions (α, Δ), we need this partially non-anticipative and non-empty-valued m/s of α. Here arises the second question: how to build such a partially non-anticipative m/s.

With the aim, for any instant $\tau \in \mathrm{T}$ we introduce a "projection" operator acting on set of all m/f with values in the set of all τ-non-anticipative m/f. Then, by means of a superposition of such "projections" (corresponding to all $\tau \in \Delta$), we get the required partially non-anticipative m/s. The procedure is completed in n_Δ steps.

It seems the idea of constructing a non-anticipative m/s of a m/f by an iterative method in order to obtain a direct solution of the dynamic optimization problem under conditions of an uncertainty appeared in [14] (see also [8, 9]). The obstacle for applications of such an iterative procedure is that in the general case it requires infinite number of iterations. For some classes of control problems (see, for example, [15, Ch. 5]), conditions are given that ensure the finiteness of iterations required to construct the function of the optimal guaranteed result, a resolving set of initial positions, or a resolving non-anticipative strategy.

The work is close to the problems considered in [7]; the constructions used are similar to those from [10] and the results supplements the results announced in [16]. In simple cases, the connection of partially non-anticipative m/s with ordinary non-anticipative m/s was noted in the course of the presentation, but is not covered in detail. The outline of the article is as follows: Sect. 1 contains basic notation and terms; in Sect. 2, a more detailed and meaningful description of the step-by-step procedure for constructing an optimal trajectory, its formalization, and the definition of the feasibility of the (α, Δ) conditions are given (see (2.3), (2.4)); in Sect. 3, the notion of a partially non-anticipative m/f (m/s) is defined and the feasibility criterion (3.10) is formulated; in Sect. 4, the above-mentioned "projection" operator on the set of all τ-non-anticipative m/f is defined (see (4.1)); then, the definition of such "projection" operator on the set of m/f that are non-anticipative for several such "moments" (see (4.18)) is given and the finite step procedure for its constructing is provided (see Theorem 4.1); in Sect. 5, illustrative examples are given: in Example 5.2, it is shown that the "projection" operators are not commutative; Example 5.3 provides the case when the feasibility of conditions (α, Δ) for arbitrary Δ not allow, nevertheless, a resolving non-anticipative strategy (a non-anticipative non-empty-valued m/s of α); in Example 5.4, an ordinal resolving non-anticipative strategy is constructed on the base of parameterized family of partially non-anticipative m/s; in Sect. 6, open formal questions and prospects for using of provided approach are briefly discussed.

§ 1. Definitions and notation

In the following, set-theoretic symbolism is used (quantifiers, propositional connectives, etc.); hereinafter $\varnothing$ is empty set, $\triangleq$ is an equality by definition, $\overset{\text{def}}{\Leftrightarrow}$ is the equivalency by definition; a family is a set all of whose elements are sets.

Let $\mathcal{P}(X)$ and $\mathcal{P}'(X)$ denote respectively the families of all (Boolean of X) and of all non-empty subsets of an arbitrary set X. If A and B are non-empty sets, then we denote by B^A the set of all mappings from A to B. If $g \in \mathcal{P}(B)^A$, then by $(\mathrm{DOM})\,[g]$ we denote the region where the m/f g takes nonempty values: $(\mathrm{DOM})\,[g] \triangleq \{a \in A \mid g(a) \neq \varnothing\}$. If $f \in B^A$ and $C \in \mathcal{P}'(A)$, then $(f|C)$, where $(f|C) \in B^C$, is the restriction of f to the set C: $(f|C)(x) \triangleq f(x)\ \forall x \in C$; in the case when $F \in \mathcal{P}'(B^A)$, we set $\boldsymbol{(}F \,|\, C\boldsymbol{)} \triangleq \{(f|C)\colon f \in F\}$.

We call a pair $(X,\prec)$ *a partially ordered set* (poset) if X is a non-empty set and $\prec\,\in\,\in \mathcal{P}(X\times X)$ is a non-strict partial order relation on X. In particular, $(\mathcal{P}(X),\subset)$ is a poset on Boolean of X with the inclusion relation $\subset$. For any poset $(X,\prec)$ and arbitrary subset $S\in\mathcal{P}(X)$ we call S a *chain* if any elements of S are comparable: $(x\prec y)\vee(y\prec x)$, $\forall x,y\in S$. If $(X,\prec)$ is a poset and $M\subset X$, denote by $\top_M$, $\top_M\in M$, the greatest element of M, if it exists: $x\prec\top_M$ for all $x\in M$.

We fix non-empty sets X, Y and T, as well as non-empty sets $\Omega\in\mathcal{P}'(\mathrm{Y}^{\mathrm{T}})$, $\mathrm{Z}\in\mathcal{P}'(\mathrm{X}^{\mathrm{T}})$ and the family $\mathcal{T}\in\mathcal{P}'(\mathcal{P}'(\mathrm{T}))$; in other words, $\mathcal{T}$ is a non-empty family of non-empty subsets of T. In order to reveal the connection with dynamic optimization problems, we will assume that T is a “time interval”: $\mathrm{T}\triangleq[t_0,\vartheta]$, $t_0,\vartheta\in\mathbb{R}$; and $\mathcal{T}$ is a chain of the form $\mathcal{T}\triangleq\{[t_0,\tau]\colon\tau\in\mathrm{T}\}$. Then Ω is a set of admissible disturbances, Z is a set of possible system trajectories.

Let $\sqsubseteq$ be a partial order on the set mappings $\mathcal{P}(\mathrm{Z})^{\Omega}$: $\forall\alpha,\beta\in\mathcal{P}(\mathrm{Z})^{\Omega}$

$$(\alpha\sqsubseteq\beta)\overset{\mathrm{def}}{\Leftrightarrow}(\alpha(\omega)\subset\beta(\omega)\ \forall\omega\in\Omega).$$

For $\omega\in\Omega$, $z\in\mathrm{Z}$ and $A\in\mathcal{P}'(\mathrm{T})$ we denote

$$\Omega(\omega|A)\triangleq\{\eta\in\Omega\mid(\eta|A)=(\omega|A)\},\qquad \mathrm{Z}(z|A)\triangleq\{g\in\mathrm{Z}\mid(g|A)=(z|A)\}.$$

Note that the family of partitions $(\{\Omega(\omega|A)\mid\omega\in\Omega)_{A\in\mathcal{T}}$ forms a chain in the set of partitions of Ω with an embedding relation (see, for example, [17, Sect. 3.1]), namely, for any $A,A'\in\mathcal{T}$, we have

$$(A\subset A')\Rightarrow((\Omega(\omega|A')\subset\Omega(\omega|A)\ \forall\omega\in\Omega).$$

Here by *partition* of a set M we call a family of its nonempty mutually disjont subsets that cover the set.

In terms of the family $\mathcal{T}$ we define the basic notion of *non-anticipatory*. Denote by $\mathbf{N}_{(\mathcal{T})}$ the set of all *non-anticipative* m/f from $\mathcal{P}(\mathrm{Z})^{\Omega}$:

$$\mathbf{N}_{(\mathcal{T})}\triangleq\left\{\mathbf{z}\in\mathcal{P}(\mathrm{Z})^{\Omega}\mid\boldsymbol{(}\mathbf{z}(\omega)\mid A\boldsymbol{)}=\boldsymbol{(}\mathbf{z}(\omega')\mid A\boldsymbol{)}\ \forall A\in\mathcal{T}\ \forall\omega\in\Omega\ \forall\omega'\in\Omega(\omega|A)\right\}.$$

By $\mathbf{N}^0_{(\mathcal{T})}$ we denote the subset of all nonempty-valued m/f:

$$\mathbf{N}^0_{(\mathcal{T})}\triangleq\{\mathbf{z}\in\mathbf{N}_{(\mathcal{T})}\mid(\mathrm{DOM})\,[\mathbf{z}]=\Omega\}.$$

The most important are m/s of a given m/f: for $\alpha\in\mathcal{P}(\mathrm{Z})^{\Omega}$ we denote

$$\mathbf{N}_{(\mathcal{T})}[\alpha]\triangleq\{\mathbf{z}\in\mathbf{N}_{(\mathcal{T})}\mid\mathbf{z}\sqsubseteq\alpha\};\tag{1.1}$$

then we define the subset of nonempty-valued m/s of m/f α:

$$\mathbf{N}^0_{(\mathcal{T})}[\alpha]\triangleq\{\mathbf{z}\in\mathbf{N}^0_{(\mathcal{T})}\mid\mathbf{z}\sqsubseteq\alpha\}.\tag{1.2}$$

Let us introduce (see [18, § 3, item V]) pointwise defined “union” and “intersection” of m/f from $\mathcal{P}(\mathrm{Z})^{\Omega}$. Let $\mathbb{M}$ be a subset of $\mathcal{P}(\mathrm{Z})^{\Omega}$. We assume that the pointwise union $\bigvee_{\mathbf{z}\in\mathbb{M}}\mathbf{z}$ and the pointwise intersection $\bigwedge_{\mathbf{z}\in\mathbb{M}}\mathbf{z}$, for each $\omega\in\Omega$ are determined by:

$$\bigvee_{\mathbf{z}\in\mathbb{M}}\mathbf{z}(\omega)\triangleq\bigcup_{\mathbf{z}\in\mathbb{M}}\mathbf{z}(\omega),\qquad\bigwedge_{\mathbf{z}\in\mathbb{M}}\mathbf{z}(\omega)\triangleq\bigcap_{\mathbf{z}\in\mathbb{M}}\mathbf{z}(\omega).$$

Note that in the form $\bigvee_{\mathbf{z}\in\mathbb{M}}\mathbf{z}$ and $\bigwedge_{\mathbf{z}\in\mathbb{M}}\mathbf{z}$ we have, respectively, supremum and infimum of the set $\mathbb{M}$ in the poset $(\mathcal{P}(\mathrm{Z})^{\Omega},\sqsubseteq)$. Unlike pointwise intersection, the result of pointwise union inherits non-anticipatory property of operands (see Lemma 3.2 below).

§ 2. Feasibility of step-by-step procedure

Let us define the property of feasibility in formal terms. Let Δ, $\Delta \subset \mathrm{T}$, be a finite set of instants: $\Delta \triangleq \{t_0 = \tau_0 < \tau_1 < \ldots < \tau_{n_\Delta} = \vartheta\}$. We will also refer to Δ as the partition of the time interval T. Denote by $\mathcal{H}_\Delta$ the subset of $\mathcal{T}$ of the form $\mathcal{H}_\Delta \triangleq \{H_i \triangleq [\tau_0, \tau_i] : i \in 1..n_\Delta\}$.

For partition Δ and a m/f $\alpha \in \mathcal{P}(\mathrm{Z})^\Omega$, denote by Ω_Δ ($\Omega_\Delta \subset \Omega^{n_\Delta}$) and Z_Δ ($\mathrm{Z}_\Delta \subset \mathcal{P}(\mathrm{Z})^{n_\Delta}$) the sets defined as follows (in expressions like Q^n, where $n \in \mathbb{N}$, the number n is not considered as a set):

$$\Omega_\Delta \triangleq \{(\omega_i)_{i\in 1..n_\Delta} \in \Omega^{n_\Delta} \mid (\omega_i|H_i) = (\omega_{i+1}|H_i),\ i \in 1..n_\Delta - 1\}, \tag{2.1}$$

$$\mathrm{Z}_\Delta \triangleq \{(\beta_i)_{i\in 1..n_\Delta} \in \mathcal{P}'(\mathrm{Z})^{n_\Delta} \mid \boldsymbol{(}\beta_i \mid H_i\boldsymbol{)} = \boldsymbol{(}\beta_{i+1} \mid H_i\boldsymbol{)},\ i \in 1..n_\Delta - 1\}. \tag{2.2}$$

Conditions (α, Δ) for the step-by-step procedure are called feasible if there exists a tuple of m/s $(\phi_i)_{i\in 1..n_\Delta}$ of m/f α,

$$\phi_i \in \mathcal{P}(\mathrm{Z})^\Omega, \quad \phi_i \sqsubseteq \alpha, \qquad i \in 1..n_\Delta, \tag{2.3}$$

such that the following inclusions are fulfilled:

$$(\phi_i(\omega_i))_{i\in 1..n_\Delta} \in \mathrm{Z}_\Delta \qquad \forall (\omega_i)_{i\in 1..n_\Delta} \in \Omega_\Delta. \tag{2.4}$$

This definition simply retells the above informal description of the step-by-step procedure in mathematical terms. Namely, let given a tuple $(\phi_i)_{i\in 1..n_\Delta}$ satisfying (2.3), (2.4), then the construction of a step-by-step response to an unknown disturbance $\bar{\omega}$, due to the (α, Δ) conditions, can be realised like this:

— we find $\omega_1 \in \Omega$ such that $(\omega_1|H_1) = (\bar{\omega}|H_1)$ and choose $h_1 \in \mathrm{Z}$ that satisfies $h_1 \in \phi_1(\omega_1)$. This can always be done, because by completing formally ω_1 to an arbitrary tuple $(\omega_1, \ldots, \omega_{n_\Delta})$ from Ω_Δ (for example, by setting $\omega_{n_\Delta} \triangleq \ldots \triangleq \omega_2 \triangleq \omega_1$), due to (2.2) and (2.4) we get the inequality $\phi_1(\omega_1) \neq \varnothing$ which enables us to choose the desired trajectory h_1;

— if there are $(\omega_i)_{i\in 1..k+1} \in \Omega^{k+1}$ and $h_k \in \mathrm{Z}$ such that

$$(\omega_{k+1}|H_{k+1}) = (\bar{\omega}|H_{k+1}),$$

$$(\omega_{i+1}|H_i) = (\omega_i|H_i), \qquad i \in 1..k, \tag{2.5}$$

$$h_i \in \phi_i(\omega_i), \qquad i \in 1..k, \tag{2.6}$$

we choose h_{k+1} from the condition

$$h_{k+1} \in \phi_{k+1}(\omega_{k+1}) \cap \mathrm{Z}(h_k|H_k).$$

Such a choice is possible: the tuple $(\omega_1, \ldots, \omega_{k+1})$ can be completed (see (2.5), (2.1)) to a tuple $(\omega_1, \ldots, \omega_{n_\Delta})$ from Ω_Δ (for example, by setting $\omega_{n_\Delta} \triangleq \ldots \triangleq \omega_{k+2} \triangleq \omega_{k+1}$). Then, by virtue of (2.2), (2.4), we get the equality

$$(\phi_{k+1}(\omega_{k+1})|H_k) = (\phi_k(\omega_k)|H_k),$$

from which, taking into account (2.6), we get $\phi(\omega_{k+1}) \cap \mathrm{Z}(h_k|H_k) \neq \varnothing$. Hence, the choice is possible to undertake. So, by induction, the step-by-step procedure can be continued up to $k + 1 = n_\Delta$.

It is clear that the non-emptiness of values and non-anticipatory property of α (that is, when $\alpha \in \mathbf{N}^0_{(\mathcal{T})}$) or, in general, the existence of a non-empty-valued non-anticipative m/s β of m/f α implies the feasibility of the conditions (α, Δ) for any partition Δ. Indeed, then it suffices to put $\phi_i \triangleq \beta$, where $\beta \in \mathbf{N}^0_{(\mathcal{T})}[\alpha]$. At the same time, it can be seen from Example 5.3 that the feasibility of the conditions (α, Δ) even for all partitions Δ does not imply existence of non-anticipative and non-empty-valued m/s of α. In following sections, we consider the property of partial non-anticipatory of a m/f that is equivalent to the feasibility property.

§ 3. Partially non-anticipative mappings: basic properties

For an arbitrary $A \in \mathcal{T}$, we introduce the notion of *A-non-anticipative* m/f: a mapping $\mathbf{z} \in \mathcal{P}(\mathrm{Z})^{\Omega}$ is called A-non-anticipative if the implication

$$((\omega_1|A) = (\omega_2|A)) \Rightarrow ((\mathbf{z}(\omega_1) \,|\, A) = (\mathbf{z}(\omega_2) \,|\, A))$$

is true for all $\omega_1, \omega_2 \in \Omega$. The subset of all A-non-anticipative m/f we denote by $\mathbf{N}_{(\{A\})}$. The subset of all non-empty-valued and A-non-anticipative m/f we denote by $\mathbf{N}^0_{(\{A\})}$:

$$\mathbf{N}^0_{(\{A\})} \triangleq \{\mathbf{z} \in \mathbf{N}_{(\{A\})} \mid (\mathrm{DOM})\,[\mathbf{z}] = \Omega\}.$$

Let $\mathcal{H} \in \mathcal{P}'(\mathcal{T})$. We call a m/f from $\mathcal{P}(\mathrm{Z})^{\Omega}$ *$\mathcal{H}$-non-anticipative* if the m/f is A-non-anticipative for all $A \in \mathcal{H}$. The set of all $\mathcal{H}$-non-anticipative (non-empty-valued and $\mathcal{H}$-non-anticipative) m/f we denote by $\mathbf{N}_{(\mathcal{H})}$ ($\mathbf{N}^0_{(\mathcal{H})}$):

$$\mathbf{N}_{(\mathcal{H})} = \bigcap_{A \in \mathcal{H}} \mathbf{N}_{(\{A\})} \qquad \Big(\mathbf{N}^0_{(\mathcal{H})} = \bigcap_{A \in \mathcal{H}} \mathbf{N}^0_{(\{A\})}\Big).$$

For any $\alpha \in \mathcal{P}(\mathrm{Z})^{\Omega}$ and $\mathcal{H} \in \mathcal{P}'(\mathcal{T})$, designations $\mathbf{N}_{(\mathcal{H})}\,[\alpha]$ and $\mathbf{N}^0_{(\mathcal{H})}\,[\alpha]$ are defined similarly to those in (1.1) and (1.2), respectively.

Note two relations which we obtain immediately from the definitions: for arbitrary $\alpha \in \mathcal{P}(\mathrm{Z})^{\Omega}$ and $\mathcal{H}, \mathcal{H}' \in \mathcal{P}(\mathcal{T})$

$$(\mathcal{H} \subset \mathcal{H}') \Rightarrow \big(\mathbf{N}^0_{(\mathcal{H}')}\,[\alpha] \subset \mathbf{N}^0_{(\mathcal{H})}\,[\alpha]\big) \,\&\, \big(\mathbf{N}^0_{(\mathcal{H}')}\,[\alpha] \subset \mathbf{N}^0_{(\mathcal{H})}\,[\alpha]\big)\,, \tag{3.1}$$

$$\big(\mathbf{N}_{(\mathcal{H}\cup\mathcal{H}')}\,[\alpha] = \mathbf{N}_{(\mathcal{H})}\,[\alpha] \cap \mathbf{N}_{(\mathcal{H}')}\,[\alpha]\big) \,\&\, \big(\mathbf{N}^0_{(\mathcal{H}\cup\mathcal{H}')}\,[\alpha] = \mathbf{N}^0_{(\mathcal{H})}\,[\alpha] \cap \mathbf{N}^0_{(\mathcal{H}')}\,[\alpha]\big)\,.$$

The following definitions and result of Lemma 3.1 is close to the idea of a monotonicity of non-anticipatory m/fs (see [12, Remark 2.8]). Denote by $\mathbf{s}_{\Omega}(\omega, \omega') \in \mathcal{P}(\mathcal{T})$ the set of the form (see [7, (3.1), (3.6)])

$$\mathbf{s}_{\Omega}(\omega, \omega') \triangleq \{H \in \mathcal{T} \mid (\omega|H) = (\omega'|H)\}, \qquad \omega, \omega' \in \Omega;$$

define the mapping $\overline{\mathbf{s}}_{\Omega}(\cdot,\cdot) \in \mathcal{T}^{\Omega\times\Omega}$ given by $\overline{\mathbf{s}}_{\Omega}(\omega, \omega') \triangleq \bigcup_{H \in \mathbf{s}_{\Omega}(\omega,\omega')} H$, for all $\omega, \omega' \in \Omega$ and the set $\mathcal{T}_{\Omega}$, $\mathcal{T}_{\Omega} \triangleq \{\overline{\mathbf{s}}_{\Omega}(\omega, \omega') \colon \omega, \omega' \in \Omega\}$.

The following lemma indicates (see (3.2)) cases of coincidence of sets of non-anticipative and partially non-anticipative m/s. Thus, in particular, for a finite set of disturbances, the problem of constructing a non-anticipative m/s can be reduced to constructing a partially non-anticipative m/s with a finite set of test moments.

Lemma 3.1. *Let the inclusions be fulfilled:*

$$\overline{\mathbf{s}}_{\Omega}(\omega, \omega') \in \mathbf{s}_{\Omega}(\omega, \omega'), \qquad \forall \omega, \omega' \in \Omega. \tag{3.2}$$

Then for any $\alpha \in \mathcal{P}(\mathrm{Z})^{\Omega}$ the equality is true:

$$\mathbf{N}_{(\mathcal{T})}[\alpha] = \mathbf{N}_{(\mathcal{T}_{\Omega})}\,[\alpha]\,. \tag{3.3}$$

In particular, condition (3.2) is fulfilled when elements of Ω are continuous from the left:

$$\lim_{\tau \to t-} \omega(\tau) = \omega(t) \;\forall t \in \mathrm{T} \;\forall \omega \in \Omega.$$

P r o o f. By definitions (see (1.1), Sect. 3, (3.1)) the inclusion $\mathbf{N}_{(\mathcal{T})}[\alpha] \subset \mathbf{N}_{(\mathcal{T}_\Omega)}[\alpha]$ is fulfilled. On the other hand, suppose $\beta \in \mathbf{N}_{(\mathcal{T}_\Omega)}[\alpha]$ and $(\omega, \omega', H) \in \Omega \times \Omega \times \mathcal{T}$. If $(\omega|H) = (\omega'|H)$, then $H \in \mathbf{s}_\Omega(\omega, \omega')$, $\overline{\mathbf{s}}_\Omega(\omega, \omega') \in \mathcal{T}_\Omega$, and

$$H \subset \overline{\mathbf{s}}_\Omega(\omega, \omega'). \tag{3.4}$$

Due to (3.2), we have $(\omega|\overline{\mathbf{s}}_\Omega(\omega, \omega')) = (\omega'|\overline{\mathbf{s}}_\Omega(\omega, \omega'))$ and by the property of $\mathcal{T}_\Omega$-non-anticipatory of β, the relation is fulfilled: $\big(\beta(\omega) \,|\, \overline{\mathbf{s}}_\Omega(\omega, \omega')\big) = \big(\beta(\omega') \,|\, \overline{\mathbf{s}}_\Omega(\omega, \omega')\big)$. Wherefrom (see (3.4)) we obtain the equality $\big(\beta(\omega) \,|\, H\big) = \big(\beta(\omega') \,|\, H\big)$. Since the choice of (ω, ω', H) was arbitrary, we have the inclusion $\beta \in \mathbf{N}_{(\mathcal{T})}[\alpha]$, i. e., $\mathbf{N}_{(\mathcal{T}_\Omega)}[\alpha] \subset \mathbf{N}_{(\mathcal{T})}[\alpha]$. Then, the equality (3.3) is true. □

Due to the arbitrary choice of α, we also have the equality

$$\mathbf{N}_{(\mathcal{T})} = \mathbf{N}_{(\mathcal{T}_\Omega)}.$$

The following is definitions and some properties of point-wise operations on non-anticipative m/f.

Lemma 3.2. *Let $\alpha \in \mathcal{P}(\mathrm{Z})^\Omega$, $\mathcal{H} \subset \mathcal{T}$ and $\mathbb{M} \in \mathcal{P}'(\mathbf{N}_{(\mathcal{H})}[\alpha])$. Then,*

$$\bigvee_{\mathbf{z} \in \mathbb{M}} \mathbf{z} \in \mathbf{N}_{(\mathcal{H})}[\alpha], \tag{3.5}$$

$$\Big(\forall \omega \in \Omega \;\exists \mathbf{z} \in \mathbb{M} : \mathbf{z}(\omega) \neq \varnothing\Big) \Rightarrow \Big(\bigvee_{\mathbf{z} \in \mathbb{M}} \mathbf{z} \in \mathbf{N}^0_{(\mathcal{H})}[\alpha]\Big), \tag{3.6}$$

$$\top_{\mathbf{N}_{(\mathcal{H})}[\alpha]} = \bigvee_{\mathbf{z} \in \mathbf{N}_{(\mathcal{H})}[\alpha]} \mathbf{z}. \tag{3.7}$$

P r o o f.
1. It is clear that

$$\Big(\bigvee_{\mathbf{z} \in \mathbb{M}} \mathbf{z} \sqsubseteq \alpha\Big) \;\&\; \Big(\bigwedge_{\mathbf{z} \in \mathbb{M}} \mathbf{z} \sqsubseteq \alpha\Big). \tag{3.8}$$

We show that the mapping $\bigvee_{\mathbf{z} \in \mathbb{M}} \mathbf{z}$ inherits the property of $\mathcal{H}$-non-anticipatory of elements from $\mathbb{M}$: let $\omega, \omega' \in \Omega$, $A \in \mathcal{H}$, $(\omega|A) = (\omega'|A)$ and $\gamma \in \big(\bigvee_{\mathbf{z} \in \mathbb{M}} \mathbf{z}(\omega) \,|\, A\big)$. Then, there are $\bar{\mathbf{z}} \in \mathbb{M}$ and $h \in \bar{\mathbf{z}}(\omega)$ such that $\gamma = (h|A)$. Therefore, taking into account the relation $\bar{\mathbf{z}} \in \mathbf{N}_{(\mathcal{H})}[\alpha]$, we have the inclusion $\gamma \in \big(\bar{\mathbf{z}}(\omega') \,|\, \mathbf{A}\big)$. Hence, there exists $h' \in \bar{\mathbf{z}}(\omega')$ such that $\gamma = (h'|A)$, whence we obtain the inclusion $h' \in \bigvee_{\mathbf{z} \in \mathbb{M}} \mathbf{z}(\omega')$ and, as a consequence, the inclusion $\gamma \in \big(\bigvee_{\mathbf{z} \in \mathbb{M}} \mathbf{z}(\omega') \,|\, A\big)$. Since γ was chosen arbitrarily, we have the relation $\big(\bigvee_{\mathbf{z} \in \mathbb{M}} \mathbf{z}(\omega) \,|\, A\big) \subset \big(\bigvee_{\mathbf{z} \in \mathbb{M}} \mathbf{z}(\omega') \,|\, A\big)$. Hence, due to symmetry of considerations and an arbitrary choice of ω, ω', the implication follows: $((\omega|A) = (\omega'|A)) \Rightarrow \big(\bigvee_{\mathbf{z} \in \mathbb{M}} \mathbf{z}(\omega) \,|\, A\big) = \big(\bigvee_{\mathbf{z} \in \mathbb{M}} \mathbf{z}(\omega') \,|\, A\big)$ for all $\omega, \omega' \in \Omega$. Due to the implication, taking into account the arbitrary choice of A, we have

$$\bigvee_{\mathbf{z} \in \mathbb{M}} \mathbf{z} \in \mathbf{N}_{(\mathcal{H})}. \tag{3.9}$$

Relations (3.8), (3.9) together give the desired inclusion (3.5).

2. Premise in (3.6) implies the equality $(\mathrm{DOM})\big[\bigvee_{\mathbf{z} \in \mathbb{M}} \mathbf{z}\big] = \Omega$, which in combination with (3.5) gives the desired assertion.

3. By construction, the expression on the right side (3.7) $\sqsubseteq$-majorizes each element of the set $\mathbf{N}_{(\mathcal{H})}[\alpha]$, while (3.5) for $\mathbb{M} \triangleq \mathbf{N}_{(\mathcal{H})}[\alpha]$ implies the inclusion $\bigvee_{\mathbf{z} \in \mathbf{N}_{(\mathcal{H})}[\alpha]} \mathbf{z} \in \mathbf{N}_{(\mathcal{H})}[\alpha]$.

The lemma is proven. □

Theorem 3.1. *For a m/f $\alpha \in \mathcal{P}(\mathrm{Z})^{\Omega}$ and a partition Δ of interval T, the conditions (α, Δ) are feasible if and only if m/f α has a non-empty-valued and $\mathcal{H}_\Delta$-non-anticipative m/s:*

$$((\alpha, \Delta) - \textit{feasible}) \Leftrightarrow \left(\mathbf{N}^0_{(\mathcal{H}_\Delta)}[\alpha] \neq \varnothing\right). \tag{3.10}$$

P r o o f. Let $\Delta \triangleq \{t_0 = \tau_0 < \tau_1 < \ldots < \tau_{n_\Delta} = \vartheta\}$. Remind that $\mathcal{H}_\Delta \triangleq \{H_i \triangleq [\tau_0, \tau_i] \colon i \in 1..n_\Delta\}$. The definitions (2.1), (2.2) imply the equalities

$$\Omega_\Delta = \{(\omega_i)_{i\in 1..n_\Delta} \in \Omega^{n_\Delta} \mid (\omega_i | H_{\overline{ij}}) = (\omega_j | H_{\overline{ij}}),\ i, j \in 1..n_\Delta\}, \tag{3.11}$$

$$\mathrm{Z}_\Delta = \{(\mathbf{z}_i)_{i\in 1..n_\Delta} \in \mathcal{P}'(\mathrm{Z})^{n_\Delta} \mid \boldsymbol{(}\mathbf{z}_i \mid H_{\overline{ij}}\boldsymbol{)} = \boldsymbol{(}\mathbf{z}_j \mid H_{\overline{ij}}\boldsymbol{)},\ i, j \in 1..n_\Delta\}, \tag{3.12}$$

where $\overline{ij} \triangleq \min\{i, j\}$ for all $i, j \in 1..n_\Delta$.

1. Let $\mathbf{N}^0_{(\mathcal{H}_\Delta)}[\alpha] \neq \varnothing$ and $\beta \in \mathbf{N}^0_{(\mathcal{H}_\Delta)}[\alpha]$. Consider a tuple $(\phi_i)_{i\in 1..n_\Delta}$ of the form $\phi_i \triangleq \beta$, $i \in 1..n_\Delta$. By definition, it satisfies the equalities $(\mathrm{DOM})[\phi_i] = \Omega$, $i \in 1..n_\Delta$, and the conditions $\phi_i \sqsubseteq \beta \sqsubseteq \alpha$, $i \in 1..n_\Delta$. This implies the fulfillment of the condition (2.3).

Let us check (2.4). Suppose a tuple $(\omega_i)_{i\in 1..n_\Delta}$ is such that $(\omega_i)_{i\in 1..n_\Delta} \in \Omega_\Delta$. Then, taking into account the $\mathcal{H}_\Delta$-non-anticipatory property of β, from the equalities $(\omega_k | H_{\overline{km}}) = (\omega_m | H_{\overline{km}})$, we get the equalities

$$\boldsymbol{(}\phi_k(\omega_k) \mid H_{\overline{km}}\boldsymbol{)} = \boldsymbol{(}\beta(\omega_k) \mid H_{\overline{km}}\boldsymbol{)} = \boldsymbol{(}\beta(\omega_m) \mid H_{\overline{km}}\boldsymbol{)} = \boldsymbol{(}\phi_m(\omega_m) \mid H_{\overline{km}}\boldsymbol{)} \quad k, m \in 1..n_\Delta.$$

Since $(\omega_i)_{i\in 1..n_\Delta}$, k and m were chosen arbitrarily, for the tuple $(\phi_i)_{i\in 1..n_\Delta}$ condition (2.4) is met. We have shown that the left side of (3.10) follows from the right side.

2. Let us show that the right side of (3.10) follows from the left side. Assume that the conditions (α, Δ) are feasible. Then (see Sect. 2) there exists a tuple $(\phi_i)_{i\in 1..n_\Delta}$ of the form (2.3) such that for any tuple $(\omega_i)_{i\in 1..n_\Delta} \in \Omega_\Delta$ (see (3.11), (3.12)):

$$(\phi_i(\omega_i))_{i\in 1..n_\Delta} \in \mathrm{Z}_\Delta. \tag{3.13}$$

Let m/f ϕ_α be defined by $\phi_\alpha \triangleq \phi_{n_\Delta}$. Then we have the comparison $\phi_\alpha \sqsubseteq \alpha$ and equality $(\mathrm{DOM})[\phi_\alpha] = \Omega$ (indeed, from (3.12), (3.13), follow the inclusions $\phi_\alpha(\omega) \in \mathcal{P}'(\mathrm{Z})$ for all $\omega \in \Omega$). To show that ϕ_α is $\mathcal{H}_\Delta$-non-anticipative, suppose $\omega, \omega' \in \Omega$ and $m \in 1..n_\Delta$ are such that

$$(\omega | H_m) = (\omega' | H_m). \tag{3.14}$$

We put

$$\omega_i \triangleq \omega, \quad i \in 1..n_\Delta, \qquad \omega'_i \triangleq \begin{cases} \omega, & i \in 1..m, \\ \omega', & i \in (m+1)..n_\Delta. \end{cases}$$

For the tuple $(\omega_i)_{i\in 1..n_\Delta}$, we obviously have the inclusion $(\omega_i)_{i\in 1..n_\Delta} \in \Omega_\Delta$.

Show the inclusion $(\omega'_i)_{i\in 1..n_\Delta} \in \Omega_\Delta$. For the tuple $(\omega'_i)_{i\in 1..n_\Delta}$ and any $i, j \in 1..n_\Delta$, we have: if $m < i \leqslant j$, then

$$(\omega'_i | H_{\overline{ij}}) = (\omega' | H_{\overline{ij}}) = (\omega'_j | H_{\overline{ij}});$$

if $i \leqslant m < j$, then (see (3.14))

$$(\omega'_i | H_{\overline{ij}}) = (\omega | H_{\overline{ij}}) = (\omega | H_i) = (\omega' | H_i) = (\omega'_j | H_i) = (\omega'_j | H_{\overline{ij}});$$

if $i \leqslant j \leqslant m$, then

$$(\omega'_i | H_{\overline{ij}}) = (\omega | H_{\overline{ij}}) = (\omega'_j | H_{\overline{ij}}).$$

Thus (see (3.11)), the inclusion $(\omega'_i)_{i\in 1..n_\Delta} \in \Omega_\Delta$ takes place.

Therefore, due to (3.13), the inclusions

$$(\phi_i(\omega_i))_{i\in 1..n_\Delta} \in \mathrm{Z}_\Delta, \qquad (\phi_i(\omega_i'))_{i\in 1..n_\Delta} \in \mathrm{Z}_\Delta,$$

considered (see (3.12)) under $i = m$ and $j = n_\Delta$, imply the equalities

$$\big(\phi_m(\omega_m) \,|\, H_{\overline{mn_\Delta}}\big) = \big(\phi_{n_\Delta}(\omega_{n_\Delta}) \,|\, H_{\overline{mn_\Delta}}\big), \quad \big(\phi_m(\omega_m') \,|\, H_{\overline{mn_\Delta}}\big) = \big(\phi_{n_\Delta}(\omega_{n_\Delta}') \,|\, H_{\overline{mn_\Delta}}\big). \tag{3.15}$$

Moreover, from the definitions of ω_m' and ω_m, we have $\omega_m' = \omega_m = \omega$ and hence the equalities

$$\big(\phi_m(\omega_m) \,|\, H_{\overline{mn_\Delta}}\big) = \big(\phi_m(\omega) \,|\, H_m\big) = \big(\phi_m(\omega_m') \,|\, H_{\overline{mn_\Delta}}\big). \tag{3.16}$$

From the given equalities and the definition of ϕ_α we obtain (the second and the fourth equalities follow from (3.12) and (3.15), the third follows from (3.16)):

$$\begin{aligned}\big(\phi_\alpha(\omega) \,|\, [\tau_0, \tau_m]\big) = \big(\phi_{n_\Delta}(\omega_{n_\Delta}) \,|\, [\tau_0, \tau_{\overline{mn_\Delta}}]\big) = \big(\phi_m(\omega_m) \,|\, [\tau_0, \tau_{\overline{mn_\Delta}}]\big) = \\ = \big(\phi_m(\omega_m') \,|\, [\tau_0, \tau_{\overline{mn_\Delta}}]\big) = \big(\phi_{n_\Delta}(\omega_{n_\Delta}') \,|\, [\tau_0, \tau_{\overline{mn_\Delta}}]\big) = \big(\phi_\alpha(\omega') \,|\, [\tau_0, \tau_m]\big).\end{aligned}$$

Since m, ω and ω' were chosen arbitrarily, from the last equalities we obtain the property of $\mathcal{H}_\Delta$-non-anticipatory of m/f ϕ_α. Taking into account the indicated properties of ϕ_α, we have the inclusion $\phi_\alpha \in \mathbf{N}^0_{(\mathcal{H}_\Delta)}[\alpha]$, i.e., the right side of (3.10) is fulfilled.

The proof is complete. □

§ 4. A construction of the partially non-anticipative multiselector

In this section, we give a description of partially non-anticipative m/s in terms of explicitly defined operators that mapping a m/f to its m/s non-anticipative at a given point of T. In general, such description is certainly non-constructive. Meanwhile, when applied to step-by-step procedures, due to the finiteness of operations, the description allows to construct and analyze corresponding partially non-anticipative m/s (see examples).

For an arbitrary $A \in \mathcal{T}$, denote by $\langle\cdot\rangle_A$ the operator that transforms the set $\mathcal{P}(\mathrm{Z})^\Omega$ and is given by:

$$\langle\alpha\rangle_A(\omega) \triangleq \Big\{h \in \alpha(\omega) \;\Big|\; (h|A) \in \bigcap_{\omega' \in \Omega(\omega|A)} \big(\alpha(\omega') \,|\, A\big)\Big\} \qquad \forall \alpha \in \mathcal{P}(\mathrm{Z})^\Omega,\ \forall \omega \in \Omega. \tag{4.1}$$

It immediately follows that $\langle\cdot\rangle_A$ is non-expansive and isotonic as an operator in the poset $(\mathcal{P}(\mathrm{Z})^\Omega, \sqsubseteq)$: for arbitrary $A \in \mathcal{T}$, $\alpha, \beta \in \mathcal{P}(\mathrm{Z})^\Omega$,

$$\langle\alpha\rangle_A \sqsubseteq \alpha, \tag{4.2}$$

$$(\alpha \sqsubseteq \beta) \Rightarrow (\langle\alpha\rangle_A \sqsubseteq \langle\beta\rangle_A)\,. \tag{4.3}$$

We also note (see Subsection 5.1) that in the general case the mapping $\mathcal{T} \ni A \mapsto \langle\alpha\rangle_A \in \mathcal{P}(\mathrm{Z})^\Omega$ is not isotonic as a mapping from poset $(\mathcal{T}, \subset)$ to poset $(\mathcal{P}(\mathrm{Z})^\Omega, \sqsubseteq)$.

Regarding the non-emptyness of values of m/f $\langle\alpha\rangle_A$, note the equivalence

$$\Big(\langle\alpha\rangle_A(\omega) \neq \varnothing\Big) \Leftrightarrow \Big(\bigcap_{\omega' \in \Omega(\omega|A)} \big(\alpha(\omega') \,|\, A\big)\Big) \qquad \forall \alpha \in \mathcal{P}(\mathrm{Z})^\Omega,\ \forall \omega \in \Omega. \tag{4.4}$$

Lemma 4.1. *For any $A \in \mathcal{T}$ and any m/f $\alpha \in \mathcal{P}(\mathrm{Z})^\Omega$:*

(i) *the set of values of the operator $\langle\cdot\rangle_A$ equals to the set of all A-non-anticipative m/f, as well as to the set of fixed points of operator $\langle\cdot\rangle_A$:*

$$\langle\mathcal{P}(\mathrm{Z})^\Omega\rangle_A = \mathbf{N}_{(\{A\})} = \mathbf{Fix}(\langle\cdot\rangle_A); \tag{4.5}$$

(ii) *the m/f* $\langle\alpha\rangle_A$ *is the* $\sqsubseteq$*-greatest* A*-non-anticipative m/s of m/f* α*:*

$$\langle\alpha\rangle_A = \top_{\mathbf{N}_{(\{A\})}[\alpha]}; \tag{4.6}$$

(iii) *the operator* $\langle\cdot\rangle_A$ *is idempotent, i. e., for any* $\alpha \in \mathcal{P}(\mathrm{Z})^\Omega$*, the equality takes place:*

$$\langle\langle\alpha\rangle_A\rangle_A = \langle\alpha\rangle_A. \tag{4.7}$$

P r o o f. 1. Let $\beta \triangleq \langle\alpha\rangle_A$. We show that $\beta \in \mathbf{N}_{(\{A\})}[\alpha]$. Due to (4.2), we have $\beta \sqsubseteq \alpha$. It remains to verify the A-non-anticipatory property of β.

Let $\omega, \omega' \in \Omega$ be such that $(\omega|A) = (\omega'|A)$ and $\xi \in \boldsymbol{(}\beta(\omega) \mid A\boldsymbol{)}$. Then (see (4.1))

$$\xi \in \bigcap_{\bar\omega \in \Omega(\omega|A)} \boldsymbol{(}\alpha(\bar\omega) \mid A\boldsymbol{)}. \tag{4.8}$$

By the choice of ω', we have $\omega' \in \Omega(\omega|A)$ and hence (see (4.8)) $\xi \in \boldsymbol{(}\alpha(\omega') \mid A\boldsymbol{)}$. Then, there exists $h' \in \alpha(\omega')$ such that $(h'|A) = \xi$. From the equality $(\omega|A) = (\omega'|A)$, it also follows that $\Omega(\omega'|A) = \Omega(\omega|A)$ and, therefore,

$$\bigcap_{\bar\omega \in \Omega(\omega|A)} \boldsymbol{(}\alpha(\bar\omega) \mid A\boldsymbol{)} = \bigcap_{\bar\omega \in \Omega(\omega'|A)} \boldsymbol{(}\alpha(\bar\omega) \mid A\boldsymbol{)}.$$

As a result, h' satisfies relations

$$h' \in \alpha(\omega'), \qquad (h'|A) \in \bigcap_{\bar\omega \in \Omega(\omega'|A)} \boldsymbol{(}\alpha(\bar\omega) \mid A\boldsymbol{)},$$

i. e., (see (4.1)), $h' \in \langle\alpha\rangle_A(\omega')$. Then $\xi \in \boldsymbol{(}\beta(\omega') \mid A\boldsymbol{)}$. Since ξ was chosen arbitrarily, we have

$$\boldsymbol{(}\beta(\omega) \mid A\boldsymbol{)} \subset \boldsymbol{(}\beta(\omega') \mid A\boldsymbol{)}.$$

From the inclusion, due to an arbitrary choice of ω, ω' and to symmetry of them in the considerations, we obtain the desired A-non-anticipatory of β.

So, taking into account the definition of β, we have the inclusions

$$\langle\alpha\rangle_A \in \mathbf{N}_{(\{A\})}[\alpha] \subset \mathbf{N}_{(\{A\})}. \tag{4.9}$$

Due to the arbitrary choice of α, (4.9) implies an embedding

$$\langle\mathcal{P}(\mathrm{Z})^\Omega\rangle_A \subset \mathbf{N}_{(\{A\})}. \tag{4.10}$$

2. Let us verify that $\langle\alpha\rangle_A$ is the $\sqsubseteq$-greatest m/f in $\mathbf{N}_{(\{A\})}[\alpha]$. Let

$$\beta \in \mathbf{N}_{(\{A\})}[\alpha], \tag{4.11}$$

$\bar\omega \in \Omega$ and $\bar h \in \beta(\bar\omega)$. Then (4.11) implies that

$$(\bar h|A) \in \boldsymbol{(}\beta(\omega) \mid A\boldsymbol{)} \subset \boldsymbol{(}\alpha(\omega) \mid A\boldsymbol{)} \quad \forall \omega \in \Omega(\bar\omega|A).$$

So, $\bar h$ satisfies the relations

$$\bar h \in \beta(\bar\omega) \subset \alpha(\bar\omega), \qquad (\bar h|A) \in \bigcap_{\omega \in \Omega(\bar\omega|A)} \boldsymbol{(}\beta(\omega) \mid A\boldsymbol{)} \subset \bigcap_{\omega \in \Omega(\bar\omega|A)} \boldsymbol{(}\alpha(\omega) \mid A\boldsymbol{)},$$

i. e., (see (4.1)), $\bar{h} \in \langle\beta\rangle_A(\bar{\omega})$ and $\bar{h} \in \langle\alpha\rangle_A(\bar{\omega})$. Since $\bar{\omega}$ and $\bar{h}$ were chosen arbitrarily, we have comparisons

$$\beta \sqsubseteq \langle\beta\rangle_A, \tag{4.12}$$

$$\beta \sqsubseteq \langle\alpha\rangle_A. \tag{4.13}$$

Comparing (4.13), due to arbitrary choice of β, gives the equality (4.6).

3. Relation (4.12) and the non-expansion property of $\langle\cdot\rangle_A$ (see (4.2)) imply the equality $\beta = \langle\beta\rangle_A$. Hence, due to arbitrary choice of β, we obtain for any $\alpha \in \mathcal{P}(\mathrm{Z})^\Omega$ the embedding $\mathbf{N}_{(\{A\})}[\alpha] \subset \mathbf{Fix}(\langle\cdot\rangle_A)$. For α of the form $\alpha(\omega) \triangleq \mathrm{Z}$, $\omega \in \Omega$, we obviously have $\mathbf{N}_{(\{A\})}[\alpha] = \mathbf{N}_{(\{A\})}$. So the following inclusion is correct:

$$\mathbf{N}_{(\{A\})} \subset \mathbf{Fix}(\langle\cdot\rangle_A). \tag{4.14}$$

For any $\beta \in \mathbf{Fix}(\langle\cdot\rangle_A)$, by the definition of a fixed point, the equality $\beta = \langle\beta\rangle_A$ is satisfied, i. e., β lies in the image of the set $\mathcal{P}(\mathrm{Z})^\Omega$ under the mapping $\langle\cdot\rangle_A$. Then the next inclusion is fulfilled:

$$\mathbf{Fix}(\langle\cdot\rangle_A) \subset \langle\mathcal{P}(\mathrm{Z})^\Omega\rangle_A. \tag{4.15}$$

From relations (4.10), (4.14) and (4.15), we get the equalities (4.5).

4. Equalities (4.5) imply the equality (4.7).

The lemma is proven. □

From the lemma, we immediately obtain a corollary that allows us to filter out m/f's that do not have a non-empty-valued and non-anticipative m/s.

Corollary 4.1. *For $\alpha \in \mathcal{P}(\mathrm{Z})^\Omega$ and $\mathcal{H} \in \mathcal{P}'(\mathcal{T})$, the following implications hold:*

$$\big(\beta \in \mathbf{N}_{(\mathcal{H})}[\alpha]\big) \Rightarrow \left(\beta \sqsubseteq \bigwedge_{A\in\mathcal{H}} \langle\alpha\rangle_A\right), \tag{4.16}$$

$$\left((\mathrm{DOM})\left[\bigwedge_{A\in\mathcal{H}} \langle\alpha\rangle_A\right] \neq \Omega\right) \Rightarrow \big(\mathbf{N}^0_{(\mathcal{H})}[\alpha] = \varnothing\big) \Rightarrow \big(\mathbf{N}^0_{(\mathcal{T})}[\alpha] = \varnothing\big). \tag{4.17}$$

P r o o f. 1. Let β satisfy the premise of (4.16). Then, from the relations

$$\beta \in \mathbf{N}_{(\{A\})}[\alpha], \qquad \forall A \in \mathcal{H},$$

and (4.6), we have

$$\beta \sqsubseteq \langle\alpha\rangle_A, \qquad \forall A \in \mathcal{H},$$

which implies the conclusion of (4.16).

2. From (4.16) and (3.1), the implications (4.17) follow.

The corollary is proven. □

Since, for arbitrary $\mathcal{H} \in \mathcal{P}'(\mathcal{T})$ and $\alpha \in \mathcal{P}(\mathrm{Z})^\Omega$, there exists the greatest (unique) element $\top_{\mathbf{N}_{(\mathcal{H})}[\alpha]}$ (see Lemma 3.2), then we introduce an operator $\langle\cdot\rangle_\mathcal{H} \colon \mathcal{P}(\mathrm{Z})^\Omega \mapsto \mathcal{P}(\mathrm{Z})^\Omega$ of the form

$$\langle\alpha\rangle_\mathcal{H} \triangleq \top_{\mathbf{N}_{(\mathcal{H})}[\alpha]}, \qquad \alpha \in \mathcal{P}(\mathrm{Z})^\Omega. \tag{4.18}$$

For $\mathcal{H} \subset \mathcal{T}$, we denote by $\mathbf{F}_\mathcal{H}$ the family of operators defined by $\mathbf{F}_\mathcal{H} \triangleq \{\langle\cdot\rangle_A \mid A \in \mathcal{H}\}$. By $\mathbf{Fix}(\mathbf{F}_\mathcal{H})$ we refer to the set of joint fixed points of the family $\mathbf{F}_\mathcal{H}$: $\mathbf{Fix}(\mathbf{F}_\mathcal{H}) \triangleq \bigcap_{A\in\mathcal{H}} \mathbf{Fix}(\langle\cdot\rangle_A)$.

Remark 4.1. It follows from the definition of $\mathbf{N}_{(\mathcal{H})}$ and the equality (4.5) that the joint fixed points of the family $\mathbf{F}_{\mathcal{H}}$ are $\mathcal{H}$-non-anticipative mappings: $\mathbf{Fix}(\mathbf{F}_{\mathcal{H}}) = \mathbf{N}_{(\mathcal{H})}$. Thus, in the case when $\mathcal{H}$ is a singleton ($\mathcal{H} = \{A\}$, $A \in \mathcal{T}$), we get the equality (see (4.6)) $\langle\cdot\rangle_{\{A\}} = \langle\cdot\rangle_A$.

We turn to the representation of the operator $\langle\cdot\rangle_{\mathcal{H}}$ in terms of superposition of operators from $\mathbf{F}_{\mathcal{H}}$. As already was noted, the representation of non-anticipative m/s in the form of fixed points of non-expansive isotonic operators in a poset and, as a consequence, in the form of limits of their iterative sequences, was proposed and studied in [8, 9, 14]. The iterative process in some cases turns out to be finite (see, for example, [15, Ch. 5]), which makes it possible to obtain efficient solutions to problems.

In this paper, despite the fact that the operators from $\mathbf{F}_{\mathcal{H}}$ are in general non-commutative (see Subsection 5.2), a finite-step construction of the operator $\langle\cdot\rangle_{\mathcal{H}}$ is given in Theorem 4.1 for all cases when the set $\mathcal{H}$ is finite. To this end, we give some definitions and auxiliary results.

Lemma 4.2. *Let $H_1, H_2 \in \mathcal{T}$, $\alpha \in \mathcal{P}(\mathrm{Z})^{\Omega}$, $\Upsilon \in \mathcal{P}'(\Omega)$ and $K \in \mathcal{P}((\mathrm{Z} \,|\, H_1))$ are such that*

$$H_1 \subset H_2, \tag{4.19}$$

$$(\alpha(\omega) \,|\, H_2) = (\alpha(\omega') \,|\, H_2), \qquad \forall \omega, \omega' \in \Upsilon. \tag{4.20}$$

Then, for $\beta \in \mathcal{P}(\mathrm{Z})^{\Omega}$ of the form

$$\beta(\omega) \triangleq \begin{cases} \{h \in \alpha(\omega) \mid (h|H_1) \in K\}, & \omega \in \Upsilon, \\ \alpha(\omega), & \omega \in \Omega \setminus \Upsilon, \end{cases} \tag{4.21}$$

the equalities (4.20) *are also satisfied:*

$$(\beta(\omega) \,|\, H_2) = (\beta(\omega') \,|\, H_2), \qquad \forall \omega, \omega' \in \Upsilon. \tag{4.22}$$

P r o o f. Let $\omega, \omega' \in \Upsilon$ and $\gamma \in (\beta(\omega) \,|\, H_2)$. Then, by the choice of γ, there exists $h \in \alpha(\omega)$ such that

$$(h|H_2) = \gamma, \tag{4.23}$$

and at the same time (see (4.21))

$$(h|H_1) \in K. \tag{4.24}$$

From (4.20) and the choice of ω, ω', it follows that there is $h' \in \alpha(\omega')$ satisfying the equality

$$(h'|H_2) = (h|H_2). \tag{4.25}$$

From (4.23), (4.25), we have

$$(h'|H_2) = \gamma. \tag{4.26}$$

In addition, (4.24), (4.25), and (4.19) imply

$$(h'|H_1) \in K. \tag{4.27}$$

From the inclusion of $h' \in \alpha(\omega')$, (4.27) and the definition of β (see (4.21)), we get $h' \in \beta(\omega')$, whence, taking into account (4.26), we have $\gamma \in (\beta(\omega') \,|\, H_2)$. Then, due to the arbitrary choice of γ, we have $(\beta(\omega) \,|\, H_2) \subset (\beta(\omega') \,|\, H_2)$. From here, in view of the symmetry of the occurrence of ω, ω', the desired equality (4.22) is extracted. The proof is complete. □

In particular, if (4.19) is true, the application of $\langle\cdot\rangle_{H_1}$ to H_2-non-anticipative m/f α does not violate this property.

Corollary 4.2. *For any $H_1, H_2 \in \mathcal{T}$ such that (4.19) is true, and any α from $\mathcal{P}(\mathrm{Z})^{\Omega}$, the implication is fulfilled:*

$$(\alpha \in \mathbf{N}_{(H_2)}) \Rightarrow (\langle\alpha\rangle_{H_1} \in \mathbf{N}_{(H_2)}).$$

P r o o f. Indeed, let $H_1, H_2 \in \mathcal{T}$ satisfy (4.19), $\alpha \in \mathbf{N}_{(H_2)}$, and $\omega \in \Omega$ is fixed. Let us define K and Υ from Lemma 4.2 as follows:

$$K \triangleq \bigcap_{\bar{\omega}\in\Omega(\omega|H_1)} \big(\alpha(\bar{\omega}) \,|\, H_1\big), \qquad \Upsilon \triangleq \Omega(\omega|H_2).$$

Then, all conditions of Lemma 4.2 are satisfied and m/f β specified in (4.21) satisfies the equalities (see (4.1)) for all $\tilde{\omega} \in \Upsilon = \Omega(\omega|H_2)$:

$$\beta(\tilde{\omega}) \triangleq \{h \in \alpha(\tilde{\omega}) \mid (h|H_1) \in K\} \triangleq \{h \in \alpha(\tilde{\omega}) \mid (h|H_1) \in \bigcap_{\bar{\omega}\in\Omega(\omega|H_1)} \big(\alpha(\bar{\omega}) \,|\, H_1\big)\} =$$
$$= \{h \in \alpha(\tilde{\omega}) \mid (h|H_1) \in \bigcap_{\bar{\omega}\in\Omega(\tilde{\omega}|H_1)} \big(\alpha(\bar{\omega}) \,|\, H_1\big)\} \triangleq \langle\alpha\rangle_{H_1}(\tilde{\omega}).$$

Using the last relations and (4.22), we obtain:

$$\big(\langle\alpha\rangle_{H_1}(\omega'') \,|\, H_2\big) = \big(\beta(\omega'') \,|\, H_2\big) = \big(\beta(\omega') \,|\, H_2\big) = \big(\langle\alpha\rangle_{H_1}(\omega') \,|\, H_2\big) \qquad \forall \omega', \omega'' \in \Upsilon. \tag{4.28}$$

Due to arbitrary choice of ω, relations (4.28) are true for all $\omega', \omega'' \in \Omega(\omega|H_2)$ and all $\omega \in \Omega$, i. e., we have inclusion $\langle\alpha\rangle_{H_1} \in \mathbf{N}_{(H_2)}$. The proof is complete. □

The following statement provides for a finite chain $\mathcal{H}$ and for an arbitrary m/f α a representation of greatest $\mathcal{H}$-non-anticipative m/s of α as a finite superposition of operators from $\mathbf{F}_{\mathcal{H}}$. The provided construction inherits the features of the backward recurrent procedures [2, 19, 20] on the one hand, and of the method of programmed iterations of A. G. Chentsov, on the other hand.

Theorem 4.1. *Let $\alpha \in \mathcal{P}(\mathrm{Z})^{\Omega}$ and $\mathcal{H} = \{H_i \in \mathcal{T} \mid i \in 1..k, k \in \mathbb{N}\}$ be a finite chain: $(i \leqslant j) \Leftrightarrow$ $\Leftrightarrow (H_i \subset H_j)$ $\forall i, j \in 1..k$. Then the equality is fulfilled:*

$$\langle \ldots \langle\alpha\rangle_{H_k} \ldots \rangle_{H_1} = \langle\alpha\rangle_{\mathcal{H}}. \tag{4.29}$$

That is, the expression on the left side gives (see (4.18)) the greatest $\mathcal{H}$-non-anticipative m/s of m/f α.

P r o o f. 1. Denote $\phi \triangleq \langle \ldots \langle\alpha\rangle_{H_k} \ldots \rangle_{H_1}$. Then, successively applying the operators $\langle\cdot\rangle_{H_k}, \ldots, \langle\cdot\rangle_{H_1}$ to the m/f α and using the property (4.2), we arrive at the inequality

$$\phi \sqsubseteq \alpha; \tag{4.30}$$

also applying successively the operators $\langle\cdot\rangle_{H_k}, \ldots, \langle\cdot\rangle_{H_1}$ to the inequality $\langle\alpha\rangle_{\mathcal{H}} \sqsubseteq \alpha$ taking into account isotonicity (see (4.3)), we get the ratio

$$\langle\alpha\rangle_{\mathcal{H}} \sqsubseteq \phi. \tag{4.31}$$

Therefore (see (ii) of Lemma 4.1), the assertion will be proven if we establish the $\mathcal{H}$-non-anticipatory property of ϕ.

2. Let's show that, for all $i \in 1..k$, m/f ϕ holds the property (4.32) of H_i-non-anticipatory:

$$\big(\phi(\omega) \,|\, H_i\big) = \big(\phi(\omega') \,|\, H_i\big), \qquad \forall \omega, \omega' \in \Omega,\ \omega' \in \Omega(\omega|H_i). \tag{4.32}$$

Case $i = k$. M/f $\langle\alpha\rangle_{H_k}$ is H_k-non-anticipative by construction (see (i) of Lemma 4.1):

$$\left(\langle\alpha\rangle_{H_k}(\omega) \mid H_k\right) = \left(\langle\alpha\rangle_{H_k}(\omega') \mid H_k\right), \qquad \forall \omega, \omega' \in \Omega,\ \omega' \in \Omega(\omega|H_k). \tag{4.33}$$

Since $\mathcal{H}$ is a chain and hence $H_{k-1} \subset H_k$, taking into account Corollary 4.2, from (4.33), we get the property of H_k-non-anticipatory of m/f $\langle\langle\alpha\rangle_{H_k}\rangle_{H_{k-1}}$:

$$\left(\langle\langle\alpha\rangle_{H_k}\rangle_{H_{k-1}}(\omega) \mid H_k\right) = \left(\langle\langle\alpha\rangle_{H_k}\rangle_{H_{k-1}}(\omega') \mid H_k\right), \qquad \forall \omega, \omega' \in \Omega,\ \omega' \in \Omega(\omega|H_k). \tag{4.34}$$

Due to relations (4.34) we can apply the reasoning to m/f $\langle\langle\alpha\rangle_{H_k}\rangle_{H_{k-1}}$ and operator $\langle\cdot\rangle_{H_{k-2}}$. Continuing these arguments up to the application of the operator $\langle\cdot\rangle_{H_1}$ inclusively, we obtain the property of H_k-non-anticipatory for the m/f ϕ, i. e., we prove statement (4.32) for the case $i = k$.

Case $i \in 1..(k-1)$. Consider m/f $\psi \triangleq \langle \ldots \langle\alpha\rangle_{H_k} \ldots \rangle_{H_i}$. Since the operator $\langle\cdot\rangle_{H_i}$ was used last, by virtue of item (i) of Lemma 4.1, ψ is a H_i-non-anticipative m/f. Further, repeating the arguments from the case $i = k$ for the operators $\langle\cdot\rangle_{H_j}$, $j \in 1..(i-1)$, we conclude that these operators, applied to m/f ψ when constructing m/f ϕ, preserve the H_i-non-anticipatory property of m/f ψ. Namely, for all $j \in 1..(i-1)$ the equalities are true:

$$\left(\langle \ldots \langle\psi(\omega)\rangle_{H_{i-1}} \ldots \rangle_{H_j} \mid H_i\right) = \left(\langle \ldots \langle\psi(\omega')\rangle_{H_{i-1}} \ldots \rangle_{H_j} \mid H_i\right), \qquad \forall \omega, \omega' \in \Omega,\ \omega' \in \Omega(\omega|H_i).$$

In particular, for $j = 1$, we have (4.32) for $i \in 1..(k-1)$. So, m/f ϕ is H_i-non-anticipative for all $i \in 1..k$:

$$\phi \in \mathbf{N}_{(\mathcal{H})}[\alpha]. \tag{4.35}$$

From the relations (4.30), (4.35) and item (ii) of Lemma 4.1, the relation $\phi \sqsubseteq \langle\alpha\rangle_{\mathcal{H}}$ follows. Together with (4.31), the relation gives us equality $\phi = \langle\alpha\rangle_{\mathcal{H}}$, i. e., the required equality (4.29) is true. The proof is complete. □

Remark 4.2. Lemma 4.2 and Theorem 4.1 remain true in general case of sets T and $\mathcal{T}$, i. e., when T is a set and $\mathcal{T}$ is a chain in poset $(\mathcal{P}'(\mathrm{T}), \subset)$.

Let us return to step-by-step finding the selector under the conditions (α, Δ). Theorem 4.1 specifies a way to construct m/f $\langle\alpha\rangle_{\mathcal{H}_\Delta}$ that is the $\sqsubseteq$-greatest $\mathcal{H}_\Delta$-non-anticipative m/s of α. If this m/s turns out to be non-empty-valued, then, by virtue of Theorem 3.1, the step-by-step procedure can be implemented by means of $\langle\alpha\rangle_{\mathcal{H}_\Delta}$ for any disturbance $\omega \in \Omega$ (see Examples 5.3, 5.4). At the same time, as in Example 5.3, an ordinary non-anticipative m/s of m/f α may be absent ($\mathbf{N}^0_{(\mathcal{T})}[\alpha] = \varnothing$).

In the case, when for some $\omega \in \Omega$ the value of the m/s is empty ($\langle\alpha\rangle_{\mathcal{H}_\Delta}(\omega) = \varnothing$) due to $\sqsubseteq$-majority of $\langle\alpha\rangle_{\mathcal{H}_\Delta}$ in $\mathbf{N}_{(\mathcal{H}_\Delta)}[\alpha]$, we obtain the fact of unrealizability (Theorem 3.1) of step by step procedure under the conditions (α, Δ) and, as a consequence (see (3.1)), under any other conditions (α, Δ') where $\Delta \subset \Delta'$.

§ 5. Examples

5.1. Example 1

Let $\mathrm{T} \triangleq [0, 3]$, $\mathcal{T} \triangleq \{[0, \tau] \mid \tau \in \mathrm{T}\}$ and $\mathrm{Y} = \mathrm{X} = \mathbb{R}$. The sets Ω and Z are shown in Figure 1. Let m/f β be of the form

$$\beta(\omega) = \begin{cases} \{h_1, h_2\}, & \omega = \omega_1, \\ \{h_1, h_2, h_3\}, & \omega = \omega_2, \\ \{h_2, h_3\}, & \omega = \omega_3. \end{cases}$$

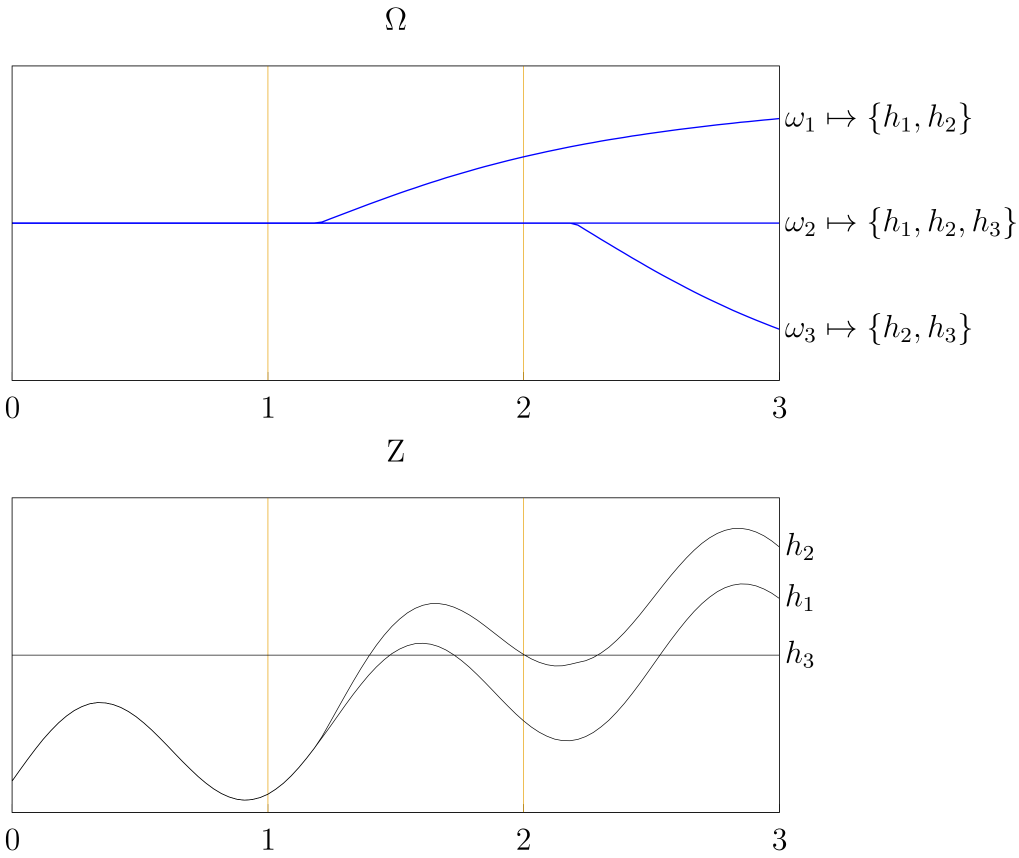

Fig. 1. Example 1

For this m/f, using definition (4.1), we get:

$$\langle\beta\rangle_{[0,1]}(\omega) = \begin{cases} \{h_1, h_2\}, & \omega = \omega_1, \\ \{h_1, h_2\}, & \omega = \omega_2, \\ \{h_2\}, & \omega = \omega_3, \end{cases} \qquad \langle\beta\rangle_{[0,2]}(\omega) = \begin{cases} \{h_1, h_2\}, & \omega = \omega_1, \\ \{h_2, h_3\}, & \omega = \omega_2, \\ \{h_2, h_3\}, & \omega = \omega_3. \end{cases}$$

It is clear that the inequalities $\langle\beta\rangle_{[0,1]} \not\sqsubseteq \langle\beta\rangle_{[0,2]}$, $\langle\beta\rangle_{[0,2]} \not\sqsubseteq \langle\beta\rangle_{[0,1]}$ are fulfilled. That is, in general, the mapping $\mathcal{T} \ni H \mapsto \langle\beta\rangle_H \in \mathcal{P}(\mathrm{Z})^{\Omega}$, considered as a mapping from poset $(\mathcal{T}, \subset)$ in poset $(\mathcal{P}(\mathrm{Z})^{\Omega}, \sqsubseteq)$ does not posses the isotonic property.

5.2. Example 2

The example shows that the operators $\langle\cdot\rangle_H$, $H \in \mathcal{T}$, can be non-commutative. Let $\mathrm{T} \triangleq [0, 3]$, $\mathcal{T} \triangleq \{[0, \tau] \mid \tau \in \mathrm{T}\}$ and $\mathrm{Y} = \mathbb{R}$, $\mathrm{X} = X_1 \times X_2 = \mathbb{R}^2$. Denote

$$\omega_{ij}(t) \triangleq (-1)^i \max\{0, t - j\}, \quad h_{ij}(t) \triangleq a_i\,(1 + \max\{0, t - j\}), \quad i \in 1..4,\ j \in 0..2,\ t \in [0, 3],$$

where $a_1 \triangleq (1, 0)$, $a_2 \triangleq (0, 1)$, $a_3 \triangleq (-1, 0)$, $a_4 \triangleq (0, -1)$. Put (see Figure 2; for reasons of symmetry of Z, the projection onto the plane $\mathrm{T} \times X_2$ is only shown)

$$\Omega \triangleq \{\omega_{ij} \mid i, j \in 1..2\}, \qquad \mathrm{Z} \triangleq \{h_{ij} \mid i \in 1..4,\ j \in 0..2\}.$$

Thus, all elements in Ω and Z are distinct. Moreover, if $k = i$, then

$$(\omega_{kl} | [0, \min\{l, j\}]) = (\omega_{ij} | [0, \min\{l, j\}]), \quad (h_{kl} | [0, \min\{l, j\}]) = (h_{ij} | [0, \min\{l, j\}]).$$

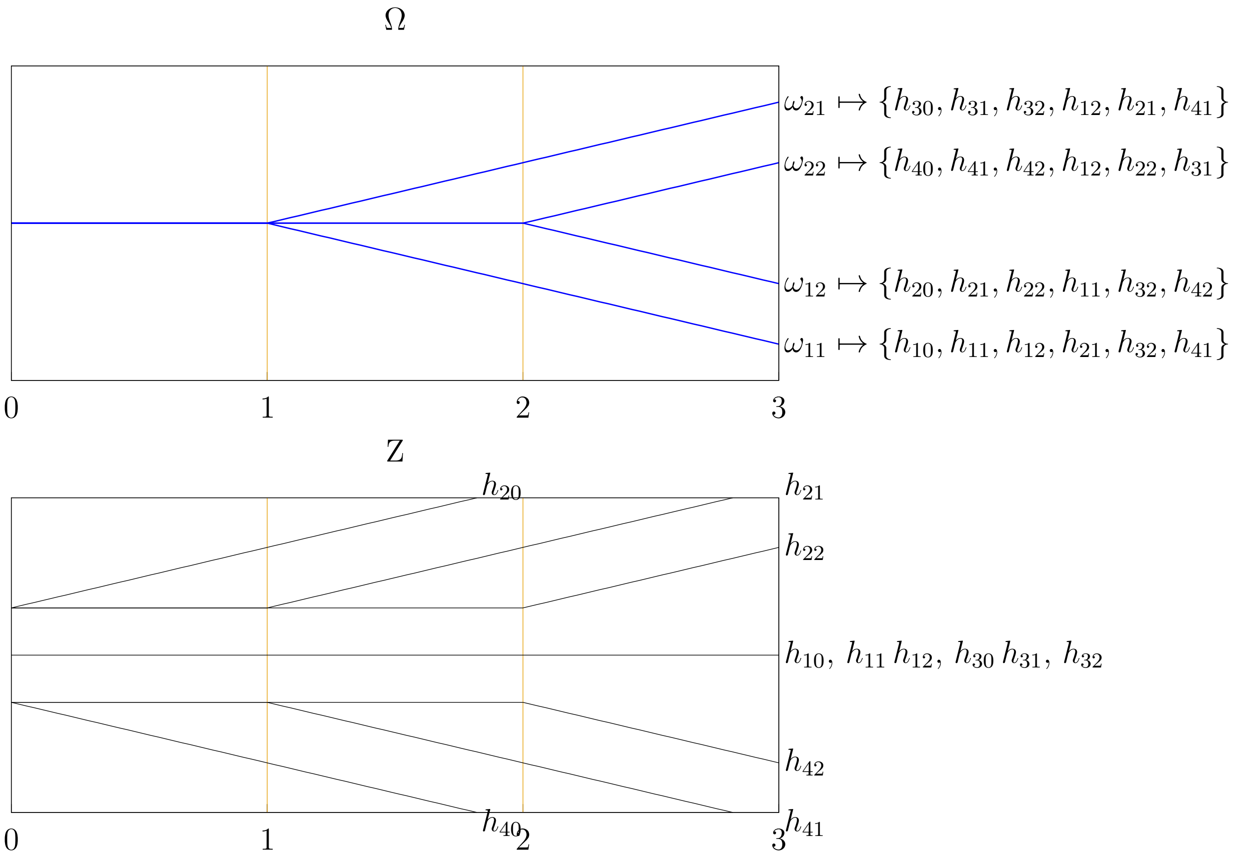

Fig. 2. Example 2

Let m/f α be given by

$$\alpha(\omega_{11}) \triangleq \{h_{10}, h_{11}, h_{12}, h_{21}, h_{32}, h_{41}\}, \quad \alpha(\omega_{21}) \triangleq \{h_{30}, h_{31}, h_{32}, h_{12}, h_{21}, h_{41}\},$$
$$\alpha(\omega_{12}) \triangleq \{h_{20}, h_{21}, h_{22}, h_{11}, h_{32}, h_{42}\}, \quad \alpha(\omega_{22}) \triangleq \{h_{40}, h_{41}, h_{42}, h_{12}, h_{22}, h_{31}\}.$$

Let $H_1 \triangleq [0,1]$. Then $\Omega = \Omega(\omega|H_1)$ for all $\omega \in \Omega$; hence, by definition (4.1), we have the equalities

$$\langle\alpha\rangle_{H_1}(\omega_{11}) = \{h_{11}, h_{12}, h_{21}, h_{32}, h_{41}\},$$
$$\langle\alpha\rangle_{H_1}(\omega_{12}) = \{h_{21}, h_{22}, h_{11}, h_{32}, h_{42}\},$$
$$\langle\alpha\rangle_{H_1}(\omega_{21}) = \{h_{31}, h_{32}, h_{12}, h_{21}, h_{41}\},$$
$$\langle\alpha\rangle_{H_1}(\omega_{22}) = \{h_{41}, h_{42}, h_{12}, h_{22}, h_{31}\}.$$

Let $H_2 \triangleq [0,2]$. Then $\Omega(\omega_{11}|H_2) = \{\omega_{11}\}$, $\Omega(\omega_{21}|H_2) = \{\omega_{21}\}$, $\Omega(\omega_{12}|H_2) = \Omega(\omega_{22}|H_2) = \{\omega_{12}, \omega_{22}\}$; and, by virtue of (4.1), we get

$$\langle\alpha\rangle_{H_2}(\omega_{11}) = \{h_{10}, h_{11}, h_{12}, h_{21}, h_{32}, h_{41}\},$$
$$\langle\alpha\rangle_{H_2}(\omega_{12}) = \{h_{22}, h_{42}\},$$
$$\langle\alpha\rangle_{H_2}(\omega_{21}) = \{h_{30}, h_{31}, h_{32}, h_{12}, h_{21}, h_{41}\},$$
$$\langle\alpha\rangle_{H_2}(\omega_{22}) = \{h_{42}, h_{22}\}.$$

Then, for m/f $\langle\langle\alpha\rangle_{H_1}\rangle_{H_2}$ we have

$$
\begin{aligned}
\langle\langle\alpha\rangle_{H_1}\rangle_{H_2}(\omega_{11}) &= \{h_{11}, h_{12}, h_{21}, h_{32}, h_{41}\},\\
\langle\langle\alpha\rangle_{H_1}\rangle_{H_2}(\omega_{12}) &= \{h_{22}, h_{42}\},\\
\langle\langle\alpha\rangle_{H_1}\rangle_{H_2}(\omega_{21}) &= \{h_{31}, h_{32}, h_{12}, h_{21}, h_{41}\},\\
\langle\langle\alpha\rangle_{H_1}\rangle_{H_2}(\omega_{22}) &= \{h_{42}, h_{22}\}.
\end{aligned}
$$

Finally, for the m/f $\langle\langle\alpha\rangle_{H_2}\rangle_{H_1}$ we have the relations:

$$
\begin{aligned}
\langle\langle\alpha\rangle_{H_2}\rangle_{H_1}(\omega_{11}) &= \{h_{21}, h_{41}\},\\
\langle\langle\alpha\rangle_{H_2}\rangle_{H_1}(\omega_{12}) &= \{h_{22}, h_{42}\},\\
\langle\langle\alpha\rangle_{H_2}\rangle_{H_1}(\omega_{21}) &= \{h_{21}, h_{41}\},\\
\langle\langle\alpha\rangle_{H_2}\rangle_{H_1}(\omega_{22}) &= \{h_{42}, h_{22}\}.
\end{aligned}
$$

It is easy to see that there is an inequality $\langle\langle\alpha\rangle_{H_2}\rangle_{H_1} \neq \langle\langle\alpha\rangle_{H_1}\rangle_{H_2}$ indicating that the operators $\langle\cdot\rangle_{H_1}$ and $\langle\cdot\rangle_{H_2}$ are not commutative.

Since the composition $\langle\langle\cdot\rangle_{H_2}\rangle_{H_1}$ corresponds to the order specified in Theorem 4.1, the result — $\langle\langle\alpha\rangle_{H_2}\rangle_{H_1}$ — presents an $\{H_1, H_2\}$-non-anticipative m/s of α.

5.3. Example 3

In this example, we consider an approaching game problem in which the m/f of optimal trajectories does not have any non-empty-valued and non-anticipative m/s, that is, the problem is not solvable in the class of non-anticipative strategies built on the base of usual (not relaxed) controls. At the same time, the m/f of optimal trajectories has the property of feasibility for all partitions of the interval T; i. e., the step-by-step procedure can be fulfilled for any partition and any disturbance.

Let the trajectories of the controlled system be given by solutions of the following Cauchy problem:

$$
\begin{cases}\dot{x}(t) = u(t) - v(t), & t \in \mathrm{T} \triangleq [0,2],\\ x(0) = 0 \in \mathbb{R}, & u \in \mathbf{U}, v \in \mathbf{V},\end{cases} \tag{5.1}
$$

$$
\mathbf{U} \triangleq \{u_i \mid i \in \mathbb{N}\}, \qquad \mathbf{V} \triangleq \{v_i \mid i \in \mathbb{N}\}, \tag{5.2}
$$

$$
u_i(t) \triangleq \begin{cases}0, & t \in [0,1],\\ 1 - 1/i, & t \in (1,2],\end{cases} \qquad v_i(t) \triangleq \begin{cases}0, & t \in [0, 1+1/i],\\ 1, & t \in (1+1/i, 2],\end{cases} \qquad i \in \mathbb{N}. \tag{5.3}
$$

For system (5.1)–(5.3), consider the problem of meeting its trajectories $x(\cdot)$ with the set $M \triangleq \{(2,x) \in \mathrm{T} \times \mathbb{R} \mid x \geqslant 0\}$ by choosing programmed control $u(\cdot) \in \mathbf{U}$ for any possible disturbances $v(\cdot) \in \mathbf{V}$. Denote by $x(\cdot, u, v)$, where $x(\cdot, u, v) \in C(\mathrm{T}, \mathbb{R})$, the solution of the Cauchy problem (5.1)–(5.3) where a control $u \in \mathbf{U}$ and a disturbance $v \in \mathbf{V}$ are given. Denote $\bar{\mathbf{U}} \triangleq \{\mathbf{u}_i \mid i \in \mathbb{N}\}$, $\bar{\mathbf{V}} \triangleq \{\mathbf{v}_i \mid i \in \mathbb{N}\}$ (see Figure 3)

$$
\mathbf{u}_i(t) \triangleq \int_0^t u_i(s)\, ds, \quad \mathbf{v}_i(t) \triangleq \int_0^t v_i(s)\, ds, \qquad i \in \mathbb{N},\ t \in \mathrm{T}.
$$

It is easy to verify that, for $x_{ij}(\cdot) \triangleq x(\cdot, u_i, v_j)$, the equalities (see (5.2), (5.3))

$$
x_{ij}(t) = \mathbf{u}_i(t) - \mathbf{v}_j(t), \quad \mathbf{u}_i(2) = \mathbf{v}_i(2) = 1 - \frac{1}{i}, \qquad i,j \in \mathbb{N}, t \in \mathrm{T},
$$

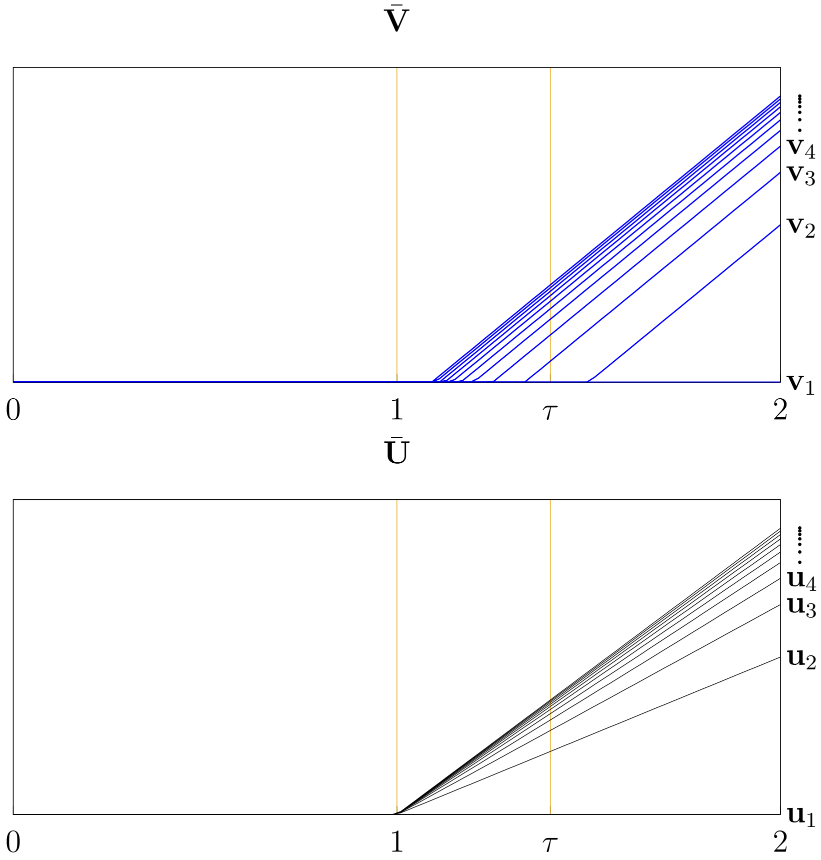

Fig. 3. Example 3

and, consequently, the equalities

$$x_{ij}(2) = \frac{1}{j} - \frac{1}{i}, \qquad i, j \in \mathbb{N},$$

are fulfilled. Hence, given a disturbance $v_j(\cdot) \in \mathbf{V}$ and a control $u_i(\cdot)$, the meeting criterion can be written as the inequality $i \geqslant j$. Therefore, for any disturbance $v_j(\cdot) \in \mathbf{V}$, denoting by $\alpha(v_j(\cdot))$ the set of all controls in $\mathbf{U}$ that solve the meeting problem, we can write

$$\alpha(v_j) = \{u_j, u_{j+1}, \ldots\} = \{u_i \in \mathbf{U} \mid i \geqslant j\}. \tag{5.4}$$

That is, in (5.4) we have the m/f of optimal answers. The family of subsets $\mathcal{T}$, implementing "flow of time", has its usual form: $\mathcal{T} \triangleq \{[0, \tau] \mid \tau \in \mathrm{T}\}$.

We choose a finite partition $\Delta \subset \mathrm{T}$ and show that conditions (α, Δ) are feasible. To this end, for $\mathcal{H}_\Delta \in \mathcal{T}$, we construct using (4.29) $\mathcal{H}_\Delta$-non-anticipative m/s of α and verify that its values be non-empty. Thus, by virtue of (3.10), the feasibility of step-by-step procedure under conditions (α, Δ) will be proven for arbitrary finite partition Δ.

1. Denote $H_\tau \triangleq [0, \tau]$. Then

$$\Omega(v|H_\tau) = \mathbf{V}, \qquad \tau \in [0, 1],\ v \in \mathbf{V};$$

$$\Omega(v_j|H_\tau) = \begin{cases} \{v_i, \ldots, v_1\}, & j \leqslant i, \\ \{v_j\}, & j > i, \end{cases} \qquad \tau \in \left(1 + \frac{1}{i+1}, 1 + \frac{1}{i}\right], \quad i, j \in \mathbb{N}.$$

Hence, by direct calculation (see (4.1), (5.4)), we obtain the representation of m/f $\langle\alpha\rangle_{H_\tau}$:

$$\langle\alpha\rangle_{H_\tau} = \alpha, \qquad \tau \in [0,1], \tag{5.5}$$

$$\langle\alpha\rangle_{H_\tau}(v_j) = \begin{cases} \alpha(v_i), & j \leqslant i, \\ \alpha(v_j), & j > i, \end{cases} \qquad \tau \in \left(1+\frac{1}{i+1}, 1+\frac{1}{i}\right], \qquad i,j \in \mathbb{N}. \tag{5.6}$$

From the equalities (5.5) and (5.6), the relations follow:

$$(\mathrm{DOM})\,[\langle\alpha\rangle_{H_\tau}] = \Omega, \qquad \forall \tau \in \mathrm{T}. \tag{5.7}$$

Moreover, it is not difficult to check that the representations (5.5) and (5.6) hold for a wider set of m/f, namely, for any $\beta \sqsubseteq \alpha$, the formulas take place:

$$\langle\beta\rangle_{H_\tau} = \beta, \qquad \tau \in [0,1],$$

and, for any β of the form $\beta \triangleq \langle\alpha\rangle_{H_\xi}$, $\xi \in \mathrm{T}$, the equalities are true:

$$\langle\beta\rangle_{H_\tau}(v_j) = \begin{cases} \beta(v_i), & i \geqslant j, \\ \beta(v_j), & i < j, \end{cases} \qquad \tau \in \left(1+\frac{1}{i+1}, 1+\frac{1}{i}\right], \qquad i,j \in \mathbb{N}. \tag{5.8}$$

2. Using presentations (5.5)–(5.8) in accordance with (4.29), we finally obtain the equality

$$\langle\alpha\rangle_{\mathcal{H}_\Delta} = \langle\alpha\rangle_{[0,\min\{\Delta\cap(1,2]\}]}.$$

Since Δ is finite and $\Delta \cap (1,2] \neq \varnothing$, we have $\langle\alpha\rangle_{\mathcal{H}_\Delta} = \langle\alpha\rangle_{H_{\bar\tau}}$ for same $H_{\bar\tau} \in \mathcal{H}_\Delta$. Then, in view of (5.7), we receive non-emptiness of values of m/f $\langle\alpha\rangle_{\mathcal{H}_\Delta}$. That is, non-emptiness of the set $\mathbf{N}^0_{(\mathcal{H}_\Delta)}[\alpha]$ of all non-empty-valued and $\mathcal{H}_\Delta$-non-anticipative m/s of α. Hence, taking into account Theorem 3.1, we conclude that conditions (α, Δ) are feasible.

3. Let us show that the problem has no solution in the class of non-anticipative strategies (quasi-strategies), that is, m/f α (5.4) does not have non-empty-valued non-anticipative m/s: $\mathbf{N}^0_{(\mathcal{T})}[\alpha] = \varnothing$.

Let's say the contrary that is m/f $\beta \in \mathbf{N}^0_{(\mathcal{T})}[\alpha]$ was found. Then (3.1) implies that $\mathbf{N}^0_{(\mathcal{T})}[\alpha] \subset \subset \mathbf{N}^0_{(\mathcal{H})}[\alpha]$, where $\mathcal{H} \triangleq \{H_k \mid k \in \mathbb{N}\}$ and $H_k \triangleq [0, 1+1/k]$. Let's use the implication (4.16); by assumption, we have $\beta \in \mathbf{N}^0_{(\mathcal{T})}[\alpha] \subset \mathbf{N}^0_{(\mathcal{H})}[\alpha]$, hence, $\beta \in \bigwedge_{k\in\mathbb{N}} \langle\alpha\rangle_{H_k}$. Then, taking into account (5.6) and (5.4), for each $j \in \mathbb{N}$ we obtain

$$\beta(v_j(\cdot)) \subset \bigcap_{k\in\mathbb{N}} \langle\alpha\rangle_{H_k}(v_j(\cdot)) = \bigcap_{k>j} \alpha(v_k(\cdot)) = \varnothing.$$

So, $(\mathrm{DOM})\,[\beta] = \varnothing$ and relation $\beta \in \mathbf{N}^0_{(\mathcal{T})}[\alpha]$ is impossible. Then, we have $\mathbf{N}^0_{(\mathcal{T})}[\alpha] = \varnothing$.

5.4. Example 4

Consider an example from [7, § 5]. We consider a construction of optimal non-anticipative strategy [7, (5.3)] by means of partially non-anticipative m/s.

As in previous example, the controlled system is given by solutions of the following Cauchy problem:

$$\dot{x}(t) = u(t) + v(t), \quad t \in \mathrm{T} \triangleq [0,3], \quad x(0) = 0,$$

where $x(t) \in \mathbb{R}$, the control u and disturbance v are Borel measurable functions subject to the instantaneous constraints $u(t) \in P \triangleq [-1,1]$, $v(t) \in Q \triangleq \{-1,0,1\}$, $t \in \mathrm{T}$.

Denote by $x(\cdot,u,v)$, where $x(\cdot,u,v)\in C(\mathrm{T},\mathbb{R})$, the solution of the Cauchy problem where a control $u\in\mathbf{U}$ and a disturbance $v\in\mathbf{V}$ are given.

Suppose that the set $\mathbf{V}$ of admissible disturbances consists of two functions, $\mathbf{V}\triangleq\{v_1,v_2\}$, where:

$$v_1(t)\triangleq\begin{cases}0, & \text{if } t\in[0,1]\cup(2,3],\\ 1, & \text{if } t\in(1,2],\end{cases}\qquad v_2(t)\triangleq\begin{cases}0, & \text{if } t\in[0,1],\\ -1, & \text{if } t\in(1,3].\end{cases}$$

Since the set of disturbances is continuous from the left, the condition (3.2) is satisfied. Then, taking into account the finite number of disturbances and by virtue of Lemma 3.1, we have (see (3.3)), for any m/f α, $\alpha\in\mathcal{P}'(\mathbf{U})^{\mathbf{V}}$, the equalities:

$$\mathbf{N}_{(\mathcal{T})}[\alpha]=\mathbf{N}_{([0,1])}\,[\alpha]\,. \tag{5.9}$$

So, the class of non-anticipative strategies in this problem coincides with the set of all $[0,1]$-non-anticipative m/f (from $\mathcal{P}'(\mathbf{U})^{\mathbf{V}}$); here $\mathbf{U}$ is the set of all possible realizations of control.

The control minimizes the following cost functional:

$$J(u,v)\triangleq-|x(3;u,v)|,\quad u\in\mathbf{U},\quad v\in\mathbf{V}.$$

Namely, the goal of control is to minimize the guaranteed result in the class of non-anticipative strategies from $\mathcal{P}'(\mathbf{U})^{\mathbf{V}}$.

Since we are considering a guaranteed state of the problem, denote by ρ (yet unknown) optimal (minimal) guaranteed result. Let us compose the m/f $\alpha_\rho\in\mathcal{P}(\mathbf{U})^{\mathbf{V}}$, which describes the ρ-optimal control responses to the realized disturbance.

Taking into account the obvious inequality $\rho\leqslant 0$, we write the values of α_ρ depending on ρ:

$$\alpha_\rho(v)=\left\{u\in\mathbf{U}\mid\left|\int_0^3\big(u(s)+v(s)\big)\,\mathrm{d}s\right|\geqslant-\rho\right\},\qquad v\in\mathbf{V}.$$

Then we get the equalities

$$\alpha_\rho(v_1)=\left\{u\in\mathbf{U}\mid\left(\int_0^3 u(s)\,\mathrm{d}s\leqslant\rho-1\right)\vee\left(-\rho-1\leqslant\int_0^3 u(s)\,\mathrm{d}s\right)\right\}, \tag{5.10}$$

$$\alpha_\rho(v_2)=\left\{u\in\mathbf{U}\mid\left(\int_0^3 u(s)\,\mathrm{d}s\leqslant\rho+2\right)\vee\left(-\rho+2\leqslant\int_0^3 u(s)\,\mathrm{d}s\right)\right\}. \tag{5.11}$$

It is clear that the set of values of the parameter ρ for which α_ρ has a non-empty-valued and non-anticipative m/s is bounded. That is, the set has infimum $\bar\rho$, which is the optimal result of the control side in the class of non-anticipative strategies. So, keeping in mind (5.9), we can write:

$$\bar\rho=\min\left\{\rho\in\mathbb{R}\mid(\mathrm{DOM})\left[\top_{\mathbf{N}_{(\mathcal{T})}[\alpha_\rho]}\right]=\mathbf{V}\right\}=\min\left\{\rho\in\mathbb{R}\mid(\mathrm{DOM})\left[\langle\alpha_\rho\rangle_{\{[0,1]\}}\right]=\mathbf{V}\right\}. \tag{5.12}$$

The m/f $\langle\alpha_\rho\rangle_{\{[0,1]\}}$ is defined by (see (4.1)):

$$\langle\alpha_\rho\rangle_{[0,1]}(v)\triangleq\left\{u\in\alpha_\rho(v)\;\middle|\;(u|[0,1])\in\bigcap_{\substack{(v'|[0,1])=(v|[0,1])\\ v'\in\mathbf{V}}}\big(\alpha_\rho(v')\mid[0,1]\big)\right\},\qquad v\in\mathbf{V}.$$

Due to this definition and (4.4) the condition $(\mathrm{DOM})\left[\langle\alpha_\rho\rangle_{\{[0,1]\}}\right]=\mathbf{V}$ is equivalent to the inequality

$$\bigcap_{v'\in\{v_1,v_2\}}\big(\alpha_\rho(v')\mid[0,1]\big)\neq\varnothing. \tag{5.13}$$

Thus, to find out the value of optimal result $\bar{\rho}$ (5.12), we have to calculate the minimum of ρ for which inequality (5.13) is true. To this end we express the sets $\boldsymbol{(}\alpha_\rho(v') \,|\, [0,1]\boldsymbol{)}$ in terms of $\alpha_\rho(v')$, $v' \in \mathbf{V}$: in accordance with (5.10), (5.11) we have

$$\boldsymbol{(}\alpha_\rho(v_1) \,|\, [0,1]\boldsymbol{)} = \Bigg\{ u \in \mathbf{U} \;\Bigg|\; \Big(\rho - 1 - \min_{u' \in \mathbf{U}} \int_1^3 u'(s)\,\mathrm{d}s \geqslant \int_0^1 u(s)\,\mathrm{d}s\Big) \vee \Big(\int_0^1 u(s)\,\mathrm{d}s \geqslant -\rho - 1 - \max_{u' \in \mathbf{U}} \int_1^3 u'(s)\,\mathrm{d}s\Big) \Bigg\},$$

$$\boldsymbol{(}\alpha_\rho(v_2) \,|\, [0,1]\boldsymbol{)} = \Bigg\{ u \in \mathbf{U} \;\Bigg|\; \Big(\rho + 2 - \min_{u' \in \mathbf{U}} \int_1^3 u'(s)\,\mathrm{d}s \geqslant \int_0^1 u(s)\,\mathrm{d}s\Big) \vee \Big(\int_0^1 u(s)\,\mathrm{d}s \geqslant -\rho + 2 - \max_{u' \in \mathbf{U}} \int_1^3 u'(s)\,\mathrm{d}s\Big) \Bigg\}.$$

Then the inequality (5.13) is true if and only if the sets $\boldsymbol{(}\alpha_\rho(v_1) \,|\, [0,1]\boldsymbol{)}$ and $\boldsymbol{(}\alpha_\rho(v_2) \,|\, [0,1]\boldsymbol{)}$ have common elements or, in other words, there exists $u \in \mathbf{U}$ for which the following statement is true:

$$\left(\Big(\rho + 1 \geqslant \int_0^1 u(s)\,\mathrm{d}s\Big) \vee \Big(\int_0^1 u(s)\,\mathrm{d}s \geqslant -\rho - 3\Big)\right) \;\&\; \left(\Big(\rho + 4 \geqslant \int_0^1 u(s)\,\mathrm{d}s\Big) \vee \Big(\int_0^1 u(s)\,\mathrm{d}s \geqslant -\rho\Big)\right). \quad (5.14)$$

Solving the minimization problem (5.12) with the constraints (5.14) leads us to the variant (see (upper-right)&(lower-left) braces in (5.14)):

$$\bar{\rho} = \min \left\{ \rho \in \mathbb{R} \;|\; \left\{ u \in \mathbf{U} \;|\; \rho + 4 \geqslant \int_0^1 u(s)\,\mathrm{d}s \geqslant -\rho - 3 \right\} \neq \varnothing \right\}$$

that implies $\rho + 4 \geqslant -\rho - 3$ or $\rho \geqslant -3.5$.

So, for the value $\bar{\rho}$ of optimal guarantee in the class of non-anticipative strategies, we have the equality $\bar{\rho} = -3.5$ ($= \Gamma$ from [7, § 5]); the greatest $[0,1]$-non-anticipative m/s of $\alpha_{\bar{\rho}}$ is non-empty-valued and given by the decision of minimization problem (5.12), (5.14):

$$\langle \alpha_{\bar{\rho}} \rangle_{[0,1]}(v_1)(t) \triangleq \begin{cases} u(t),\ u \in \mathbf{U}_{([0,1],0.5)}, & t \in [0,1], \\ 1 = \operatorname{argmax}_{w \in P} w, & t \in (1,3], \end{cases}$$

$$\langle \alpha_{\bar{\rho}} \rangle_{[0,1]}(v_2)(t) \triangleq \begin{cases} u(t),\ u \in \mathbf{U}_{([0,1],0.5)}, & t \in [0,1], \\ -1 = \operatorname{argmin}_{w \in P} w, & t \in (1,3], \end{cases}$$

where

$$\mathbf{U}_{([0,1],0.5)} \triangleq \left\{ u \in \mathbf{U} \;|\; (u|[0,1]) \in \bigcap_{v \in \{v_1, v_2\}} \boldsymbol{(}\alpha_{\bar{\rho}}(v) \,|\, [0,1]\boldsymbol{)} \right\} = \left\{ u \in \mathbf{U} \;|\; \int_0^1 u(s)\,\mathrm{d}s = \bar{\rho} + 4 = -\bar{\rho} - 3 = 0.5 \right\}.$$

In accordance with (ii) from Lemma 4.6, for the optimal non-anticipative strategy α_0 from [7, § 5], we have the inclusion $\alpha_0 \sqsubseteq \langle \alpha_{\bar{\rho}} \rangle_{[0,1]}$.

§ 6. Conclusion

Since the topic of the paper is at the initial stage of development, many details and even essential issues remain unexplored. Here, we make some remarks on them. Expression (4.29), in view of the maximality properties of (4.6) and (4.18), gives hope for new results on existence of a m/f non-anticipative m/s. Concerning applications, an interesting question arises about the convergence of guaranteed results for $\mathcal{H}_\Delta$-non-anticipatory m/s of a m/f α to the guaranteed result of the non-anticipative m/s of α as the step of the partition Δ tends to zero. Another question is the implementation of the proposed constructions in a solution of dynamic optimization problem. Namely, we need a systematic approach to resolving of the mathematical programming and the parametric optimization problems that arise when "calculating" partially non-anticipatory m/s (see the last example).

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Д. А. Серков

О построении частично неупреждающего мультиселектора и его использовании в задачах динамической оптимизации

Ключевые слова: неупреждающие мультиселекторы, многозначные стратегии, оптимизация гарантированного результата.

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В контексте задач гарантированного управления рассматриваются следующие вопросы: связь возможности пошагового (на заданном разбиении Δ) вычисления селектора мультифункции (м/ф) α для неизвестного, восстанавливаемого по шагам Δ, аргумента с существованием у α мультиселектора (м/с) со специальным свойством (названым здесь Δ-неупреждаемостью или частичной неупреждаемостью); второй вопрос — способы построение такого м/с для произвольной пары (α, Δ); и последний — поиск эффективно проверяемых условий, обеспечивающих совпадение Δ-неупреждающего м/с с неупреждающим.

Мотивом к рассмотрению этих вопросов послужила схема управления, возникающая, например, в методе альтернированного интеграла, при использовании в управлении контрстратегий, или в некоторых задачах при использовании метода управления с поводырём.

В работе показано, что рассматриваемая пошаговая схема управления реализуема тогда и только тогда, когда м/ф α имеет Δ-неупреждающий и непустозначный м/с. Дана конечношаговая процедура построения такого м/с. Указаны эффективно проверяемые условия, обеспечивающие неупреждаемость частично неупреждающего м/с. Рассмотрены иллюстрирующие примеры.

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