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COMPLETE CHARACTERIZATION OF BRIDGE GRAPHS WITH LOCAL ANTIMAGIC CHROMATIC NUMBER 2

An edge labeling of a connected graph $G = (V, E)$ is said to be local antimagic if it is a bijection $f\colon E \to \{1, \ldots, |E|\}$ such that for any pair of adjacent vertices x and y, $f^+(x) \neq f^+(y)$, where the induced vertex label $f^+(x) = \sum f(e)$, with e ranging over all the edges incident to x. The local antimagic chromatic number of G, denoted by $\chi_{la}(G)$, is the minimum number of distinct induced vertex labels over all local antimagic labelings of G. In this paper, we characterize s-bridge graphs with local antimagic chromatic number 2.

Keywords: local antimagic labeling, local antimagic chromatic number, s-bridge graphs.

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Introduction

In 1994, Hartsfield and Ringer [3] introduced the concept of antimagic labeling of a graph $G(V, E)$. A bijective edge labeling $f\colon E \to \{1, \ldots, |E|\}$ is called an antimagic labeling of G if for any two distinct vertices u and v, $w(u) \neq w(v)$, where $w(u) = \sum f(e)$ with e ranging over all the edges incident to u. The most famous unsolved problems are the following conjectures [3].

Conjecture 1. Every connected graph other than K_2 is antimagic.

Conjecture 2. Every tree other than K_2 is antimagic.

Arumugam et al. [1] introduced the concept of local antimagic labeling and local antimagic chromatic number. A connected graph G is said to be *local antimagic* if it admits a *local antimagic edge labeling*, i. e., a bijection $f\colon E \to \{1, \ldots, |E|\}$ such that the induced vertex labeling $f^+\colon V \to \mathbb{Z}$ given by $f^+(u) = \sum f(e)$ (with e ranging over all the edges incident to u) has the property that any two adjacent vertices have distinct induced vertex labels. Thus, f^+ is a coloring of G. Clearly, the order of G must be at least 3. The vertex label $f^+(u)$ is called the *induced color* of u under f (the *color* of u, for short, if no ambiguous occurs). The number of distinct induced colors under f is denoted by $c(f)$, and is called the *color number* of f. The *local antimagic chromatic number* of G, denoted by $\chi_{la}(G)$, is $\min\{c(f) \mid f \text{ is a local antimagic labeling of } G\}$. Clearly, $2 \leq \chi_{la}(G) \leq |V(G)|$. In [4], Haslegrave proved that the local antimagic chromatic number is well-defined for every connected graph except K_2.

In [1], the authors determined the local antimagic chromatic number of many families of standard graphs including paths, cycles, certain complete bipartite graphs and wheel graphs. In [5], Lau et al. gave a sharp lower bound of $\chi_{la}(G \vee O_n)$, where $G \vee O_n$ is the join product of G and the null graph of order n. They also completely settled the local antimagic chromatic number of wheels and complete bipartite graphs. In [6], the authors also determined the local antimagic chromatic number of the join product of many cycle-related graphs. However, very little is known about graphs with local antimagic chromatic number 2 (see [1, Theorem 2.11] and [7, Theorem 2.4]).

§ 1. Bridge graphs

A graph consisting of s paths joining two vertices is called an *s-bridge graph*, which is denoted by $\theta(a_1, \dots, a_s)$, where $s \geq 2$ and $1 \leq a_1 \leq a_2 \leq \dots \leq a_s$ are the lengths of the s paths. For convenience, we shall let $\theta_s = \theta(a_1, a_2, \dots, a_s)$ if there is no confusion. In this paper, we shall characterize θ_s with $\chi_{la}(\theta_s) = 2$.

The contrapositive of the following lemma in [6, Lemma 2.1] or [7, Lemma 2.3] gives a sufficient condition for a bipartite graph G to have $\chi_{la}(G) \geq 3$.

Lemma 1.1 ([7, Lemma 2.3]). *Let G be a graph of size q. Suppose there is a local antimagic labeling of G inducing a 2-coloring of G with colors x and y, where $x < y$. Let X and Y be the sets of vertices colored x and y, respectively. Then G is a bipartite graph with bipartition (X, Y) and $|X| > |Y|$. Moreover, $x|X| = y|Y| = \frac{q(q+1)}{2}$.*

Clearly, $2 \leq \chi(\theta(a_1, a_2, \dots, a_s)) \leq 3$ and the lower bound holds if and only if $a_1 \equiv \dots \equiv a_s \pmod 2$. By Lemma 1.1, we immediately have the following lemma.

Lemma 1.2. *For $s \geq 2$ and $1 \leq i \leq s$, if $\chi_{la}\big(\theta(a_1, a_2, \dots, a_s)\big) = 2$, then $a_i \equiv 0 \pmod 2$. Otherwise, $\chi_{la}\big(\theta(a_1, a_2, \dots, a_s)\big) \geq 3$.*

Throughout this paper, we shall use $a^{[n]}$ to denote a sequence of length n in which all terms are a, where $n \geq 2$. For integers $1 \leq a < b$, we let $[a, b]$ denote the set of integers from a to b. Interested readers may refer to [2, 8, 9] for more results related to local antimagic chromatic number of graphs.

§ 2. Main result

In this section, we assume $\chi_{la}(\theta_s) = 2$. So, by Lemma 1.2, $\theta_s = \theta(a_1, \dots, a_s)$ is bipartite and all a_i are even. When $s = 2$, θ_s is a cycle, whose local antimagic chromatic number is 3. Thus, $s \geq 3$.

For integers i and d and positive integer s, let $A_s(i; d)$ be the arithmetic progression of length s with common difference d and first term i. We first have two useful lemmas.

Lemma 2.1. *Suppose $s, d \in \mathbb{N}$.*

(a) *For $i, j \in \mathbb{Z}$, the sum of the k-th term of $A_s(i; d)$ and that of $A_s(j; -d)$ is $i + j$ for $k \in [1, s]$; and the sum of the k-th term of $A_s(i; d)$ and the $(k-1)$-st term of $A_s(j; -d)$ is $i + j + d$ for $k \in [2, s]$.*

(b) *If $0 < |i_1 - i_2| < d$, then $A_s(i_1; d) \cap A_s(i_2, \pm d) = \varnothing$.*

P r o o f. It is easy to obtain (a). We prove the contrapositive of (b). Suppose $A_s(i_1; d) \cap A_s(i_2, \pm d) \neq \varnothing$. Let $a \in A_s(i_1; d) \cap A_s(i_2, \pm d)$. Now, $a = i_1 + j_1 d = i_2 + j_2 d$ for some integers j_1, j_2. Thus, $|i_1 - i_2| = d|j_2 - j_1| \geq d$ if $j_2 \neq j_1$ or else $|i_1 - i_2| = 0$ if $j_2 = j_1$. ☐

Lemma 2.2. *Suppose $\delta \in [0, n^2] \setminus \{2, n^2 - 2\}$ for some integer $n \geq 2$. There is a subset B of $A_n(1; 2)$ such that the sum of integers in B is δ.*

P r o o f. If $\delta = 0$, then we may choose $B = \varnothing$. Suppose $1 \leq \delta \leq 2n - 1$ and $\delta \neq 2$. If δ is odd, then choose $B = \{\delta\}$. If δ is even, then $\delta \geq 4$. We may choose $B = \{1, \delta - 1\}$.

Suppose $\delta > 2n - 1$, then we may choose the largest k such that $\kappa = \sum\limits_{j=n-k+1}^{n} (2j - 1) \leq \delta$. Let $\tau = \delta - \kappa$. By the choice of k, $0 \leq \tau < 2n - 2k - 1$. There are 3 cases.

1. Suppose $\tau = 0$. $B = A_k(2n-2k+1;2)$ is the required subset.
2. Suppose τ is odd. $B = A_k(2n-2k+1;2) \cup \{\tau\}$ is the required subset.
3. Suppose τ is even. If $\tau \geq 4$, then we may choose $B = A_k(2n-2k+1;2) \cup \{\tau-1, 1\}$. If $\tau = 2$, then $2 = \tau < 2n-2k-1$. We have $k \leq n-2$. If $k \leq n-3$, then choose $B = A_{k-1}(2n-2k+3;2) \cup \{2n-2k-1, 3, 1\}$. If $k = n-2$, then $\kappa = n^2-4$ and hence $\delta = n^2-2$ which is not a case. □

Let A_1 and A_2 be two sequences of length n. We combine these two sequences as a sequence of length $2n$, denoted $A_1 \diamond A_2$, whose $(2i-1)$-st term is the i-th term of A_1 and the $(2i)$-th term is the i-th term of A_2, $1 \leq i \leq n$.

Theorem 2.1. *For $s \geq 3$, $\chi_{la}(\theta_s) = 2$ if and only if $\theta_s = K_{2,s}$ with even $s \geq 4$ or the size m of θ_s is greater than $2s+2$ and θ_s is one of the following graphs:*

(1) $\theta(4l^{[3l+2]}, (4l+2)^{[l]})$, $l \geq 1$;

(2a) $\theta(2l-2, (4l-2)^{[3l-1]})$, $l \geq 2$;

(2b) $\theta(2, 4^{[3]}, 6)$; $\theta(4, 8^{[5]}, 10^{[2]})$; $\theta(6, 12^{[7]}, 14^{[3]})$;

(3a) $\theta(4l-2-2t, 2t, (4l-4)^{[l]}, (4l-2)^{[l-2]})$, $2 \leq l \leq t \leq \frac{5l-2}{4}$;

(3b) $\theta(4l-2-2t, 2t-2, (4l-4)^{[l-1]}, (4l-2)^{[l-1]})$, $2 \leq l \leq t \leq \frac{5l}{4}$;

(4) $\theta(2t, 4s-6-2t, 2s-4, (4s-6)^{[s-3]})$, $\frac{2s-3}{8} \leq t \leq \frac{6s-5}{8}$, $s \geq 4$.

P r o o f. Note that $K_{2,s} = \theta(2^{[s]})$. In [1, Theorems 2.11 and 2.12], the authors obtained

$$\chi_{la}(K_{2,s}) = \begin{cases} 2, & \text{if } s \geq 4 \text{ is even,} \\ 3, & \text{otherwise.} \end{cases}$$

We only consider $\theta_s \neq K_{2,s}$, $s \geq 3$. Throughout the proof, we let u and v be the vertices of θ_s of degree s. We shall call the $2s$ edges incident to u or else to v as *end-edges*. An integer labeled to an end-edge is called an *end-edge label*. A path that starts at u and ends at v is called a *(u,v)-path*.

Suppose $\chi_{la}(\theta_s) = 2$. Since each a_i is even, θ_s has even size $m = \sum\limits_{i=1}^{s} a_i \geq 2s+2 \geq 8$ edges and order $m-s+2$. Let f be a local antimagic labeling that induces a 2-coloring of θ_s with colors x and y. Without lost of generality, we may assume $f^+(u) = f^+(v) = y$. Let X and Y be the sets of vertices with colors x and y, respectively. It is easy to get that $|Y| = m/2 - s + 2$ and $|X| = m/2$. By Lemma 1.1, we have $x|X| = y|Y| = m(m+1)/2$. Hence, $x = m+1 \geq 2s+3 \geq 9$ is odd, $y = m(m+1)/(m-2s+4)$ and $y \geq (1+2+\ldots+2s)/2 = (2s^2+s)/2$.

Note that θ_s has at least 2 adjacent non-end-edges. Suppose z_1z_2 is not an end-edge with $f(z_1z_2) = l$. Without loss of generality, we assume $f^+(z_1) = x$, $f^+(z_2) = y$. Since z_1z_2 is not an end-edge, there is another vertex z_3 such that $z_1z_2z_3$ forms a path. So, $f(z_2z_3) = y - l$. Since $1 \leq y - l \leq m$, we have $l \geq y - m = y - x + 1$. Consequently, all integers in $[1, y-x]$ must be assigned to end-edges. So, $y - x \leq 2s$. Moreover, since $l \neq y - l$, we get $l \neq y/2$ so that $y/2$ must be an end-edge label when y is even.

Solving for m, we get $m = \frac{1}{2}(y - 1 \pm \sqrt{y^2+14y-8ys+1})$. Hence, $y^2+14y-8ys+1 = t^2 \geq 0$, where t is a nonnegative integer. This gives $(y+7-4s)^2+1-(7-4s)^2 = t^2$

or $(y+7-4s-t)(y+7-4s+t) = 8(s-2)(2s-3)$. By letting $a = y+7-4s-t$ and $b = y+7-4s+t$, we have $2y+14-8s = a+b$ with $ab = 8(2s^2-7s+6) = 8(s-2)(2s-3)$. Clearly, $b \ge a > 0$. Since a, b must be of same parity, we have both a, b are even.

Recall that $y - (2s^2+s)/2 \ge 0$. Now

$$\begin{aligned} y - (2s^2+s)/2 &= 4s - 7 + \frac{a+b}{2} - \frac{2s^2+s}{2} \\ &= \frac{a+b}{2} - \frac{2s^2-7s+6}{2} - 4 = \frac{a+b}{2} - \frac{ab}{16} - 4 \\ &= \frac{8a+8b-ab-64}{16} = -\frac{(a-8)(b-8)}{16}. \end{aligned} \tag{2.1}$$

This implies that $a \le 8$.

We shall need the following claim which is easy to obtain. Throughout the proof, by symmetry, we always assume $\alpha_1 < \beta_r$.

Claim: *Let ϕ be a labeling of a path $P_{2r+1} = v_1v_2\dots v_{2r+1}$ with $\phi(v_{2i-1}v_{2i}) = \alpha_i$ and $\phi(v_{2i}v_{2i+1}) = \beta_i$ for $1 \le i \le r$. Suppose $\phi^+(v_{2j}) = x$ for $1 \le j \le r$ and $\phi^+(v_{2k+1}) = y$ for $0 \le k \le r$, where $y > x$. Then $\alpha_1 + \beta_1 = x$, $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$ is an increasing sequence with common difference $y - x$ while $\{\beta_1, \beta_2, \dots, \beta_r\}$ is a decreasing sequence with common difference $y - x$.*

We shall consider 4 cases for $a = 8, 6, 4, 2$ respectively.

Case (1). Suppose $a = 8$. By (2.1) we have $y = (2s^2+s)/2$ which implies s is even. Express t and y in terms of s. This gives (i) $m = s^2 - 3s/2 - 1$ which implies $s \equiv 2 \pmod 4$ and $x = s^2 - 3s/2$ or (ii) $m = 2s$. Since $m \ge 2s+2$, (ii) is not a case. In (i), $y - x = 2s$ so that all integers in $[1, 2s]$ are end-edge labels.

Let P be a (u,v)-path of θ_s with length $2r$ whose end-edges are labeled by integers in $[1, 2s]$. Suppose one of its end-edges is labeled by α_1. By the claim, another end-edge is labeled by $\beta_r = \beta_1 - (r-1)(y-x) = x - \alpha_1 - 2rs + 2s \le 2s$. So

$$2r \ge \frac{x-\alpha_1}{s} \ge \frac{s^2 - 3s/2 - 2s}{s} = s - \frac{7}{2}.$$

Since s and $2r$ are even, $2r \ge s-2$. Since $\beta_r \ge 2$, we have $2r \le \frac{1}{s}(x - \alpha_1 + 2s - 2) < s + \frac{1}{2}$. Thus, each (u,v)-path of θ_s is of length s or $s-2$. Suppose θ_s has h path(s) of length s and $(s-h)$ path(s) of length $s-2$. We now have $sh + (s-h)(s-2) = m\,(= s^2 - 3s/2 - 1)$. Therefore, $h = (s-2)/4$. Thus, $\theta_s = \theta\big((s-2)^{[(3s+2)/4]}, s^{[(s-2)/4]}\big)$ for $s \equiv 2 \pmod 4$.

Let $s = 4l+2$, $l \ge 1$. We now show that $\theta\big((s-2)^{[(3s+2)/4]}, s^{[(s-2)/4]}\big) = \theta\big((4l)^{[3l+2]}, (4l+2)^{[l]}\big)$ admits a local antimagic 2-coloring. Recall that $m = 16l^2 + 10l$, $x = 16l^2 + 10l + 1$, $y = 16l^2 + 18l + 5$ and $y - x = 8l + 4$.

Step 1: Label the edges of the path of length $4l+2$, denoted $R_i, 1 \le i \le l$, by using the sequence $A_{2l+1}(i; 8l+4) \diamond A_{2l+1}(x-i; -8l-4)$ in order. Note that, as a set $A_{2l+1}(x-i; -8l-4) = A_{2l+1}(2l+1-i; 8l+4)$. So, by Lemma 2.1(b), $A_{2l+1}(i; 8l+4) \diamond A_{2l+1}(x-i; -8l-4)$ for all $i \in [1, l]$, denoted U_1, form a partition of $\bigcup_{j=0}^{2l} [(8l+4)j+1, (8l+4)j+2l]$. By Lemma 2.1(a), we see that all induced labels of internal vertices are x and y alternatively. Now, integers in $[1, 2l]$ are end-edge labels.

Step 2: Label the edges of the path of length $4l$, denoted $Q_j, 1 \le j \le 3l+2$, by the sequence $A_{2l}(\alpha; 8l+4) \diamond A_{2l}(x-\alpha; -8l-4)$, where α is the j-th integer of the sequence $[3l+1, 4l+1] \cup [4l+3, 5l+1] \cup \{5l+3, 6l+3\} \cup [7l+5, 8l+4]$, denoted U_2, in order. Note again, $A_{2l}(\alpha; 8l+4) \diamond A_{2l}(x-\alpha; -8l-4)$ for all $\alpha \in U_2$ form a partition of $\bigcup_{j=0}^{2l-1}[(8l+4)j+2l+1, (8l+4)j+8l+4]$. By Lemma 2.1(a), we see that all induced labels of internal vertices are x and y alternatively. Now, integers in $[2l+1, 8l+4]$ are end-edge labels.

Step 3: We now merge the end-vertices with end-edge labels in $U_1 \cup U_2$ to get the vertex u. We then merge the other end-vertices with end-edge labels in $[1, 8l+4] \setminus (U_1 \cup U_2)$ to get the vertex v. Clearly, both u and v have induced vertex label y.

Note that

$$\left(\bigcup_{j=0}^{2l}[(8l+4)j+1, (8l+4)j+2l]\right) \cup \left(\bigcup_{j=0}^{2l-1}[(8l+4)j+2l+1, (8l+4)j+8l+4]\right) = [1, 16l^2+10l].$$

So the labeling defined above is a local antimagic 2-coloring for $\theta((4l)^{[3l+2]}, (4l+2)^{[l]})$.

Case (2). Suppose $a = 6$. Now, $b = \frac{4}{3}(s-2)(2s-3)$. By (2.1), we have $y = 2s(2s-1)/3$ and, hence, $s \equiv 0, 2 \pmod 3$. Similar to Case (1), since $m \geq 2s+2 \geq 8$, we must have $m = (4s^2-8s)/3$ and $s \geq 5$. Now $y - x = 2s-1$. So integers in $[1, 2s-1] \cup \{y/2 = (2s^2-s)/3\}$ are end-edge labels.

Note that there are $s-1$ paths in θ_s with both end-edges labeled with integers in $[1, 2s-1]$. Suppose P_{2r+1} is one of these $s-1$ paths. Since $\alpha_1 < \beta_r$, we have $\alpha_1 \in [1, 2s-2]$. Now, $\beta_r = (x-\alpha_1) - (r-1)(y-x) \leq 2s-1 = y-x$. Since $x = (4s^2-8s+3)/3$ and $y-x = 2s-1$, we have that

$$(2s-6)(2s-1)/3 + 1 = (4s^2-14s+9)/3 \leq x - \alpha_1 \leq r(y-x) = r(2s-1).$$

Thus, $r > (2s-6)/3 \geq \frac{4}{3}$, i.e., $r \geq 2$. Hence, β_{r-1} is labeled at a non-end-edge so that $\beta_{r-1} = (x-\alpha_1) - (r-2)(y-x) \geq 2s$. Therefore,

$$(r-2)(2s-1) \leq x - \alpha_1 - 2s \leq (4s^2-14s)/3 = (2s-6)(2s-1)/3 - 2 < (2s-6)(2s-1)/3.$$

Consequently, $r - 2 < (2s-6)/3 = 2s/3 - 2$, i.e., $r < 2s/3$. Combining the aboves, we have $2s/3 - 2 < r < 2s/3$ so that $2s-6 < 3r < 2s$. This implies that $3r \in [2s-5, 2s-1]$. Since $s \not\equiv 1 \pmod 3$, we have the following two cases.

a) Consider $s = 3l$, $l \geq 2$. Since $3r \equiv 0 \pmod 3$, we have $3r = 2s-3$, i.e., $r = 2l-1$. Thus, the s-th path must have length $m - (3l-1)(4l-2) = 2l-2$. Consequently, $\theta_{3l} = \theta\big(2l-2, (4l-2)^{[3l-1]}\big)$.

We now show that $\theta_{3l} = \theta\big(2l-2, (4l-2)^{[3l-1]}\big)$ admits a local antimagic 2-coloring. For $l = 2$, $\theta_6 = \theta\big(2, 6^{[5]}\big)$ with induced labels $y = 44, x = 33$ and the paths have vertex labels

22, 11;	1, 32, 12, 21, 23, 10;	3, 30, 14, 19, 25, 8;
4, 29, 15, 18, 26, 7;	5, 28, 16, 17, 27, 6;	9, 24, 20, 13, 31, 2.

All the left (respectively right) end vertices are merged to get the degree 6 vertex with induced label 44.

For $l \geq 3$, we apply the following steps.

Step 1: Label the edges of the path R_i of length $4l-2$ by the sequence $A_{2l-1}(i; 6l-1) \diamond A_{2l-1}((6l-1)(2l-1) - i; -6l+1)$ in order, $1 \leq i \leq 3l-1$.

Step 2: Label the path Q of length $2l-2$ by the sequence

$$A_{l-1}(6l-1;6l-1)\diamond A_{l-1}((6l-1)(l-2);-6l+1)$$

in order. By Lemma 2.1, one may check that all integers in $[1,4l(3l-2)]$ are assigned after the step.

Step 3: If we merge the end-vertices with end-edge labels in $[1,3l-1]\cup\{y/2\}$ as u, then the induced label of u is $\frac{1}{2}(9l^2-3l)+(6l^2-l)=\frac{1}{2}(21l^2-5l)$. Clearly it is less than $y=12l^2-2l$. The difference is $\delta=\frac{l}{2}(3l+1)$.

Step 4: Consider the set of differences of two end-edge labels in R_i, $1\le i\le 3l-1$, which is $D=\{1,3,5,\ldots,6l-3\}=A_{3l-1}(1;2)$. Clearly, $3<\delta<(3l-1)^2-3$. By Lemma 2.2, we have a subset B of D such that the sum of numbers in B is δ.

Step 5: Label all end-edges incident to u by

$$([1,3l-1]\setminus\{\frac{6l-1-i}{2}\mid i\in B\})\cup\{\frac{6l-1+i}{2}\mid i\in B\}\cup\{6l^2-l\}.$$

We have a local antimagic 2-coloring for $\theta_{3l}=\theta\big(2l-2,(4l-2)^{[3l-1]}\big)$.

b) Consider $s=3l-1$, $l\ge 2$. Now, $3r=2s-4$ or $2s-1$ so that $r\in\{2l-2,2l-1\}$. Note that $r\ge 2$.

Let the path with an end-edge label $y/2=(2s^2-s)/3$ be of length $2q$. Since $y/2\notin[1,2s-1]$ and we assume $\alpha_1<\beta_q$, this means $\beta_q=(2s^2-s)/3=(3l-1)(2l-1)$.

If $q=1$, then $\alpha_1+\beta_1=x$. This implies $\alpha_1+(3l-1)(2l-1)=(2l-1)(6l-5)$ and hence $\alpha_1=6l^2-11l+4$. Since $\alpha_1\le 2s-1=6l-3$, we get $6l^2-17l+7=(2l-1)(3l-7)\le 0$. The only solution is $l=2$ so that $s=5$. Note that $q=l-1$.

Suppose $q\ge 2$. Now $\alpha_q+\beta_q=x$ and $\alpha_q=\alpha_1+(q-1)(y-x)$ implies that $\alpha_1=x-\beta_q-(q-1)(2s-1)\le(2s-1)$. So $x-\beta_q\le q(2s-1)$. In terms of l, we have $(2l-1)(6l-5)-(3l-1)(2l-1)\le q(6l-3)$. Thus, $3l-4\le 3q$. This implies $q\ge l-1$. Also note that $\beta_1=\beta_q+(2s-1)(q-1)\le m=\frac{1}{3}(4s^2-8s)$. In terms of l we will obtain $(6l-3)q\le 6l^2-5l$. This implies $q\le l-\frac{2l}{6l-3}<l$. Thus, $q\le l-1$. Combining the aboves, we have $q=l-1$, as in $q=1$ above.

Now, suppose there are k paths of length $4l-4$ and $3l-2-k$ paths of length $4l-2$. We then have $(2l-2)+k(4l-4)+(3l-2-k)(4l-2)=4(3l-1)(l-1)=m$. Solving this, we get $k=2l-1$. Consequently, $\theta_{3l-1}=\theta\big(2l-2,(4l-4)^{[2l-1]},(4l-2)^{[l-1]}\big)$ for $l\ge 2$.

Recall that $y=12l^2-10l+2$, $x=12l^2-16l+5$, $y-x=6l-3$. Using the claim, we now have the followings.

- Consider the $l-1$ path(s) of length $4l-2$. We have $\alpha_1=i<\beta_{2l-1}=x-i-(y-x)\cdot\cdot(2l-2)=2l-1-i$. So $1\le i\le l-1$. Thus, numbers in $[1,l-1]$ must serve as α_1 for these $l-1$ path(s). Hence, numbers in $[l,2l-2]$ must serve as β_{2l-1} for these $l-1$ path(s). Thus, numbers in $[1,2l-2]$ are assigned to these $l-1$ paths.
- Consider the $2l-1$ paths of length $4l-4$. We have $2l-1\le\alpha_1=i<\beta_{2l-2}==x-i-(y-x)(2l-3)=8l-4-i$. So $2l-1\le i\le 4l-3$. Thus, numbers in $[2l-1,4l-3]$ must serve as α_1 for these $2l-1$ path(s). Hence, numbers in $[4l-1,6l-3]$ must serve as β_{2l-2} for these $2l-1$ path(s). Thus, numbers in $[2l-1,6l-3]\setminus\{4l-2\}$ are assigned to these $2l-1$ paths.

- Consider the path of length $2l-2$. This path must have $\alpha_1 = 4l-2$ and $\beta_{l-1} = 6l^2-5l+1 = y/2$.

Since $y/2$ is assigned to an end-edge incident to w, say, at the path of length $2l-2$, we have

$$\frac{1}{2}(25l^2-25l+6) = \sum_{i=1}^{l-1} i + \sum_{j=2l-1}^{4l-3} j + (6l^2-5l+1) \le f^+(w) = 12l^2-10l+2.$$

We get $l = 2, 3, 4$, which implies $s = 5, 8, 11$, respectively.

For $s = 5$, we get $\theta_5 = \theta(2, 4^{[3]}, 6)$ with induced vertex labels $y = 30$, $x = 21$. The labels of the paths are

$$15, 6;\quad 3, 18, 12, 9;\quad 4, 17, 13, 8;\quad 7, 14, 16, 5;\quad 1, 20, 10, 11, 19, 2.$$

For $s = 8$, we get $\theta_8 = \theta\big(4, 8^{[5]}, 10^{[2]}\big)$ with induced vertex labels $y = 80$, $x = 65$. The labels of the paths are

$40, 25, 55, 10;$ $5, 60, 20, 45, 35, 30, 50, 15;$ $6, 59, 21, 44, 36, 29, 51, 14;$
$7, 58, 22, 43, 37, 28, 52, 13;$ $8, 57, 23, 42, 38, 27, 53, 12;$ $11, 54, 26, 39, 41, 24, 56, 9;$
$1, 64, 16, 49, 31, 34, 46, 19, 61, 4;$ $2, 63, 17, 48, 32, 33, 47, 18, 62, 3.$

For $s = 11$, we get $\theta_{11} = \theta\big(6, 12^{[7]}, 14^{[3]}\big)$ with induced vertex labels $y = 154$, $x = 133$. The labels of the paths are

$77, 56, 98, 35, 119, 14;$ $7, 126, 28, 105, 49, 84, 70, 63, 91, 42, 112, 21;$
$8, 125, 29, 104, 50, 83, 71, 62, 92, 41, 113, 20;$ $9, 124, 30, 103, 51, 82, 72, 61, 93, 40, 114, 19;$
$10, 123, 31, 102, 52, 81, 73, 60, 94, 39, 115, 18;$ $11, 122, 32, 101, 53, 80, 74, 59, 95, 38, 116, 17;$
$12, 121, 33, 100, 54, 79, 75, 58, 96, 37, 117, 16;$ $13, 120, 34, 99, 55, 78, 76, 57, 97, 36, 118, 15;$
$1, 132, 22, 111, 43, 90, 64, 69, 85, 48, 106, 27, 127, 6;$ $2, 131, 23, 110, 44, 89, 65, 68, 86, 47, 107, 26, 128, 5;$
$4, 129, 25, 108, 46, 87, 67, 66, 88, 45, 109, 24, 130, 3.$

Case (3). Suppose $a = 4$. In this case, $b = 2(2s^2-7s+6)$ and $2y+14-8s = 4s^2-14s+16$. So $y = 2s^2-3s+1$. Similar to the previous cases, $m = 2s^2-5s+2$ only. Hence s is even, $x = 2s^2-5s+3$ and $y-x = 2s-2$. So, integers in $[1, 2s-2]$ must be assigned to $2s-2$ end-edges. Let the remaining two end-edges be labeled by γ_1 and γ_2. We have $4s^2-6s+2 = 2y = f^+(u)+f^+(v) = \sum\limits_{i=1}^{2s-2} i + \gamma_1 + \gamma_2 = (s-1)(2s-1)+\gamma_1+\gamma_2$. Thus, $\gamma_1+\gamma_2 = 2s^2-3s+1 = y$.

Suppose γ_1 and γ_2 are labeled at the end-edges of the same path of length $2q$. Without loss of generality, $\alpha_1 = \gamma_1$ and $\beta_q = \gamma_2$ so that $y = \alpha_1+\beta_q = \alpha_1+(x-\alpha_1)-(q-1)(y-x)$. We have $q(y-x) = 0$ which is impossible. Therefore, γ_1 and γ_2 are labeled at different paths. Thus, there are $s-2$ paths whose end-edges are labeled by integers in $[1, 2s-2]$ and exactly two paths, say Q_i with an end-edge label in $[1, 2s-2]$ and another end-edge label $\gamma_i \ge 2s-1$, $i = 1, 2$.

Suppose P_{2r+1} is a path with both end-edges labeled with integers in $[1, 2s-2]$. By the assumption $1 \le \alpha_1 < \beta_r \le 2s-2$ and the claim, we have $\beta_r = (x-\alpha_1)-(r-1)(y-x) \le 2s-2$. So,

$$(2s-2)(s-3) = 2s^2-8s+6 < 2s^2-7s+5 \le x-\alpha_1 \le r(y-x) = r(2s-2).$$

Thus, $r \ge s-2 \ge 2$. So β_{r-1} is labeled at a non-end-edge. Therefore, $\beta_{r-1} = (x-\alpha_1)-(r-2)\cdot$

$\cdot (y-x) \geq 2s-1$. We have

$$(r-2)(2s-2) \leq x-\alpha_1-2s+1 \leq 2s^2-7s+3 < 2s^2-6s+4 = (2s-2)(s-2).$$

So, $r < s$. Thus, $r \in \{s-2, s-1\}$.

Suppose Q_i is of length $2r_i$ whose end-edges are labeled by $\alpha_{1,i} \in [1, 2s-2]$ and $\beta_{r_i,i} = \gamma_i$. So, $\beta_{r_i,i} = \gamma_i = x-\alpha_{1,i}-(r_i-1)(y-x)$. Since $\gamma_1+\gamma_2 = 2s^2-3s+1$ is odd, $\gamma_2 \geq \frac{1}{2}(2s^2-3s+2)$ and $\gamma_1 \leq \frac{1}{2}(2s^2-3s)$. Now

$$\begin{aligned}(r_2-1)(2s-2) &= x-\alpha_{1,2}-\gamma_2 \leq 2s^2-5s+3-1-\frac{1}{2}(2s^2-3s+2)\\ &= (2s^2-7s+2)/2 = [(2s-2)(s-2)-s-2]/2 < (2s-2)(s-2)/2.\end{aligned}$$

We have $2r_2-2 < s-2$ and hence $2r_2 \leq s-2$.

Now $y = \gamma_1+\gamma_2 = 2x-\alpha_{1,1}-\alpha_{1,2}-(r_1+r_2-2)(y-x)$ or $(r_1+r_2-1)(2s-2) = = (r_1+r_2-1)(y-x) = x-\alpha_{1,1}-\alpha_{1,2}$. Since $\alpha_{1,1}, \alpha_{1,2} \in [1, 2s-2]$,

$$\begin{aligned}(s-1)(2s-2) &> (s-1)(2s-2)-s-2 = 2s^2-5s = x-3 \geq (r_1+r_2-1)(2s-2)\\ &\geq x-(4s-5) = 2s^2-9s+8 = (s-4)(2s-2)+s > (s-4)(2s-2).\end{aligned}$$

So, $s > r_1+r_2 > s-3$ or $2r_1+2r_2 \in \{2s-2, 2s-4\}$. Thus, $2r_1+s-2 \geq 2r_1+2r_2 \geq 2s-4$. So, we have $2r_1 \geq s-2 \geq 2r_2$. Since $2r_1+2r_2 \leq 2s-2$ and $2r_2 \geq 2$, $2r_1 \leq 2s-4$.

Without loss of generality, we may always assume that γ_1 is labeled at the end-edge of Q_1 incident to u. Since $s \geq 4$ and $f^+(u) = y$, γ_2 must be labeled at the end-edge of Q_2 incident to v. Suppose there are k paths of length $2s-4$ and $s-k-2$ paths of length $2s-2$. Therefore, $2(r_1+r_2)+k(2s-4)+(s-k-2)(2s-2) = 2s^2-5s+2$. So, $2(r_1+r_2) = s-2+2k$. For convenience, we write $s = 2l$ for $l \geq 2$.

(a) Suppose $2r_1+2r_2 = 4l-2$. Now, $k = l$ and $\theta_{2l} = \theta\big(4l-2-2r_1, 2r_1, (4l-4)^{[l]}, (4l-2)^{[l-2]}\big)$ for $l \leq r_1 \leq 2l-2$. Since $l-1 \geq r_2 = 2l-1-r_1$, $r_1 \geq l$. Rewriting r_1 as t, we have $\theta_{2l} = \theta\big(4l-2-2t, 2t, (4l-4)^{[l]}, (4l-2)^{[l-2]}\big)$ for $l \leq t \leq 2l-2$. Here Q_2 and Q_1 are (u,v)-paths of length $4l-2-2t$ and $2t$, respectively.

Next, we consider all (u,v)-paths of θ_s. Let the (u,v)-paths of length $4l-4$ be R_i, $1 \leq i \leq l$, and the (u,v)-path(s) of length $4l-2$ be T_j, $1 \leq j \leq l-2$. Let T_{l-1} be the path obtained from Q_2 and Q_1 by merging the vertex v of Q_2 and the vertex u of Q_1. Hence, T_{l-1} is a (u,v)-path of length $4l-2$. Under the labeling f, the end-edge labels are in $[1, 4l-2]$ and the induced vertex labels of all internal vertices of T_{l-1} are x and y alternatively.

(b) Suppose $2r_1+2r_2 = 4l-4$. Now, $2r_1 = 4l-4-2r_2 \leq 4l-6$ so that $k = l-1$ and $\theta_{2l} = \theta\big(4l-4-2r_1, 2r_1, (4l-4)^{[l-1]}, (4l-2)^{[l-1]}\big)$ for $l-1 \leq r_1 \leq 2l-3$. Rewriting r_1 as $t-1$, we have $\theta_{2l} = \theta\big(4l-2-2t, 2t-2, (4l-4)^{[l-1]}, (4l-2)^{[l-1]}\big)$ for $l \leq t \leq 2l-2$. Here Q_2 and Q_1 are (u,v)-paths of length $4l-2-2t$ and $2t-2$, respectively.

Next, we consider all (u,v)-paths of θ_s. Let the path(s) of length $4l-4$ be R_i, $1 \leq i \leq l-1$, and the path(s) of length $4l-2$ be T_j, $1 \leq j \leq l-2$. Let R_l be the path obtained from Q_2 and Q_1 by merging the vertex v of Q_2 and the vertex u of Q_1. Hence R_l is a (u,v)-path of length $4l-4$. Under the labeling f, the end-edge labels are in $[1, 4l-2]$ and the induced vertex labels of all internal vertices of R_l are x and y alternatively.

For each case, after the merging, we have l paths R_i of length $4l-4$, $1 \leq i \leq l$ and $l-1$ paths T_j of length $4l-2$, $1 \leq j \leq l-1$, where $l \geq 2$. All the end-edge labels are in $[1, 4l-2]$ under the labeling f. Consider the (u,v)-path R_i of length $2s-4=4l-4$. Suppose $x_i=\alpha_1$ is an end-edge label, then another end-edge label is $\beta_{s-2}=(x-\alpha_1)-(s-3)(2s-2)\le 2s-2$. We have $\alpha_1\ge s-1$. By symmetry, $\beta_{s-2}\ge s-1$. So, all the l paths R_i have their end-edges labeled by integers in $[2l-1,4l-2]$. Thus, all (u,v)-paths T_j have their end-edges labeled by integers in $[1,2l-2]$.

Let the label assigned to the end-edge of T_j incident to u be y_j.

(a) For the case $\theta_{2l}=\theta\big(4l-2-2t,2t,(4l-4)^{[l]},(4l-2)^{[l-2]}\big)$, $2\le l\le t\le 2l-2$, γ_1 is the $(4l-2-2t+1)$-st edge label of T_{l-1} so that $\gamma_1=y_{l-1}+(2l-1-t)(4l-2)$. Hence,

$$(4l-1)(2l-1)=f^+(u)=\gamma_1+\sum_{j=1}^{l-1}y_j+\sum_{i=1}^{l}x_i=y_{l-1}+(2l-1-t)(4l-2)+\sum_{j=1}^{l-1}y_j+\sum_{i=1}^{l}x_i.$$

We have

$$\begin{aligned}(2l-1-t)(4l-2)&=(4l-1)(2l-1)-y_{l-1}-\sum_{j=1}^{l-1}y_j-\sum_{i=1}^{l}x_i\\&\ge(4l-1)(2l-1)-(2l-2)-\frac{(l-1)(3l-2)}{2}-\frac{l(7l-3)}{2}\\&=3l^2-4l+2.\end{aligned}$$

This means

$$t(4l-2)\le 2(2l-1)^2-(3l^2-4l+2)=5l^2-4l=\frac{1}{4}[(5l-1)(4l-2)-2l-2]<\frac{1}{4}(5l-1)(4l-2).$$

Therefore, $t<\frac{5l-1}{4}$, i.e., $t\le\frac{5l-2}{4}$. Thus, $l\le t\le\frac{5l-2}{4}$.

(b) For the case $\theta_{2l}=\theta\big(4l-2-2t,2t-2,(4l-4)^{[l-1]},(4l-2)^{[l-1]}\big)$ for $2\le l\le t\le 2l-2$, similarly, we have

$$\begin{aligned}(2l-1-t)(4l-2)&=(4l-1)(2l-1)-x_l-\sum_{j=1}^{l-1}y_j-\sum_{i=1}^{l}x_i\\&\ge(4l-1)(2l-1)-(4l-2)-\frac{(l-1)(3l-2)}{2}-\frac{l(7l-3)}{2}\\&=3l^2-6l+2.\end{aligned}$$

This means

$$t(4l-2)\le 2(2l-1)^2-(3l^2-6l+2)=5l^2-2l=\frac{1}{4}[(5l+1)(4l-2)-2l+2]<\frac{1}{4}(5l+1)(4l-2).$$

Therefore, $t<\frac{5l+1}{4}$, i.e., $t\le\frac{5l}{4}$. Thus, $l\le t\le\frac{5l}{4}$.

Consequently, we have the following two cases.

(a) $\theta_{2l}=\theta\big(4l-2-2t,2t,(4l-4)^{[l]},(4l-2)^{[l-2]}\big)$ for $2\le l\le t\le\frac{5l-2}{4}$, or else

(b) $\theta_{2l}=\theta\big(4l-2-2t,2t-2,(4l-4)^{[l-1]},(4l-2)^{[l-1]}\big)$ for $2\le l\le t\le\frac{5l}{4}$.

Now, we are going to find a local antimagic 2-coloring for the above graphs.

(a) $\theta_{2l} = \theta\big(4l-2-2t, 2t, (4l-4)^{[l]}, (4l-2)^{[l-2]}\big)$ for $2 \le l \le t \le \frac{5l-2}{4}$.

Step 1: Label the edges of T_j by the sequence $A_{2l-1}(l-1+j; 4l-2) \diamond A_{2l-1}(x-l+1-j; -4l+2)$, $1 \le j \le l-1$. Note that we choose $\alpha_1 = l-1+j$. This gives $\beta_{2l-1} = l-j$. So, as a set $A_{2l-1}(x-(l-1+j); -4l+2) = A_{2l-1}(l-j; 4l-2)$. Thus, integers in $[1, 2l-2]$ are end-edge labels of all path(s) T_j and integers in $\bigcup_{j=1}^{l-1}[(j-1)(4l-2)+1, (j-1)(4l-2)+(2l-2)]$ are assigned.

Step 2: Label the edges of the (u,v)-path R_i by the sequence $A_{2l-2}(2l-2+i; 4l-2) \diamond \diamond A_{2l-2}(x-2l+2-i; -4l+2)$, $1 \le i \le l$. Note that we choose $\alpha_1 = 2l-2+i$. This gives $\beta_{2l-2} = 6l-3-(2l-2+i) = 4l-1-i$. So, as a set $A_{2l-2}(x-2l+2-i; -4l+2) = = A_{2l-2}(4l-1-i; 4l-2)$. Thus, integers in $[2l-1, 4l-2]$ are end-edge labels of all path(s) R_i and integers in $\bigcup_{i=1}^{l}[(i-1)(4l-2)+(2l-1), (i-1)(4l-2)+(4l-2)]$ are assigned. The set of difference between the two end-edge labels of a path R_i is $D_2 = \{1, 3, \ldots, 2l-1\} = A_l(1; 2)$.

Step 3: Pick the (u,v)-path T_{l-1} and separate it into two paths. Note that the end-edge labels of T_{l-1} are $2l-2$ and 1. The first $4l-2-2t$ edges form a (u,v)-path Q_2 and the remaining $2t$ edges form a (u,v)-path Q_1. Note that the label of $(4l-1-2t)$-th edge of T_{l-1} is $\gamma_1 = (2l-1-t)(4l-2)+(2l-2)$.

Thus, the above labeling is a local antimagic labeling. Under this labeling, the induced vertex label of u is

$$\begin{aligned}&\sum_{j=1}^{l-1}(l-1+j) + \sum_{i=1}^{l}(2l-2+i) + \gamma_1\\ &= \frac{(l-1)(3l-2)}{2} + \frac{l(5l-3)}{2} + (2l-1-t)(4l-2) + (2l-2)\\ &= 12l^2 + 2t - 10l - 4lt + 1.\end{aligned}$$

The difference from $y = 8l^2-6l+1$ is $\delta(t) = 4lt+4l-4l^2-2t = (4l-2)(t-l)+2l$. Clearly, $2 < \delta(t) \le (4l-2)\frac{l-2}{4} + 2l \le l^2$. Suppose $\delta(t) = l^2-2$, then $t = \frac{5l^2-4l-2}{4l-2} = \frac{5l-2}{4} + \frac{l-6}{2(4l-2)}$. Since $t \le \frac{5l-2}{4}$, $2 \le l \le 6$. Since $t \in \mathbb{Z}$, $l = 6$ and, hence, $t = 7$. Thus, by Lemma 2.2, we may choose $B \subset D_2$ to obtain a local antimagic 2-coloring of $\theta\big(4l-2-2t, 2t, (4l-4)^{[l]}, (4l-2)^{[l-2]}\big)$ for $2 \le l \le t \le \frac{5l-2}{4}$ and $(l,t) \ne (6,7)$. We shall provide a local antimagic 2-coloring for the special case $(l,t) = (6,7)$ in Example 3.3(a)(ii).

(b) $\theta_{2l} = \theta\big(4l-2-2t, 2t-2, (4l-4)^{[l-1]}, (4l-2)^{[l-1]}\big)$ for $2 \le l \le t \le \frac{5l}{4}$.

Step 1: Label the edges of T_j by the sequence $A_{2l-1}(j; 4l-2) \diamond A_{2l-1}(x-j; -4l+2)$, $1 \le j \le l-1$. The set of difference between the last label and the first label of a paths T_j's is $D_1 = \{1, 3, \ldots, 2l-3\} = A_{l-1}(1; 2)$.

Step 2: Label the edges of R_i by the sequence $A_{2l-2}(3l-2+i; 4l-2) \diamond A_{2l-2}(x-3l+2-i; -4l+2)$, $1 \le i \le l$. The set of difference between the last label and the first label of a paths R_i's, $1 \le i \le l-1$, is $D_2 = \{-1, -3, \ldots, -(2l-3)\} = A_{l-1}(-1; -2)$.

Step 3: Pick the (u,v)-path R_l and separate it into two paths. Note that the end-edge labels of R_l are $4l-2$ and $2l-1$. The first $4l-2-2t$ edges form a (u,v)-path Q_2 and the remaining $2t-2$ edges form a (u,v)-path Q_1. Note that the label of $(4l-1-2t)$-th edge of R_l is $\gamma_1 = (2l-1-t)(4l-2)+(4l-2)$.

Similar to the previous case, the above labeling is a local antimagic labeling. Under this labeling, the induced vertex label of u is

$$\sum_{j=1}^{l-1} j + \sum_{i=1}^{l}(3l-2+i) + \gamma_1 = \frac{(l-1)l}{2} + \frac{l(7l-3)}{2} + (2l-1-t)(4l-2) + (4l-2)$$
$$= 12l^2 + 2t - 6l - 4lt.$$

The difference from $y = 8l^2 - 6l + 1$ is $\delta(t) = -4l^2 - 2t + 4lt + 1$. Clearly, $\delta(t)$ is an increasing function of t. It is easy to show that $3 \le 2l-1 \le \delta(t) \le l^2 - \frac{5l}{2} + 1 \le$ $\le (l-1)^2 - 1$ when $l+1 \le t \le \frac{5l}{4}$. We need to show that $\delta(t) \ne (l-1)^2 - 2$. Now $\delta((5l-1)/4)) = \frac{2l^2-7l+3}{2} = (l-1)^2 - \frac{3l-1}{2} < (l-1)^2 - 2$. If $\frac{5l}{4} \in \mathbb{Z}$, then $l \ge 4$. So, $\delta(5l/4) = \frac{2l^2-5l+1}{2} = (l-2)^2 - \frac{l+1}{2} < (l-1)^2 - 2$. Thus, $3 \le \delta(t) \le l^2 - \frac{5l}{2} + 1 \le (l-1)^2 - 2$ when $l+1 \le t \le \frac{5l}{4}$. By Lemma 2.2, we may choose $B \subset D_1$ and then we obtain a local antimagic 2-coloring for $\theta(4l-2-2t, 2t-2, (4l-4)^{[l-1]}, (4l-2)^{[l-1]})$ for $l+1 \le t \le \frac{5l}{4}$.

The remaining case is $t = l$. For this case, $\delta(l) = -2l + 1$. If $l \ne 3$, then we may choose $B = \{-(2l-3), -3, 1\} \subset D_1 \cup D_2$. When $l = 3$, we have $t = 3$. This is a special case with solution given in Example 3.4(b).

Case (4). Suppose $a = 2$. In this case, $b = 4(2s^2 - 7s + 6)$ and $2y + 14 - 8s = 8s^2 - 28s + 26$. So, $y = 4s^2 - 10s + 6$. Similar to the previous cases we have $m = 4s^2 - 12s + 8$. Hence, $x = 4s^2 - 12s + 9$.

Suppose $s = 3$. We get $m = 8$, $x = 9$ and $y = 12$. Thus, $\theta_3 = \theta(2, 2, 4)$. The sequences we can use are $3, 6$; $1, 8$ and $4, 5, 7, 2$ or else $3, 6$; $1, 8, 4, 5$ and $7, 2$, both of which give no solution. We now assume $s \ge 4$.

Note that $y - x = 2s - 3$, y is even and $y/2 > 2s - 3$. Recall that if y is even, then $y/2$ is an end-edge label. Thus, integers in $[1, 2s-3] \cup \{y/2\}$ are end-edge labels.

There are only 3 end-edge labels greater than $2s - 3$. So, there are at least $s - 3$ paths with both end-edges labeled by integers in $[1, 2s-3]$. Suppose P_{2r+1} is one of these $s - 3$ paths. Keep the notation defined in the claim and the assumption $\alpha_1 < \beta_r$. So, $\alpha_1 \in [1, 2s-4]$.

Now $\beta_r = (x - \alpha_1) - (r-1)(y-x) \le 2s - 3$. Since $x = 4s^2 - 12s + 9$ and $y - x = 2s - 3$, we have

$$(2s-3)(2s-4) < 4s^2 - 14s + 13 \le x - \alpha_1 \le r(y-x) = r(2s-3)$$

Thus, $r \ge 2s - 3$.

Since $r \ge 4$, β_{r-1} is labeled at a non-end-edge. So, $\beta_{r-1} = (x - \alpha_1) - (r-2)(y-x) \ge 2s - 2$ so that

$$(r-2)(2s-3) \le x - \alpha_1 - 2s + 2 \le 4s^2 - 14s + 10 < (2s-3)(2s-4).$$

So, $r - 2 \le 2s - 5$ or $r \le 2s - 3$. Thus, $r = 2s - 3$. Note that $\beta_{2s-3} = 2s - 3 - \alpha_1$.

Suppose $y/2 = 2s^2 - 5s + 3$ is labeled at an end-edge of a path Q. Let the length of Q be $2q$. So, we have $\alpha_1 \le 2s - 3$, $\beta_q = y/2$ and $\beta_1 = y/2 + (q-1)(2s-3)$. Now $x = \alpha_1 + \beta_1 =$ $= \alpha_1 + y/2 + (q-1)(y-x)$ so that $2x > y + (2q-2)(y-x)$. We have $(2s-3)^2 = x >$ $> (2q-1)(y-x) = (2q-1)(2s-3)$. Thus, $2q - 1 < 2s - 3$, i. e., $q \le s - 2$.

On the other hand, $2x = 2\alpha_1 + y + (2q-2)(y-x) \le 2(2s-3) + y + (2q-2)(y-x) =$ $= y + 2q(y-x)$ so that $(2s-3)^2 = x \le (2q+1)(y-x) = (2q+1)(2s-3)$. This means $2q + 1 \ge 2s - 3$, i. e., $q \ge s - 2$. Thus, $q = s - 2$. Consequently, θ_s contains a path of length $2s - 4$ with an end-edge label $\beta_{s-2} = 2s^2 - 5s + 3 = y/2$ so that $\alpha_i = i(2s-3)$ and $\beta_i = 4s^2 - 14s + 12 - (i-1)(2s-3) = (2s-3)(2s-3-i) \ge (2s-3)(s-1)$ for $1 \le i \le s - 2$.

Let the remaining two end-edge labels be γ_1 and γ_2. Thus, $2y = f^+(u) + f^+(v) = \gamma_1 + \gamma_2 + y/2 + (2s-3)(s-1)$. So, $\gamma_1 + \gamma_2 = 4s^2 - 10s + 6 = y$.

Suppose γ_1 and γ_2 are labeled at the same path of length $2q$. By a similar proof of Case (3), we have $4s^2 - 10s + 6 = \gamma_1 + \gamma_2 = \gamma_1 + (x - \gamma_1) - (q-1)(y-x) = 4s^2 - 12s + 9 - (q-1)(2s-3)$ which is impossible.

As a conclusion, there are exactly $s-3$ paths of length $4s-6$ whose end-edges are labeled by integers in $[1, 2s-4]$, one path of length $2s-4$ whose end-edges are labeled by $2s-3$ and $y/2$, two paths Q_i of length s_i whose end-edges are labeled by $\alpha_{1,i} \in [1, 2s-4]$ and γ_i, $i = 1, 2$. By counting the number of edges of the graph, we have $s_1 + s_2 = 4s - 6$. Thus, $\theta_s = \theta\big(2t, 4s-6-2t, 2s-4, (4s-6)^{[s-3]}\big)$ for some $t \geq 1$.

Let us rename all (u, v)-paths.

- Let $R_1, \ldots R_{s-3}$ be the (u, v)-paths in θ_s of length $4s-6$. Let the end-edge label of R_i incident to u be x_i, $1 \leq i \leq s-3$.

- Let P be the (u, v)-path of length $2s-4$ whose end-edge labels are $2s-3$ and $(s-1)(2s-3)$.

- Let Q_1 be (u, v)-path of length $4s-6-2t$ whose end-edge labels are γ_1 and x_{s-1}. Let Q_2 be (u, v)-path of length $2t$ whose end-edge labels are x_{s-2} and γ_2. Without loss of generality, we may assume that $\gamma_1 < \gamma_2$. Since $\gamma_1 + \gamma_2 = y$, $\gamma_1 < y/2 < \gamma_2$. Also, without loss of generality, we may always assume that γ_1 is labeled at the end-edge incident to u. Thus, x_{s-2} is labeled at the end-edge of Q_2 incident to u.

  Let R_{s-2} be the labeled (u, v)-path obtained from Q_2 and Q_1 by merging the end vertex v of Q_2 with the end vertex u of Q_1. Therefore, R_{s-2} satisfies the assumption of the Claim. Thus, x_{s-2} is labeled at the end-edge of R_{s-2} incident to u. Now $\gamma_1 = t(2s-3) + x_{s-2}$.

Suppose $2s-3$ is labeled at the end-edge of P incident to u, then

$$2(s-1)(2s-3) = f^+(u) = \sum_{i=1}^{s-3} x_i + (2s-3) + x_{s-2} + \gamma_1$$
$$= \sum_{i=1}^{s-2} x_i + (2s-3) + [t(2s-3) + x_{s-2}] = \sum_{i=1}^{s-2} x_i + (t+1)(2s-3) + x_{s-2}.$$

This means $(2s-t-3)(2s-3) = x_{s-2} + \sum\limits_{i=1}^{s-2} x_i \leq (2s-4) + \frac{(s-2)(3s-5)}{2}$. Since $1 \leq t \leq s-2$, $(s-1)(2s-3) \leq (2s-4) + \frac{(s-2)(3s-5)}{2} = \frac{3s^2-7s+2}{2}$ which is impossible. Thus, $(s-1)(2s-3)$ must be a label of the end-edge of P incident to u. Consequently, we have

$$2(s-1)(2s-3) = f^+(u) = \sum_{i=1}^{s-3} x_i + (s-1)(2s-3) + x_{s-2} + \gamma_1$$
$$= \sum_{i=1}^{s-2} x_i + (s-1)(2s-3) + [t(2s-3) + x_{s-2}] = \sum_{i=1}^{s-2} x_i + (s-1+t)(2s-3) + x_{s-2}.$$

This means $(s-t-1)(2s-3) = x_{s-2} + \sum\limits_{i=1}^{s-2} x_i \geq 1 + \frac{(s-2)(s-1)}{2} = \frac{s^2-3s+4}{2} = \frac{(2s-3)^2}{8} + \frac{7}{8} > \frac{(2s-3)^2}{8}$. Solving this inequality, we have $t < \frac{6s-5}{8}$.

Similarly, we have $(s-t-1)(2s-3) = x_{s-2} + \sum_{i=1}^{s-2} x_i \le \frac{3s^2-7s+2}{2} = \frac{(6s-5)(2s-3)}{8} - \frac{7}{8} <$ $< \frac{(6s-5)(2s-3)}{8}$. Solving this inequality, we have $t > \frac{2s-3}{8}$.

Hence,

$$t \in \begin{cases} [2j-1, 6j-4], & \text{if } s = 8j-4; \\ [2j-1, 6j-3], & \text{if } s = 8j-3; \\ [2j, 6j-3], & \text{if } s = 8j-2; \\ [2j, 6j-2], & \text{if } s = 8j-1; \\ [2j, 6j-1], & \text{if } s = 8j; \\ [2j, 6j], & \text{if } s = 8j+1; \\ [2j+1, 6j], & \text{if } s = 8j+2; \\ [2j+1, 6j+1], & \text{if } s = 8j+3; \end{cases} \iff t \in \begin{cases} [k, 3k-1], & \text{if } s = 4k; \\ [k, 3k], & \text{if } s = 4k+1; \\ [k+1, 3k], & \text{if } s = 4k+2; \\ [k+1, 3k+1], & \text{if } s = 4k+3; \end{cases}$$

where $j, k \ge 1$.

We now show that $\theta_s = \theta\big(2t, 4s-6-2t, 2s-4, (4s-6)^{[s-3]}\big)$, for $s \ge 4$ and $\frac{2s-3}{8} < t < \frac{6s-5}{8}$, admits a local antimagic 2-coloring. We keep the notation defined above. Following is a general approach:

Step 1: Label the edges of the path R_j of length $4s-6$ by the sequence $A_{2s-3}(j; 2s-3) \diamond$ $\diamond A_{2s-3}(x-j; -(2s-3))$ in order, for $1 \le j \le s-2$.

Step 2: For convenience, write $x_{s-2} = \alpha$. Separate R_{s-2} into two paths. The first $2t$ edges form the path Q_2 and the rest form the path Q_1. So α and γ_1 are labeled at the end-edges incident to u. Recall that $\gamma_1 = t(2s-3) + \alpha$.

Step 3: Label the edges of the (u, v)-path P of length $2s-4$ by the reverse of the sequence $A_{s-2}(2s-3; 2s-3) \diamond A_{s-2}((2s-3)(2s-4); -2s+3)$, i. e., $A_{s-2}((s-1)(2s-3);$ $2s-3) \diamond A_{s-2}((s-2)(2s-3); -2s+3)$.

Clearly, by the construction above, it induces a local antimagic labeling for $\theta\big(2t, 4s-6-2t,$ $2s-4, (4s-6)^{[s-3]}\big)$. Under this labeling, the induced vertex label for u is

$$(s-1)(2s-3) + \sum_{i=1}^{s-2} i + \gamma_1 = (2s-3)(s-1+t) + \frac{s^2-3s+2}{2} + \alpha.$$

The difference from $y = (2s-3)(2s-2)$ is $\delta(t) = (2s-3)(s-1-t) - \frac{s^2-3s+2}{2} - \alpha$. Clearly, $\delta(t)$ is a decreasing function of t.

Now, if we choose $\alpha = 1$, then $\delta(t) = \frac{3s^2-7s-4st+6t+2}{2}$, where $\frac{2s-3}{8} < t < \frac{6s-5}{8}$. So,

$$\left.\begin{array}{r} 16k^2-11k+1 \\ 16k^2-k-1 \\ 16^2+k-1 \\ 16k^2+11k+1 \end{array}\right\} \ge \delta(t) \ge \begin{cases} 3k-2, & \text{if } s = 4k; \\ k-1, & \text{if } s = 4k+1; \\ 7k, & \text{if } s = 4k+2; \\ 5k+1, & \text{if } s = 4k+3. \end{cases}$$

The set of differences of two end-edge labels in R_j, $2 \le j \le s-2$, is $D = \{1, 3, 5, \dots, 2s-7\} =$ $= A_{s-3}(1; 2)$.

Clearly, $\delta(t) = 2$ only when $(s,t) = (13,9)$. Also the maximum value of $\delta(t)$ for each case of s is greater than $(s-3)^2$. Let us look at the second and third largest values δ_2 and δ_3 of $\delta(t)$ if any:

$$\delta_2 = \begin{cases} 16k^2 - 19k + 4, & \text{if } s = 4k; \\ 16k^2 - 9k, & \text{if } s = 4k+1; \\ 16k^2 - 7k - 2, & \text{if } s = 4k+2; \\ 16k^2 + 3k - 2, & \text{if } s = 4k+3; \end{cases} \qquad \delta_3 = \begin{cases} 16k^2 - 27k + 7, & \text{if } s = 4k; \\ 16k^2 - 17k + 1, & \text{if } s = 4k+1; \\ 16k^2 - 15k - 3, & \text{if } s = 4k+2; \\ 16k^2 - 5k - 5, & \text{if } s = 4k+3. \end{cases}$$

Clearly, $0 \le \delta_3 < (s-3)^2 - 2$. So, by Lemma 2.2, there is a subset B of D such that the sum of integers in B is $\delta(t)$ when $\frac{2s-3}{8} + 2 < t < \frac{6s-5}{8}$ except the case $(s,t) = (13,9)$. Similar to Case (2), we find a local antimagic 2-coloring for $\theta\big(2t, 4s-6-2t, 2s-4, (4s-6)^{[s-3]}\big)$ according to the above range of t.

For the case $(s,t) = (13,9)$, $y = 552$. Under the proposed labeling, we can see that the induced label for u is $549 + \alpha$. So, we may choose $a = 3$.

The remaining cases are when $\frac{2s-3}{8} < t \le \frac{2s-3}{8} + 2$. When $s = 4$, we have $\delta_2 = 1$ and δ_3 does not exist. We shall modify our proposed labeling. Now, we choose $\alpha = 2s-4$. In this case, 1 is not labeled at the end-edge incident to u so that the set of labels of the end-edges incident to u is $\{(s-1)(2s-3), \gamma_1\} \cup [2, s-2] \cup \{2s-4\}$. Thus, the sum is $(s-1)(2s-3) + (2s-4) + + \sum\limits_{i=2}^{s-2} i + \gamma_1 = (2s-3)(s-1+t) + \frac{s^2+5s-16}{2}$. The difference from $y = (2s-3)(2s-2)$ is $\delta^*(t) = \frac{3s^2-15s-4st+6t+22}{2}$. One may easily check that $3 \le \delta^*(t) \le (s-3)^2 - 3$ for $\frac{2s-3}{8} < t \le \le \frac{2s-3}{8} + 2$, except $(s,t) = (4,2), (5,2), (6,3), (7,3)$. Thus, we have a local antimagic 2-coloring for $\theta\big(2t, 4s-6-2t, 2s-4, (4s-6)^{[s-3]}\big)$ when $\frac{2s-3}{8} < t \le \frac{2s-3}{8} + 2$.

For those exceptional cases, we have

1. $(s,t) = (4,2)$. Now $\delta(2) = 1$. We may apply the original approach.

2. $(s,t) = (5,2)$, $\theta_5 = \theta(4, 6, 10, 14, 14)$ with edge labels
   $39, 10, 46, 3;$
   $7, 42, 14, 35, 21, 28;$
   $4, 45, 11, 38, 18, 31, 25, 24, 32, 17;$
   $1, 48, 8, 41, 15, 34, 22, 27, 29, 20, 36, 13, 43, 6;$
   $5, 44, 12, 37, 19, 30, 26, 23, 33, 16, 40, 9, 47, 2.$

3. $(s,t) = (6,3)$. Now $\delta(3) = 7 < 3^2$. We may apply the original approach.

4. $(s,t) = (7,3)$. Now $x = 121$, $y = 132$, $\theta_7 = \theta(6, 10, 16, 22, 22, 22, 22)$ with sequences
   $4, 117, 15, 106, 26, 95;$
   $66, 55, 77, 44, 88, 33, 99, 22, 110, 11;$
   $37, 84, 48, 73, 59, 62, 70, 51, 81, 40, 92, 29, 103, 18, 114, 7;$
   $2, 119, 13, 108, 24, 97, 35, 86, 46, 75, 57, 64, 68, 53, 79, 42, 90, 31, 101, 20, 112, 9;$
   $5, 116, 16, 105, 27, 94, 38, 83, 49, 72, 60, 61, 71, 50, 82, 39, 93, 28, 104, 17, 115, 6;$
   $8, 113, 19, 102, 30, 91, 41, 80, 52, 69, 63, 58, 74, 47, 85, 36, 96, 25, 107, 14, 118, 3;$
   $10, 111, 21, 100, 32, 89, 43, 78, 54, 67, 65, 56, 76, 45, 87, 34, 98, 23, 109, 12, 120, 1.$

So we have a local antimagic 2-coloring for $\theta(2t, 4s-6-2t, 2s-4, (4s-6)^{[s-3]})$ when $s \ge 4$ and $\frac{2s-3}{8} < t < \frac{6s-5}{8}$.

Note that, one may see from each case that $m > 2s+2$. This completes the proof. $\square$

§ 3. Examples

In this section, we shall provide example(s) to illustrate the construction of each case and also provide solutions for the exceptional cases raised in the proof of Theorem 2.1.

Example 3.1. The aim of this example is to illustrate the construction showed in Case (1).

Take $s = 6$ (i. e., $k = 1$), we have $\theta_6 = \theta(4, 4, 4, 4, 4, 6)$ with $m = 26$, $x = 27$, $y = 39$, $U_1 = \{1\}$, $U_2 = \{4, 5, 8, 9, 12\}$, $[1, 12] \setminus (U_1 \cup U_2) = \{2, 3, 6, 7, 10, 11\}$.

$A_3(1; 12) = (1, 13, 25)$ and $A_3(26; -12) = (26, 14, 2)$. So, $A_3(1; 12) \diamond A_3(26; -12) = = (1, 26, 13, 14, 25, 2)$.

Similarly, $A_2(4; 12) = (4, 16)$ and $A_2(23, -12) = (23, 11)$, $A_2(5; 12) = (5, 17)$ and $A_2(22; -12) = (22, 10)$, $A_2(8; 12) = (8, 20)$ and $A_2(19; -12) = (19, 7)$, $A_2(9; 12) = (9, 21)$ and $A_2(18; -12) = (18, 6)$, $A_2(12; 12) = (12, 24)$ and $A_2(15; -12) = (15, 3)$.

So, the paths of length 4 and 6 have edge labels

$$4, 23, 16, 11; \quad 5, 22, 17, 10; \quad 8, 19, 20, 7; \quad 9, 18, 21, 6; \quad 12, 15, 24, 3; \quad 1, 26, 13, 14, 25, 2.$$

All the left (respectively right) end vertices are merged to get the degree 6 vertex with induced label 39. ■

Example 3.2. The aim of this example is to illustrate the construction showed in Case (2).

Take $s = 9$ (i. e., $l = 3$), we get $\theta\big(4, 10^{[8]}\big)$ with $y = 102$, $x = 85$. Keep the notation defined in Lemma 2.2 and the proof of Theorem 2.1. Since $\delta = 15$, $n = 8$, we choose $\kappa = 15$ with $\tau = 0$. By Lemma 2.2, we have $B = \{15\}$. So we replace 1 by 16 as a label of end-edge incident to u. Thus u is incident to end-edge labels in $\{16, 2, 3, 4, 5, 6, 7, 8, 51\}$. The paths labels are
$51, 34, 68, 17$: $A_2(51; 17) \diamond A_2(34; -17)$;
$16, 69, 33, 52, 50, 35, 67, 18, 84, 1$: the *reverse of* $A_5(1; 17) \diamond A_5(84; -17)$;
$2, 83, 19, 66, 36, 49, 53, 32, 70, 15$: $A_5(2; 17) \diamond A_5(83; -17)$;
$3, 82, 20, 65, 37, 48, 54, 31, 71, 14$: $A_5(3; 17) \diamond A_5(82; -17)$;
$4, 81, 21, 64, 38, 47, 55, 30, 72, 13$: $A_5(4; 17) \diamond A_5(81; -17)$;
$5, 80, 22, 63, 39, 46, 56, 29, 73, 12$: $A_5(5; 17) \diamond A_5(80; -17)$;
$6, 79, 23, 62, 40, 45, 57, 28, 74, 11$: $A_5(6; 17) \diamond A_5(79; -17)$;
$7, 78, 24, 61, 41, 44, 58, 27, 75, 10$: $A_5(7; 17) \diamond A_5(78; -17)$;
$8, 77, 25, 60, 42, 43, 59, 26, 76, 9$: $A_5(8; 17) \diamond A_5(77; -17)$.

Using $s = 12$ (i. e., $l = 4$), we get $\theta(6, 14^{[11]})$ with $y = 184$, $x = 161$. Since $\delta = 26$. We choose $\kappa = 21$ (i. e., $k = 1$) with $\tau = 5$. By Lemma 2.2, we have $B = \{21, 5\}$. So, we replace 1 by 22 and 9 by 14 as labels of end-edges incident to u. Thus, u is incident to end-edge labels in $\{22, 2, 3, 4, 5, 6, 7, 8, 14, 10, 11, 92\}$. The paths labels are
$92, 69, 115, 46, 138, 23$: $A_3(92; 23) \diamond A_3(69; -23)$;
$22, 139, 45, 116, 68, 93, 91, 70, 114, 47, 137, 24, 160, 1$: the reverse of $A_7(1; 23) \diamond A_7(160; -23)$;
$2, 159, 25, 136, 48, 113, 71, 90, 94, 67, 117, 44, 140, 21$: $A_7(2; 23) \diamond A_7(159; -23)$;
$3, 158, 26, 135, 49, 112, 72, 89, 95, 66, 118, 43, 141, 20$: $A_7(3; 23) \diamond A_7(158; -23)$;
$4, 157, 27, 134, 50, 111, 73, 88, 96, 65, 119, 42, 142, 19$: $A_7(4; 23) \diamond A_7(157; -23)$;
$5, 156, 28, 133, 51, 110, 74, 87, 97, 64, 120, 41, 143, 18$: $A_7(5; 23) \diamond A_7(156; -23)$;
$6, 155, 29, 132, 52, 109, 75, 86, 98, 63, 121, 40, 144, 17$: $A_7(6; 23) \diamond A_7(155; -23)$;
$7, 154, 30, 131, 53, 108, 76, 85, 99, 62, 122, 39, 145, 16$: $A_7(7; 23) \diamond A_7(154; -23)$;
$8, 153, 31, 130, 54, 107, 77, 84, 100, 61, 123, 38, 146, 15$: $A_7(8; 23) \diamond A_7(153; -23)$;
$14, 147, 37, 124, 60, 101, 83, 78, 106, 55, 129, 32, 152, 9$: the reverse of $A_7(9; 23) \diamond A_7(152; -23)$;
$10, 151, 33, 128, 56, 105, 79, 82, 102, 59, 125, 37, 148, 13$: $A_7(10; 23) \diamond A_7(151; -23)$;
$11, 150, 34, 127, 57, 104, 80, 81, 103, 58, 126, 36, 149, 12$: $A_7(11; 23) \diamond A_7(150; -23)$. ■

Example 3.3. The aim of this example is to illustrate the construction showed in Case (3) and provide a local antimagic 2-coloring for the exceptional case $(l, t) = (6, 7)$.

Let $s = 12$, i. e., $l = 6$. Now, $x = 231$ and $y = 253$.

(a) The graph is $\theta_{12} = \theta\big(22 - 2t, 2t, 20^{[6]}, 22^{[4]}\big)$, where $t = 6, 7$. Begin with the sequences

$A_{11}(6;22) \diamond A_{11}(225;-22)$: 6, 225, 28, 203, 50, 181, 72, 159, 94, 137, 116, 115, 138, 93, 160, 71, 182, 49, 204, 27, 226, 5
$A_{11}(7;22) \diamond A_{11}(224;-22)$: 7, 224, 29, 202, 51, 180, 73, 158, 95, 136, 117, 114, 139, 92, 161, 70, 183, 48, 205, 26, 227, 4
$A_{11}(8;22) \diamond A_{11}(223;-22)$: 8, 223, 30, 201, 52, 179, 74, 157, 96, 135, 118, 113, 140, 91, 162, 69, 184, 47, 206, 25, 228, 3
$A_{11}(9;22) \diamond A_{11}(222;-22)$: 9, 222, 31, 200, 53, 178, 75, 156, 97, 134, 119, 112, 141, 90, 163, 68, 185, 46, 207, 24, 229, 2
$A_{11}(10;22) \diamond A_{11}(221;-22)$: 10, 221, 32, 199, 54, 177, 76, 155, 98, 133, 120, 111, 142, 89, 164, 67, 186, 45, 208, 23, 230, 1

$A_{10}(11;22) \diamond A_{10}(220;-22)$: 11, 220, 33, 198, 55, 176, 77, 154, 99, 132, 121, 110, 143, 88, 165, 66, 187, 44, 209, 22
$A_{10}(12;22) \diamond A_{10}(219;-22)$: 12, 219, 34, 197, 56, 175, 78, 153, 100, 131, 122, 109, 144, 87, 166, 65, 188, 43, 210, 21
$A_{10}(13;22) \diamond A_{10}(218;-22)$: 13, 218, 35, 196, 57, 174, 79, 152, 101, 130, 123, 108, 145, 86, 167, 64, 189, 42, 211, 20
$A_{10}(14;22) \diamond A_{10}(217;-22)$: 14, 217, 36, 195, 58, 173, 80, 151, 102, 129, 124, 107, 146, 85, 168, 63, 190, 41, 212, 19
$A_{10}(15;22) \diamond A_{10}(216;-22)$: 15, 216, 37, 194, 59, 172, 81, 150, 103, 128, 125, 106, 147, 84, 169, 62, 191, 40, 213, 18
$A_{10}(16;22) \diamond A_{10}(215;-22)$: 16, 215, 38, 193, 60, 171, 82, 149, 104, 127, 126, 105, 148, 83, 170, 61, 192, 39, 214, 17

Now the difference sets are $D_1 = A_5(-1;-2)$ and $D_2 = A_6(1;2)$.

i) $t = 6$. So, $\theta_{12} = \theta\big(10, 12, 20^{[6]}, 22^{[4]}\big)$. Initially, we use the first five sequences above to label the (u, v)-paths T_j and the last six sequences above to label the (u, v)-paths R_i. We then break T_5 into two parts such that the first 10 edges form the (u, v)-path Q_2 and the remaining 12 edges form the (u, v)-path Q_1. Now, the induced vertex label for u is $\sum_{j=6}^{16} j + 120 = 241$. Thus, $\delta(6) = 12$. So, we choose $B = \{1, 11\} \subset D_2$. Therefore, the actual assignment for each (u, v)-path is to label:
T_1 by $A_{11}(6;22) \diamond A_{11}(225;-22)$; T_2 by $A_{11}(7;22) \diamond A_{11}(224;-22)$; T_3 by $A_{11}(8;22) \diamond A_{11}(223;-22)$; T_4 by $A_{11}(9;22) \diamond A_{11}(222;-22)$;
Q_2 by 10, 221, 32, 199, 54, 177, 76, 155, 98, 133;
Q_1 by 120, 111, 142, 89, 164, 67, 186, 45, 208, 23, 230, 1;
R_1 by the reverse of $A_{10}(11;22) \diamond A_{10}(220;-22)$; R_2 by $A_{10}(12;22) \diamond A_{10}(219;-22)$; R_3 by $A_{10}(13;22) \diamond A_{10}(218;-22)$; R_4 by $A_{10}(14;22) \diamond A_{10}(217;-22)$; R_5 by $A_{10}(15;22) \diamond A_{10}(216;-22)$; R_6 by the reverse of $A_{10}(16;22) \diamond A_{10}(215;-22)$.
Thus,

$$f^+(u) = 6 + 7 + 8 + 9 + 10 + 120 + 22 + 12 + 13 + 14 + 15 + 17 = 253.$$

ii) $t = 7$. So, $\theta_{12} = \theta\big(8, 14, 20^{[6]}, 22^{[4]}\big)$. Initially, we use the first five sequences above to label the (u, v)-paths T_j and the last six sequences above to label the (u, v)-paths R_i. We then break T_5 into two parts such that the first 8 edges form the (u, v)-path Q_2 and the remaining 14 edges form the (u, v)-path Q_1. Now, the induced vertex label for u is $\sum_{j=6}^{16} j + 98 = 219$. Thus, $\delta(7) = 34$. For this case, we do not have $B \subset D_2$. So, we choose $B = \{-1, 3, 5, 7, 9, 11\} \subset D_1 \cup D_2$. Thus the actual assignment for each (u, v)-path is to label:
T_1 by the reverse of $A_{11}(6;22) \diamond A_{11}(225;-22)$; T_2 by $A_{11}(7;22) \diamond A_{11}(224;-22)$; T_3 by $A_{11}(8;22) \diamond A_{11}(223;-22)$; T_4 by $A_{11}(9;22) \diamond A_{11}(222;-22)$;
Q_2 by 10, 221, 32, 199, 54, 177, 76, 155;
Q_1 by 98, 133, 120, 111, 142, 89, 164, 67, 186, 45, 208, 23, 230, 1;
R_1 by the reverse of $A_{10}(11;22) \diamond A_{10}(220;-22)$; R_2 by the reverse of $A_{10}(12;22) \diamond A_{10}(219;-22)$; R_3 by the reverse of $A_{10}(13;22) \diamond A_{10}(218;-22)$; R_4 by the reverse of

$A_{10}(14;22) \diamond A_{10}(217;-22)$; R_5 by the reverse of $A_{10}(15;22) \diamond A_{10}(216;-22)$; R_6 by $A_{10}(16;22) \diamond A_{10}(215;-22)$.
Thus,

$$f^+(u) = 5+7+8+9+10+98+22+21+20+19+18+16 = 253.$$

(b) The graph is $\theta_{12} = \theta\big(22-2t, 2t-2, 20^{[5]}, 22^{[5]}\big)$, where $t = 6, 7$. We begin with the following sequences that are the reverse of the initial sequences in Case (a):
$A_{11}(1;22) \diamond A_{11}(230;-22)$, $A_{11}(2;22) \diamond A_{11}(229;-22)$, $A_{11}(3;22) \diamond A_{11}(228;-22)$, $A_{11}(4;22) \diamond A_{11}(227;-22)$, $A_{11}(5;22) \diamond A_{11}(226;-22)$, $A_{10}(17;22) \diamond A_{10}(214;-22)$, $A_{10}(18;22) \diamond A_{10}(213;-22)$, $A_{10}(19;22) \diamond A_{10}(212;-22)$, $A_{10}(20;22) \diamond A_{10}(211;-22)$, $A_{10}(21;22) \diamond A_{10}(210;-22)$, $A_{10}(22;22) \diamond A_{10}(209;-22)$.

Now, the difference sets are $D_1 = A_5(1;2)$ and $D_2 = A_6(-1,-2)$.

i) $t = 6$. So $\theta_{12} = \theta\big(10, 10, 20^{[5]}, 22^{[5]}\big)$. Initially, we use the first five sequences above to label the (u,v)-paths T_j and the last six sequences above to label the (u,v)-paths R_i. We then break R_6 into two parts such that the first 10 edges form the (u,v)-path Q_2 and the remaining 10 edges form the (u,v)-path Q_1. Now, the induced vertex label of u is $\sum\limits_{j=1}^{5} j + \sum\limits_{i=17}^{22} i + 132 = 264$. So, we choose $B = \{-9,-3,1\} \subset D_1 \cup D_2$.
Thus the actual assignment for each (u,v)-path is to label:
T_1 by $A_{11}(1;22) \diamond A_{11}(230;-22)$; T_2 by $A_{11}(2;22) \diamond A_{11}(229;-22)$; T_3 by $A_{11}(3;22) \diamond$ $\diamond A_{11}(228;-22)$; T_4 by $A_{11}(4;22) \diamond A_{11}(227;-22)$; T_5 by the reverse of $A_{11}(5;22) \diamond$ $\diamond A_{11}(226;-22)$;
R_1 by $A_{10}(17;22) \diamond A_{10}(214;-22)$; R_2 by the reverse of $A_{10}(18;22) \diamond A_{10}(213;-22)$; R_3 by $A_{10}(19;22) \diamond A_{10}(212;-22)$; R_4 by $A_{10}(20;22) \diamond A_{10}(211;-22)$; R_5 by the reverse of $A_{10}(21;22) \diamond A_{10}(210;-22)$;
Q_2 by 22, 209, 44, 187, 66, 165, 88, 143, 110, 121;
Q_1 by 132, 99, 154, 77, 176, 55, 198, 33, 220, 11.
Thus,

$$f^+(u) = 1+2+3+4+6+17+15+19+20+12+22+132 = 253.$$

ii) $t = 7$. So $\theta_{12} = \theta(8, 12, 20^{[5]}, 22^{[5]})$. Initially, we use the first five sequences above to label the (u,v)-paths T_j and the last six sequences above to label the (u,v)-paths R_i. We then break R_6 into two parts such that the first 8 edges form the (u,v)-path Q_2 and the remaining 12 edges form the (u,v)-path Q_1. Now, the induced vertex label of u is $\sum\limits_{j=1}^{5} j + \sum\limits_{i=17}^{22} i + 110 = 242$. Now $\delta(6) = 11$. So, we may choose $B = \{1,3,7\}$.
Thus the actual assignment for each (u,v)-path is to label:
T_1 by $A_{11}(1;22) \diamond A_{11}(230;-22)$; T_2 by the reverse of $A_{11}(2;22) \diamond A_{11}(229;-22)$; T_3 by $A_{11}(3;22) \diamond A_{11}(228;-22)$; T_4 by the reverse of $A_{11}(4;22) \diamond A_{11}(227;-22)$; T_5 by the reverse of $A_{11}(5;22) \diamond A_{11}(226;-22)$;
R_1 by $A_{10}(17;22) \diamond A_{10}(214;-22)$; R_2 by $A_{10}(18;22) \diamond A_{10}(213;-22)$;
R_3 by $A_{10}(19;22) \diamond A_{10}(212;-22)$; R_4 by $A_{10}(20;22) \diamond A_{10}(211;-22)$;
R_5 by $A_{10}(21;22) \diamond A_{10}(210;-22)$;
Q_2 by 22, 209, 44, 187, 66, 165, 88, 143;
Q_1 by 110, 121, 132, 99, 154, 77, 176, 55, 198, 33, 220, 11.

Thus,

$$f^+(u) = 1 + 9 + 3 + 7 + 6 + 17 + 18 + 19 + 20 + 21 + 22 + 110 = 253.$$

■

Example 3.4. The aim of this example is to illustrate the construction showed in Case (3) and provide a local antimagic 2-coloring for the exceptional case $(l, t) = (3, 3)$. Let $s = 6$, i. e., $l = 3$. Now, $x = 45$ and $y = 55$. The sequences are
$A_5(1; 10) \diamond A_5(44; -10)$: 1, 44, 11, 34, 21, 24, 31, 14, 41, 4
$A_5(2; 10) \diamond A_5(43; -10)$: 2, 43, 12, 33, 22, 23, 32, 13, 42, 3
$A_4(5; 10) \diamond A_4(40; -10)$: 5, 40, 15, 30, 25, 20, 35, 10
$A_4(6; 10) \diamond A_4(39; -10)$: 6, 39, 16, 29, 26, 19, 36, 9
$A_4(7; 10) \diamond A_4(38; -10)$: 7, 38, 17, 28, 27, 18, 37, 8

(a) $t = l = 3$. So $\theta_6 = \theta(4, 6, 8^{[3]}, 10)$.

(u, v)-path T_1 is labeled by 4, 41, 14, 31; 24, 21, 34, 11, 44, 1. So
(u, v)-path Q_2 is labeled by 4, 41, 14, 31 and
(u, v)-path Q_1 is labeled by 24, 21, 34, 11, 44, 1.

(u, v)-path T_2 is labeled by 3, 42, 13, 32, 23, 22, 33, 12, 43, 2.

(u, v)-path R_1 is labeled by 10, 35, 20, 25, 30, 15, 40, 5.

(u, v)-path R_3 is labeled by 8, 37, 18, 27, 28, 17, 38, 7.

(u, v)-path R_2 is labeled by 6, 39, 16, 29, 26, 19, 36, 9

Thus, $f^+(u) = 4 + 24 + 3 + 10 + 8 + 6 = 55$.

(b) $t = l = 3$. So $\theta_6 = \theta(4, 4, 8^{[2]}, 10^{[2]})$.

(u, v)-path Q_2 is labeled by 8, 37, 18, 27.

(u, v)-path Q_1 is labeled by 28, 17, 38, 7.

(u, v)-path R_1 is labeled by 6, 39, 16, 29, 26, 19, 36, 9.

(u, v)-path R_2 is labeled by 10, 35, 20, 25, 30, 15, 40, 5.

(u, v)-path T_1 is labeled by 1, 44, 11, 34, 21, 24, 31, 14, 41, 4.

(u, v)-path T_2 is labeled by 2, 43, 12, 33, 22, 23, 32, 13, 42, 3.

Thus, $f^+(u) = 8 + 28 + 6 + 10 + 1 + 2 = 55$. ■

Example 3.5. The aim of this example is to illustrate the construction given in Case (4). Take $s = 7$ so that $\theta_7 = \theta(2t, 22 - 2t, 10, 22^{[4]})$, $2 \le t \le 4$. We have $x = 121$, $y = 132$ and $y - x = 11$.

$A_{11}(1; 11) \diamond A_{11}(120; -11) = 1, 120, 12, 109, 23, 98, 34, 87, 45, 76, 56, 65, 67, 54, 78, 43, 89, 32, 100, 21, 111, 10;$

$A_{11}(2; 11) \diamond A_{11}(119; -11) = 2, 119, 13, 108, 24, 97, 35, 86, 46, 75, 57, 64, 68, 53, 79, 42, 90, 31, 101, 20, 112, 9;$ [7]

$A_{11}(3; 11) \diamond A_{11}(118; -11) = 3, 118, 14, 107, 25, 96, 36, 85, 47, 74, 58, 63, 69, 52, 80, 41, 91, 30, 102, 19, 113, 8;$ [5]

$A_{11}(4; 11) \diamond A_{11}(117; -11) = 4, 117, 15, 106, 26, 95, 37, 84, 48, 73, 59, 62, 70, 51, 81, 40, 92, 29, 103, 18, 114, 7;$ [3]

$A_{11}(5; 11) \diamond A_{11}(116; -11) = 5, 116, 16, 105, 27, 94, 38, 83, 49, 72, 60, 61, 71, 50, 82, 39, 93, 28, 104, 17, 115, 6.$ [1]

$A_5(66; 11) \diamond A_5(55; -11) = 66, 55, 77, 44, 88, 33, 99, 22, 110, 11 \leftarrow$ this sequence is for the (u, v)-path P.

Note that $(s - 3)^2 = 16$. The number with a bracket behind the sequence is the difference between the last and the first terms. Hence, $D = \{1, 3, 5, 7\}$.

1. When $t = 4$. We have $\delta(4) = 6 < 16$. First, we separate $A_{11}(1; 11) \diamond A_{11}(120; -11)$ into two sequences: 1, 120, 12, 109, 23, 98, 34, 87; and 45, 76, 56, 65, 67, 54, 78, 43, 89, 32, 100, 21, 111, 10. Since $\delta(4) < 7$, by Lemma 2.2, we choose $B = \{1, 5\}$. So, we reverse the order of $A_{11}(5; 11) \diamond A_{11}(116; -11)$ and $A_{11}(3; 11) \diamond A_{11}(118; -11)$, i. e., the end-edge labels for u are 1, $45 = \gamma_1$, 2, 8, 4, 6, 66.

2. When $t = 3$. We have $\delta(3) = 17 > 16$ and $\delta^*(3) = -1$. We must use an ad hoc method which is shown in the proof.

3. When $t = 2$. We have $\delta(2) = 28 > 16$. $\delta^*(2) = 10 < 16$. First, we separate the reverse of $A_{11}(1; 11) \diamond A_{11}(120; -11)$ into two sequences: 10, 111, 21, 100; and 32, 89, 43, 78, 54, 67, 65, 56, 76 45, 87, 34, 98, 23, 109, 12, 120, 1. Since $\delta^*(2) = 10$, we choose $B = \{7, 3\}$. So, we reverse the order of $A_{11}(2; 11) \diamond A_{11}(119; -11)$ and $A_{11}(4; 11) \diamond A_{11}(117; -11)$, i. e., the end-edge labels for u are 10, $32 = \gamma_1$, 9, 3, 7, 5, 66. ■

§ 4. Conjecture and Open Problem

We have completely characterized s-bridge graphs θ_s with $\chi_{la}(\theta_s) = 2$. We note that the only other known results on s-bridge graphs are (i) $\chi_{la}(\theta(a, b)) = 3$ for $a, b \geq 1$ and $a + b \geq 3$; and (ii) $\theta(2^{[s]}) = 3$ for odd $s \geq 3$. We end with the following conjecture and open problem.

Conjecture 3. If θ_s is not a graph in Theorem 2.1, then $\chi_{la}(\theta_s) = 3$.

Problem 4.1. Characterize graph G with $\chi_{la}(G) = 2$.

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Полная характеризация мостовых графов с локальным антимагическим хроматическим числом 2

Ключевые слова: локальная антимагическая разметка, локальное антимагическое хроматическое число, s-мостовые графы.

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Разметка ребер связного графа $G = (V, E)$ называется локальной антимагической, если она является биекцией $f\colon E \to \{1, \dots, |E|\}$ такой, что для любой пары смежных вершин x и y выполнено $f^+(x) \neq f^+(y)$, где $f^+(x) = \sum f(e)$ — индуцированная метка вершины, а e пробегает все ребра, инцидентные x. Локальное антимагическое хроматическое число графа G, обозначаемое $\chi_{la}(G)$, — это минимальное число различных индуцированных меток вершин среди всех локальных антимагических разметок G. В данной статье мы охарактеризуем s-мостовые графы с локальным антимагическим хроматическим числом 2.

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