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ON ARBITRARY MATRIX COEFFICIENT ASSIGNMENT FOR THE CHARACTERISTIC MATRIX POLYNOMIAL OF BLOCK MATRIX LINEAR CONTROL SYSTEMS

For block matrix linear control systems, we study the property of arbitrary matrix coefficient assignability for the characteristic matrix polynomial. This property is a generalization of the property of eigenvalue spectrum assignability or arbitrary coefficient assignability for the characteristic polynomial from system with scalar ($s = 1$) block matrices to systems with block matrices of higher dimensions ($s > 1$). Compared to the scalar case ($s = 1$), new features appear in the block cases of higher dimensions ($s > 1$) that are absent in the scalar case. New properties of arbitrary (upper triangular, lower triangular, diagonal) matrix coefficient assignability for the characteristic matrix polynomial are introduced. In the scalar case, all the described properties are equivalent to each other, but in block matrix cases of higher dimensions this is not the case. Implications between these properties are established.

Keywords: linear time-invariant control system, eigenvalue spectrum assignment, linear static feedback, block matrix system.

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§ 1. Introduction

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$; $\mathbb{K}^n := \{x = \operatorname{col}(x_1, \dots, x_n) : x_i \in \mathbb{K},\ i = \overline{1, n}\}$; $M_{r,q}(\mathbb{K})$ is a space of $r \times q$-matrices with elements of $\mathbb{K}$, $M_r(\mathbb{K}) := M_{r,r}(\mathbb{K})$ (we will denote $M_{r,q} := M_{r,q}(\mathbb{K})$, $M_r := M_r(\mathbb{K})$, if the set $\mathbb{K}$ is predefined); $I_r \in M_r(\mathbb{K})$ is the identity matrix (we will omit the index r in the matrix I_r when it does not cause confusion); $J := \{\vartheta_{ij}\} \in M_r(\mathbb{K})$ where $\vartheta_{ij} = 1$ for $j = i + 1$ and $\vartheta_{ij} = 0$ for $j \neq i + 1$; $\operatorname{Sp} A$ and $\chi(A, \lambda)$ are the trace and the characteristic polynomial of a matrix $A \in M_r(\mathbb{K})$, respectively; $A \otimes B$ is the Kronecker product of matrices A and B.

Consider a linear control system

$$\dot{x} = Fx + Gu, \quad y = Hx. \tag{1}$$

Here $x \in \mathbb{K}^n$ is a state vector, $u \in \mathbb{K}^m$ is a control vector, $y \in \mathbb{K}^k$ is an output vector, $F \in M_n(\mathbb{K})$, $G \in M_{n,m}(\mathbb{K})$, $H \in M_{k,n}(\mathbb{K})$. Suppose that the control in system (1) has the form of linear static output feedback (LSOF):

$$u = Qy. \tag{2}$$

Here $Q \in M_{m,k}(\mathbb{K})$. The closed-loop system has the form

$$\dot{x} = (F + GQH)x, \quad x \in \mathbb{K}^n. \tag{3}$$

If $k = n$ and

$$H = I \in M_n(\mathbb{K}), \tag{4}$$

then $y = x$, that is (2) is a linear static state feedback (LSSF) control

$$u = Qx, \tag{5}$$

and the closed-loop system has the form

$$\dot{x} = (F + GQ)x, \quad x \in \mathbb{K}^n. \tag{6}$$

The classical problem of eigenvalue spectrum assignment for system (1) by LSOF (2) (or by LSSF (5)) is as follows. Let an arbitrary set $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{C}$ of numbers $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ be given (if $\mathbb{K} = \mathbb{R}$, then the set σ should be a set of real type, that is invariant under the complex conjugation operation). One needs to construct a gain matrix $Q \in M_{m,k}(\mathbb{K})$ such that the eigenvalue spectrum of the closed-loop system (3) coincides with the pregiven set σ. The spectrum σ is one-to-one determined by the coefficients $\gamma_i \in \mathbb{K}$, $i = \overline{1,n}$, of the characteristic polynomial

$$\chi(F + GQH, \lambda) = \lambda^n + \gamma_1 \lambda^{n-1} + \dots + \gamma_{n-1}\lambda + \gamma_n \tag{7}$$

of the matrix $F + GQH$ of system (3). Therefore, the eigenvalue spectrum assignment problem for system (1) by LSOF (2) is equivalent to the problem of assigning arbitrary coefficients of the characteristic polynomial (7) in the following formulation.

Definition 1. It is said that, for system (1), the problem of *arbitrary coefficient assignment (ACA) for the characteristic polynomial (CP) by linear static output feedback (LSOF) is resolvable*, if for any $\gamma_i \in \mathbb{K}$, $i = \overline{1,n}$, there exists a gain matrix $Q \in M_{m,k}(\mathbb{K})$ such that the characteristic polynomial $\chi(F + GQH, \lambda)$ of the matrix $F + GQH$ of system (3) satisfies equality (7).

In partial case, when (4) is fulfilled and the closed-loop system has the form (6), it is said that, for system (1), the problem of *ACA for CP by LSSF is resolvable*.

For system (1), the problem of ACA for CP by LSSF has been solved in [1] for $\mathbb{K} = \mathbb{C}$ and in [2] for $\mathbb{K} = \mathbb{R}$. The following proposition was proven.

Proposition 1. *The following statements are equivalent.*

1. $\operatorname{rank}[G, FG, \dots, F^{n-1}G] = n$.

2. *For arbitrary* $\gamma_i \in \mathbb{K}$, $i = \overline{1,n}$, *there exists* $Q \in M_{m,n}(\mathbb{K})$ *such that the matrix* $F + GQ$ *is similar to the matrix*

$$\Phi = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\gamma_n & -\gamma_{n-1} & -\gamma_{n-2} & \dots & -\gamma_1 \end{bmatrix}. \tag{8}$$

The matrix (8) is a companion matrix of the polynomial

$$\varphi(\lambda) = \lambda^n + \gamma_1 \lambda^{n-1} + \dots + \gamma_{n-1}\lambda + \gamma_n.$$

The following equality holds: $\chi(\Phi, \lambda) = \varphi(\lambda)$. A matrix of the form (8) is also called *a lower Frobenius matrix*.

Remark 1. Note that, if some matrix A is similar to some lower Frobenius matrix (8) (i.e., $A = S\Phi S^{-1}$ for some $S \in M_n(\mathbb{K})$), then this lower Frobenius matrix Φ is uniquely defined, that is, if the matrix (8) is similar to another lower Frobenius matrix

$$\widetilde{\Phi} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\widetilde{\gamma}_n & -\widetilde{\gamma}_{n-1} & -\widetilde{\gamma}_{n-2} & \dots & -\widetilde{\gamma}_1 \end{bmatrix},$$

then $\Phi = \widetilde{\Phi}$ (i. e., $\gamma_i = \widetilde{\gamma}_i$, $i = \overline{1,n}$).

For system (1), the problem of ACA for CP by LSOF is much more difficult than by LSSF, and it still does not have a complete constructive solution. The most essential results have been obtained in [3–5]; see also reviews [6–8] and references in [9].

This work is devoted to the study of the above problem for systems of a more general form. Let $s \in \mathbb{N}$ be given. Consider an input-output linear control system with block matrix coefficients with $s \times s$-blocks:

$$\dot{x} = Fx + Gu, \quad y = Hx, \tag{9}$$

$$F = \begin{bmatrix} F_{11} & \dots & F_{1n} \\ \vdots & & \vdots \\ F_{n1} & \dots & F_{nn} \end{bmatrix}, \quad G = \begin{bmatrix} G_{11} & \dots & G_{1m} \\ \vdots & & \vdots \\ G_{n1} & \dots & G_{nm} \end{bmatrix}, \quad H = \begin{bmatrix} H_{11} & \dots & H_{1n} \\ \vdots & & \vdots \\ H_{k1} & \dots & H_{kn} \end{bmatrix}. \tag{10}$$

Here $x \in \mathbb{K}^{ns}$ is a state vector, $u \in \mathbb{K}^{ms}$ is a control vector, $y \in \mathbb{K}^{ks}$ is an output vector; $F_{ij}, G_{j\alpha}, H_{\beta i} \in M_s(\mathbb{K})$, $i, j = \overline{1,n}$, $\alpha = \overline{1,m}$, $\beta = \overline{1,k}$. Suppose that the control in system (9), (10) is a LSOF control:

$$u = Qy. \tag{11}$$

Here $Q = \{Q_{\alpha\beta}\} \in M_{ms,ks}(\mathbb{K})$, $Q_{\alpha\beta} \in M_s(\mathbb{K})$, $\alpha = \overline{1,m}$, $\beta = \overline{1,k}$. The closed-loop system has the form

$$\dot{x} = (F + GQH)x, \quad x \in \mathbb{K}^{ns}. \tag{12}$$

If $k = n$ and

$$H = I \in M_{ns}(\mathbb{K}), \tag{13}$$

then $y = x$, that is (11) is a LSSF control

$$u = Qx,$$

and the closed-loop system has the form

$$\dot{x} = (F + GQ)x, \quad x \in \mathbb{K}^{ns}. \tag{14}$$

We would like to study a generalization of the problem of assigning eigenvalue spectrum (or problem of ACA for CP) by LSOF (or by LSSF) to block matrix systems. Here difficulties arise already at the level of problem formulation. Let a matrix polynomial be given

$$\Psi(\lambda) = I\lambda^n + \Gamma_1\lambda^{n-1} + \ldots + \Gamma_{n-1}\lambda + \Gamma_n, \quad I, \Gamma_i \in M_s(\mathbb{K}), \quad i = \overline{1,n}. \tag{15}$$

The question of what is called the roots of this equation is not clear-cut. There are roots, which are called *left solvents* (L_j), and accordingly, *right solvents* (R_j) [10]. They satisfy equations

$$L_j^n + L_j^{n-1}\Gamma_1 + L_j^{n-2}\Gamma_2 + \ldots + L_j\Gamma_{n-1} + \Gamma_n = 0, \quad j = 1, \ldots, n,$$

and

$$R_j^n + \Gamma_1 R_j^{n-1} + \Gamma_2 R_j^{n-2} + \ldots + \Gamma_{n-1}R_j + \Gamma_n = 0, \quad j = 1, \ldots, n,$$

respectively, and $L_i \neq R_j$, in general. The fundamental theorem of algebra for matrix polynomials (15) does not hold. There exists a matrix polynomial with no solvents [11, Theorem 2.6], e.g.:

$$\Psi(\lambda) = I\lambda^2 - 2I\lambda + A, \quad I \in M_2, \quad A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}.$$

Conversely, there are matrices that cannot be solvents of any equation. More precisely [10, Corollary 6.1], there exist sets containing n matrices which are not a set of left (or right) solvents for any monic matrix polynomial of degree n. Example [10, Sect. 6]: $n = 2$, $s = 2$,

$$X_1 = \begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 4 & 2 \\ 0 & 3 \end{bmatrix}.$$

Thus, in contrast to the case $s = 1$, there is no bijection between the set σ of "roots" of the polynomial (15) and the (ordered) set of (matrix) coefficients $(\Gamma_1, \ldots, \Gamma_n)$ of the polynomial (15). Therefore, a question on generalizing the formulation of the problem of assigning eigenvalue spectrum or problem of ACA for CP by LSOF (or by LSSF) to block matrix systems is not evident.

An attempt to generalize the formulation of the problem of assigning coefficients of the characteristic polynomial to block-matrix systems was made in [ZK–2024][1]. It is based on the property described in Proposition 1. Using polynomial (15), we construct the block companion matrix $\Theta \in M_{ns}(\mathbb{K})$ associated to this polynomial:

$$\Theta = \begin{bmatrix} 0 & I & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & I \\ -\Gamma_n & -\Gamma_{n-1} & -\Gamma_{n-2} & \ldots & -\Gamma_1 \end{bmatrix}. \tag{16}$$

We will call such a matrix *a block lower Frobenius matrix*. The following definition was given in the paper [ZK–2024].

Definition 2. We say that, for system (9), (10), the problem of *arbitrary matrix coefficient assignment* (*AMCA*) *for the characteristic matrix polynomial* (*CMP*) *by linear static output feedback* (*LSOF*) *is resolvable* if for any $\Gamma_i \in M_s(\mathbb{K})$, $i = \overline{1, n}$, there exists a gain matrix $Q \in M_{ms,ks}(\mathbb{K})$ such that the closed-loop system (12) is reducible by some change of variables $z = Sx$ to the system

$$\dot{z} = \Theta z, \quad z \in \mathbb{K}^{ns}, \tag{17}$$

with the matrix (16), that is the matrix $F + GQH$ of the system (12) is similar to the matrix (16): $S(F + GQH)S^{-1} = \Theta$.

In partial case, when (13) is fulfilled and the closed-loop system has the form (14), it is said that, for system (9), (10), the problem of *AMCA for CMP by LSSF is resolvable*.

If $s = 1$, then the problem of AMCA for CMP by LSOF (or by LSSF) formulated in Definition 2, due to Remark 1, is more general than the problem of ACA for CP by LSOF (or by LSSF) formulated in Definition 1. So, we are considering a generalization of the problem of eigenvalue spectrum assignment by LSOF (or by LSSF) to systems with block matrix coefficients.

Note that for $s > 1$ new difficulties arise, among other things. In particular, an analogue of Remark 1 no longer holds. If some matrix A is similar to certain block lower Frobenius matrix (16), this does not imply that this matrix Θ is uniquely defined. In other words, there exist block lower Frobenius matrices $\Theta_1, \Theta_2 \in M_{ns}$ such that $\Theta_1 \sim \Theta_2$ but $\Theta_1 \neq \Theta_2$ (note that this cannot happen if $s = 1$, by Remark 1). This is confirmed by the following example.

[1][ZK–2024] Zaitsev V., Kim I. Arbitrary matrix coefficient assignment for block matrix linear control systems by static output feedback, *European Journal of Control*, 2024. (Submitted 10 May 2024).

Example 1. Let $n = 2$, $s = 2$. Consider

$$\Theta_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & 3 & 0 \\ 0 & -12 & 0 & 7 \end{bmatrix}, \qquad \Theta_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 5 & 0 \\ 0 & -6 & 0 & 5 \end{bmatrix}.$$

The eigenvalues of the matrix Θ_1 are $\lambda_{1,2,3,4}(\Theta_1) = \{1, 2, 3, 4\}$. The eigenvalues of the matrix Θ_2 are $\lambda_{1,2,3,4}(\Theta_2) = \{1, 2, 3, 4\}$. All the eigenvalues are pairwise distinct. Hence, the matrices Θ_1 and Θ_2 are similar. □

The purpose of this work is a more detailed study of the property given in Definition 2.

§ 2. Main Results

In Definition 2, matrices Γ_i, $i = \overline{1, n}$, are arbitrary matrices from the matrix space $M_s(\mathbb{K})$. In $M_s(\mathbb{K})$, consider the subspace $UT_s(\mathbb{K}) \subset M_s(\mathbb{K})$ of the upper triangular matrices, the subspace $LT_s(\mathbb{K}) \subset M_s(\mathbb{K})$ of the lower triangular matrices, and the subspace $D_s(\mathbb{K}) \subset M_s(\mathbb{K})$ of the diagonal matrices. Let us give the following definitions similar to Definition 2.

Definition 3. We say that, for system (9), (10), the problem of
(a) *arbitrary upper triangular matrix coefficient assignment* (*AUTMCA*),
(b) *arbitrary lower triangular matrix coefficient assignment* (*ALTMCA*),
(c) *arbitrary diagonal matrix coefficient assignment* (*ADMCA*),
for CMP by LSOF is resolvable if
(a) for any $\Gamma_i \in UT_s(\mathbb{K})$, $i = \overline{1, n}$,
(b) for any $\Gamma_i \in LT_s(\mathbb{K})$, $i = \overline{1, n}$,
(c) for any $\Gamma_i \in D_s(\mathbb{K})$, $i = \overline{1, n}$,
there exists a gain matrix $Q \in M_{ms,ks}(\mathbb{K})$ such that the closed-loop system (12) is reducible by some change of variables $z = Sx$ to the system (17) with the matrix (16), that is the matrix $F + GQH$ of the system (12) is similar to the matrix (16): $S(F + GQH)S^{-1} = \Theta$.

In partial case, when (13) is fulfilled and the closed-loop system has the form (14), it is said that, for system (9), (10), the problem of (a) *AUTMCA* (b) *ALTMCA* (c) *ADMCA for CMP by LSSF is resolvable.*

The main problem that we explore in this work is the question on relationship between all above definitions in Definitions 3 and 2. If $s = 1$, then, obviously, all these definitions are equivalent between themselves (by LSOF or by LSSF, respectively). In the case when $s > 1$, this problem has not been studied explicitly before. Obviously, the following implications hold:

$$\begin{array}{ccc} AMCA & \Rightarrow & AUTMCA \\ \Downarrow & & \Downarrow \\ ALTMCA & \Rightarrow & ADMCA \end{array} \tag{18}$$

The question arises about other implications in this diagram. We will give a partial answer to this question.

Theorem 1. *1. For any matrix*

$$\Omega = \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I \\ U_1 & U_2 & U_3 & \dots & U_n \end{bmatrix}, \tag{19}$$

where $0, I, U_i \in UT_s(\mathbb{K})$, $i = \overline{1,n}$, *there exists a non-degenerate matrix* $\mathcal{S} \in M_{ns}(\mathbb{K})$ *such that the matrix* $\Xi := \mathcal{S}\Omega\mathcal{S}^{-1}$ *has the form*

$$\Xi = \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I \\ L_1 & L_2 & L_3 & \dots & L_n \end{bmatrix}, \tag{20}$$

where $0, I, L_i \in LT_s(\mathbb{K})$, $i = \overline{1,n}$.

2. Conversely, for any matrix (20), *where* $0, I, L_i \in LT_s(\mathbb{K})$, $i = \overline{1,n}$, *there exists a non-degenerate matrix* $\mathcal{S} \in M_{ns}(\mathbb{K})$ *such that the matrix* $\Omega := \mathcal{S}\Xi\mathcal{S}^{-1}$ *has the form* (19), *where* $0, I, U_i \in UT_s(\mathbb{K})$, $i = \overline{1,n}$.

Theorem 1 implies the following theorem.

Theorem 2. *The following statements are equivalent.*

1. For system (9), (10), *the problem of ALTMCA for CMP by LSOF is resolvable.*

2. For system (9), (10), *the problem of AUTMCA for CMP by LSOF is resolvable.*

So, the implications $ALTMCA \Longleftrightarrow AUTMCA$ in the diagram (18) take place.

Next, the following theorem takes place.

Theorem 3. *Let* $\mathbb{K} = \mathbb{C}$, $n = 2$, *and* $s = 2$. *For any matrix* Θ *of* (16), *where* $0, I, \Gamma_i \in M_s(\mathbb{K})$, $i = \overline{1,n}$, *there exists a matrix* Ω *of* (19), *where* $0, I, U_i \in UT_s(\mathbb{K})$, $i = \overline{1,n}$, *such that* $\Omega \sim \Theta$.

Theorem 3 implies the following theorem.

Theorem 4. *Let* $\mathbb{K} = \mathbb{C}$, $n = 2$, *and* $s = 2$. *Suppose that, for system* (9), (10), *the problem of AUTMCA for CMP by LSOF is resolvable. Then, for system* (9), (10), *the problem of AMCA for CMP by LSOF is resolvable.*

So, taking into account Theorem 2, the implications

$$ALTMCA \Longrightarrow AMCA \quad \text{and} \quad AUTMCA \Longrightarrow AMCA \tag{21}$$

in the diagram (18) take place under the conditions that $\mathbb{K} = \mathbb{C}$, $n = 2$, and $s = 2$. We believe that the statements of Theorem 3 and Theorem 4 are true, for $\mathbb{K} = \mathbb{C}$, for arbitrary $n, s \in \mathbb{N}$. But this hypothesis has not yet been proven for $n > 2$ or $s > 2$ and remains open.

Further, it turns out that the statement of Theorem 3 ceases to be true in the case when $\mathbb{K} = \mathbb{R}$, for arbitrary $s > 1$ and $n \in \mathbb{N}$.

Theorem 5. *Let* $\mathbb{K} = \mathbb{R}$ *and* $s > 1$. *Not for any matrix* Θ *of* (16), *where* $0, I, \Gamma_i \in M_s(\mathbb{K})$, $i = \overline{1,n}$, *there exists a matrix* Ω *of* (19), *where* $0, I, U_i \in UT_s(\mathbb{K})$, $i = \overline{1,n}$, *such that* $\Omega \sim \Theta$.

From Theorem 5, it follows that we cannot assert the truth of the implications (21) in the case when $\mathbb{K} = \mathbb{R}$ and $s > 1$ (although their falsity also does not yet follow from anywhere, since the statement of Theorem 3 is sufficient but not necessary for the conclusion of Theorem 4).

Finally, the following statement is true for both cases $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{R}$.

Theorem 6. *Let* $s > 1$. *Not for any matrix* Ω *of* (19), *where* $0, I, U_i \in UT_s(\mathbb{K})$, $i = \overline{1, n}$, *there exists a matrix*

$$\Delta = \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I \\ R_1 & R_2 & R_3 & \dots & R_n \end{bmatrix}, \tag{22}$$

where $0, I, R_i \in D_s(\mathbb{K})$, $i = \overline{1, n}$, *such that* $\Delta \sim \Omega$.

From Theorem 6, it follows that we cannot assert the truth of the implications

$$ADMCA \Longrightarrow ALTMCA \quad \text{and} \quad ADMCA \Longrightarrow AUTMCA$$

in the diagram (18) for the case $s > 1$.

Remark 2. We conducted a study of the issue of implications in the diagram (18) only on the basis of the type of matrices Γ_i in Definitions 3 and 2, but did not touch here on the type of feedback (LSOF or LSSF) and other conditions on the feedback coefficients. For example, if $m = k = n$ and $G = H = I \in M_{ns}$ then all properties in the diagram (18) are equivalent to each other. These questions will be the subject of other research.

§ 3. Proofs of Theorems 1 and 2

Lemma 1. *For any matrix* $U \in UT_s(\mathbb{K})$ *there exist* $L \in LT_s(\mathbb{K})$ *and non-degenerate* $S \in M_s(\mathbb{K})$ *such that* $L = SUS^{-1}$.

P r o o f. Let

$$U = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1s} \\ 0 & u_{22} & \dots & u_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{ss} \end{bmatrix}.$$

Set

$$S := \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix} \in M_s(\mathbb{K}). \tag{23}$$

Then

$$SUS^{-1} = \begin{bmatrix} u_{ss} & 0 & \dots & 0 \\ u_{s-1,s} & u_{s-1,s-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_{1s} & u_{1,s-1} & \dots & u_{11} \end{bmatrix}.$$

□

P r o o f o f T h e o r e m 1. Let Ω have the form (19) where $U_i \in UT_s(\mathbb{K})$, $i = \overline{1, n}$. Construct

$$\mathcal{S} = \operatorname{diag}\{S, \dots, S\} := \begin{bmatrix} S & 0 & \dots & 0 \\ 0 & S & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & S \end{bmatrix} \in M_{ns}(\mathbb{K}),$$

where $S \in M_s(\mathbb{K})$ is defined by (23). Then

$$\mathcal{S}\Omega\mathcal{S}^{-1} = \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I \\ SU_1S^{-1} & SU_2S^{-1} & SU_3S^{-1} & \dots & SU_nS^{-1} \end{bmatrix}.$$

Denote $L_i := SU_iS^{-1}$, $i = \overline{1,n}$. Then, by Lemma 1, $L_i \in LT_s(\mathbb{K})$, $i = \overline{1,n}$. So, the first part of the theorem has been proven.

Carrying out the reasoning in Lemma 1 and in the first part of the proof of Theorem 1 in reverse order, and taking into account that $S^{-1} = S$, we obtain the proof of the second part of the theorem. □

P r o o f o f T h e o r e m 2. Suppose that, for system (9), (10), the problem of $ALTMCA$ for CMP by $LSOF$ is resolvable. Let us prove that, for system (9), (10), the problem of $AUTMCA$ for CMP by $LSOF$ is resolvable. Let a matrix Ω of (19) be given, where $U_i \in UT_s(\mathbb{K})$, $i = \overline{1,n}$, are arbitrary. By Theorem 1, part 1, there exists a matrix $\mathcal{S} \in M_{ns}(\mathbb{K})$ such that the matrix $\Xi := \mathcal{S}\Omega\mathcal{S}^{-1}$ has the form (20), where $L_i \in LT_s(\mathbb{K})$, $i = \overline{1,n}$. For the matrix Ξ, by the condition of the theorem, there exists a gain matrix $Q \in M_{ms,ks}(\mathbb{K})$ such that the matrix $F + GQH$ of the system (12) is similar to the matrix Ξ, i. e., $P(F+GQH)P^{-1} = \Xi$, for some $P \in M_{ns}(\mathbb{K})$. Hence, $P(F+GQH)P^{-1} = \mathcal{S}\Omega\mathcal{S}^{-1}$. It follows that $\mathcal{S}^{-1}P(F+GQH)P^{-1}\mathcal{S} = \Omega$, i. e., $F + GQH$ is similar to Ω. This means that the problem of $AUTMCA$ for CMP by $LSOF$ is resolvable. So, the implication $1 \Rightarrow 2$ is proven. The implication $2 \Rightarrow 1$ can be proved in a similar way, using the second part of Theorem 1. □

§ 4. Proofs of Theorems 3 and 4

Let us prove Theorem 3. Let $\mathbb{K} = \mathbb{C}$, $n = 2$, and $s = 2$. One needs to prove the following assertion: for any matrix

$$\Theta = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \theta_{31} & \theta_{32} & \theta_{33} & \theta_{34} \\ \theta_{41} & \theta_{42} & \theta_{43} & \theta_{44} \end{bmatrix}, \tag{24}$$

there exists a matrix

$$\Omega = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \omega_{31} & \omega_{32} & \omega_{33} & \omega_{34} \\ 0 & \omega_{42} & 0 & \omega_{44} \end{bmatrix} \tag{25}$$

such that $\Omega \sim \Theta$. We will prove this statement by brute force as follows. We will go through all possible options for what elementary divisors the matrix Θ can have, and then, for the given set of elementary divisors of the matrix Θ, we will build a matrix Ω of form (25), which has the same set of elementary divisors. From here the similarity of the matrices Θ and Ω will follow.

Next, different letters will mean different numbers.

1. Elementary divisors: $(\lambda - a)$, $(\lambda - b)$, $(\lambda - c)$, $(\lambda - d)$.

$$\Omega = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -ab & 0 & a+b & 0 \\ 0 & -cd & 0 & c+d \end{bmatrix}.$$

2. Elementary divisors: $(\lambda-a)^2$, $(\lambda-b)$, $(\lambda-c)$.

$$\Omega=\begin{bmatrix}0&0&1&0\\0&0&0&1\\-a^2&0&2a&0\\0&-bc&0&b+c\end{bmatrix}.$$

3. Elementary divisors: $(\lambda-a)$, $(\lambda-a)$, $(\lambda-b)$, $(\lambda-c)$.

$$\Omega=\begin{bmatrix}0&0&1&0\\0&0&0&1\\-ab&0&a+b&0\\0&-ac&0&a+c\end{bmatrix}.$$

4. Elementary divisors: $(\lambda-a)^2$, $(\lambda-b)^2$.

$$\Omega=\begin{bmatrix}0&0&1&0\\0&0&0&1\\-a^2&0&2a&0\\0&-b^2&0&2b\end{bmatrix}.$$

5. Elementary divisors: $(\lambda-a)^2$, $(\lambda-b)$, $(\lambda-b)$.

$$\Omega=\begin{bmatrix}0&0&1&0\\0&0&0&1\\-ab&-b&a+b&1\\0&-ab&0&a+b\end{bmatrix}.$$

6. Elementary divisors: $(\lambda-a)$, $(\lambda-a)$, $(\lambda-b)$, $(\lambda-b)$.

$$\Omega=\begin{bmatrix}0&0&1&0\\0&0&0&1\\-ab&0&a+b&0\\0&-ab&0&a+b\end{bmatrix}.$$

7. Elementary divisors: $(\lambda-a)^3$, $(\lambda-b)$.

$$\Omega=\begin{bmatrix}0&0&1&0\\0&0&0&1\\-a^2&1&2a&0\\0&-ab&0&a+b\end{bmatrix}.$$

8. Elementary divisors: $(\lambda-a)^2$, $(\lambda-a)$, $(\lambda-b)$.

$$\Omega=\begin{bmatrix}0&0&1&0\\0&0&0&1\\-a^2&0&2a&0\\0&-ab&0&a+b\end{bmatrix}.$$

9. Elementary divisors: $(\lambda-a)$, $(\lambda-a)$, $(\lambda-a)$, $(\lambda-b)$.

In fact, a matrix Θ of the form (24) cannot have such elementary divisors. Let us prove it by contradiction. Let the matrix Θ have such elementary divisors. Then the matrix Θ is similar to the matrix

$$A = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{bmatrix}.$$

Then $\Theta - aI \sim A - aI$. But $\operatorname{rank}(A - aI) = 1$ while $\operatorname{rank}(\Theta - aI) \geq 2$ since the first two rows of the matrix $(\Theta - aI)$ are linearly independent. We have come to a contradiction.

10. Elementary divisors: $(\lambda - a)^4$.

$$\Omega = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a^2 & 1 & 2a & 0 \\ 0 & -a^2 & 0 & 2a \end{bmatrix}.$$

11. Elementary divisors: $(\lambda - a)^3$, $(\lambda - a)$.

$$\Omega = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a^2 & -a & 2a & 1 \\ 0 & -a^2 & 0 & 2a \end{bmatrix}.$$

12. Elementary divisors: $(\lambda - a)^2$, $(\lambda - a)^2$.

$$\Omega = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a^2 & 0 & 2a & 0 \\ 0 & -a^2 & 0 & 2a \end{bmatrix}.$$

13. Elementary divisors: $(\lambda - a)^2$, $(\lambda - a)$, $(\lambda - a)$.

In fact, a matrix Θ of the form (24) cannot have such elementary divisors. Let us prove it by contradiction. Let the matrix Θ have such elementary divisors. Then the matrix Θ is similar to the matrix

$$A = \begin{bmatrix} a & 1 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix}.$$

Then $\Theta - aI \sim A - aI$. But $\operatorname{rank}(A - aI) = 1$ while $\operatorname{rank}(\Theta - aI) \geq 2$ since the first two rows of the matrix $(\Theta - aI)$ are linearly independent. We have come to a contradiction.

14. Elementary divisors: $(\lambda - a)$, $(\lambda - a)$, $(\lambda - a)$, $(\lambda - a)$.

In fact, a matrix Θ of the form (24) cannot have such elementary divisors. Let us prove it by contradiction. Let the matrix Θ have such elementary divisors. Then the matrix Θ is similar to the matrix $A = aI$. Then $\Theta - aI \sim A - aI = 0$. But this is obviously false.

We went through all possible cases of elementary divisors of the matrix Θ. So, Theorem 3 is proven. □

Theorem 4 obviously follows from Theorem 3 using arguments similar to the proof of Theorem 2.

§ 5. Proof of Theorem 5

We will carry out the proof from particular cases to the general case.

Case 1. Let $s = 2$ and $n = 1$. Set

$$N := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{26}$$

Set $\Theta := N$. It is clear that there is no real upper triangular 2×2-matrix U such that $U \sim \Theta$.

Case 2. Let $s = 2$ and $n = 2$. Set

$$\Theta := \begin{bmatrix} 0 & I \\ 0 & N \end{bmatrix}, \quad 0, I, N \in M_2(\mathbb{R}),$$

where N is defined by (26). The matrix Θ has the characteristic polynomial

$$\chi(\Theta, \lambda) = \lambda^4 + \lambda^2, \tag{27}$$

and the eigenvalues $\lambda_1 = i$, $\lambda_2 = -i$, $\lambda_3 = \lambda_4 = 0$.

Let us prove that there is no matrix Ω of the form (25) with $\omega_{ij} \in \mathbb{R}$ such that $\Omega \sim \Theta$. Let us prove it by contradiction. Suppose that there exists a matrix Ω of the form (25) with $\omega_{ij} \in \mathbb{R}$ such that $\Omega \sim \Theta$. Then, in particular,

$$\chi(\Omega, \lambda) = \chi(\Theta, \lambda). \tag{28}$$

We have $\operatorname{rank} \Theta = 2$, hence,

$$\operatorname{rank} \Omega = 2. \tag{29}$$

From (29), it follows that

$$\omega_{31} = 0, \quad \omega_{32} = 0, \quad \omega_{42} = 0. \tag{30}$$

Further, the following equality holds:

$$\operatorname{Sp} \Omega = \operatorname{Sp} \Theta = 0. \tag{31}$$

From (31), it follows that

$$\omega_{33} = -\omega_{44}. \tag{32}$$

From (30) and (32), it follows that

$$\chi(\Omega, \lambda) = \lambda^4 - \omega_{44}^2 \lambda^2. \tag{33}$$

From (27), (28), and (33), it follows that $\omega_{44}^2 = -1$. This contradicts the fact that ω_{44} is real.

Case 3. Let $s = 2$ and $n = 3$. Set

$$\Theta := \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & N \end{bmatrix}, \quad 0, I, N \in M_2(\mathbb{R}),$$

where N is defined by (26). The matrix Θ has the characteristic polynomial

$$\chi(\Theta, \lambda) = \lambda^6 + \lambda^4, \tag{34}$$

and the eigenvalues $\lambda_1 = i$, $\lambda_2 = -i$, $\lambda_{3,4,5,6} = 0$.

Let us prove that there is no matrix

$$\Omega = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ \Omega_1 & \Omega_2 & \Omega_3 \end{bmatrix}, \quad 0, I, \Omega_i \in UT_2(\mathbb{R}), \quad i = \overline{1,3}, \tag{35}$$

such that $\Omega \sim \Theta$. Let us prove it by contradiction. Suppose that there exists a matrix Ω of the form (35) such that

$$\Omega \sim \Theta. \tag{36}$$

Then, in particular,

$$\chi(\Omega, \lambda) = \chi(\Theta, \lambda). \tag{37}$$

From (36), it follows that $\Omega^2 \sim \Theta^2$. We have

$$\Theta^2 = \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & N \\ 0 & 0 & N^2 \end{bmatrix}.$$

Hence, $\operatorname{rank} \Theta^2 = 2$. Therefore,

$$\operatorname{rank} \Omega^2 = 2 \tag{38}$$

as well. We have

$$\Omega^2 = \begin{bmatrix} 0 & 0 & I \\ \Omega_1 & \Omega_2 & \Omega_3 \\ * & * & * \end{bmatrix}. \tag{39}$$

From (38) and (39) it follows that

$$\Omega_1 = \Omega_2 = 0 \in M_2(\mathbb{R}). \tag{40}$$

From (35) and (40), it follows that

$$\Omega = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & \Omega_3 \end{bmatrix}, \quad \Omega_3 = \begin{bmatrix} \omega_{55} & \omega_{56} \\ 0 & \omega_{66} \end{bmatrix}. \tag{41}$$

Further, the following equality holds:

$$\operatorname{Sp} \Omega = \operatorname{Sp} \Theta = 0. \tag{42}$$

From (42), it follows that

$$\omega_{55} = -\omega_{66}. \tag{43}$$

From (41) and (43), it follows that

$$\chi(\Omega, \lambda) = \lambda^6 - \omega_{66}^2 \lambda^4. \tag{44}$$

From (34), (37), and (44), it follows that $\omega_{66}^2 = -1$. This contradicts the fact that ω_{66} is real.

Case 4. Let $s = 2$ and $n > 3$. Set

$$\Theta := \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I \\ 0 & 0 & 0 & \dots & N \end{bmatrix} \in M_{2n}(\mathbb{R}), \quad 0, I, N \in M_2(\mathbb{R}),$$

where N is defined by (26). The matrix Θ has the characteristic polynomial

$$\chi(\Theta,\lambda)=\lambda^{2n}+\lambda^{2(n-1)}, \tag{45}$$

and the eigenvalues $\lambda_1=i$, $\lambda_2=-i$, $\lambda_3=\ldots=\lambda_{2n}=0$.

Let us prove that there is no matrix

$$\Omega=\begin{bmatrix}0 & I & 0 & \ldots & 0\\ 0 & 0 & I & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \ldots & I\\ \Omega_1 & \Omega_2 & \Omega_3 & \ldots & \Omega_n\end{bmatrix}, \quad 0, I, \Omega_i \in UT_2(\mathbb{R}), \quad i=\overline{1,n}, \tag{46}$$

such that $\Omega\sim\Theta$. Let us prove it by contradiction. Suppose that there exists a matrix Ω of the form (46) such that

$$\Omega\sim\Theta. \tag{47}$$

Then, in particular,

$$\chi(\Omega,\lambda)=\chi(\Theta,\lambda). \tag{48}$$

From (47), it follows that $\Omega^{n-1}\sim\Theta^{n-1}$. We have

$$\Theta^{n-1}=\begin{bmatrix}0 & \ldots & 0 & I\\ 0 & \ldots & 0 & N\\ \vdots & & \vdots & \vdots\\ 0 & \ldots & 0 & *\end{bmatrix}.$$

Hence, $\operatorname{rank}\Theta^{n-1}=2$. Therefore,

$$\operatorname{rank}\Omega^{n-1}=2 \tag{49}$$

as well. We have

$$\Omega^{n-1}=\begin{bmatrix}0 & \ldots & 0 & I\\ \Omega_1 & \ldots & \Omega_{n-1} & \Omega_n\\ * & \ldots & * & *\\ \vdots & & \vdots & \vdots\\ * & \ldots & * & *\end{bmatrix}. \tag{50}$$

From (49) and (50) it follows that

$$\Omega_1=\ldots=\Omega_{n-1}=0\in M_2(\mathbb{R}). \tag{51}$$

From (46) and (51), it follows that

$$\Omega=\begin{bmatrix}0 & I & 0 & \ldots & 0\\ 0 & 0 & I & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \ldots & I\\ 0 & 0 & 0 & \ldots & \Omega_n\end{bmatrix}, \quad \Omega_n=\begin{bmatrix}\omega_{2n-1,2n-1} & \omega_{2n-1,2n}\\ 0 & \omega_{2n,2n}\end{bmatrix}. \tag{52}$$

Further, the following equality holds:

$$\operatorname{Sp}\Omega=\operatorname{Sp}\Theta=0. \tag{53}$$

From (53), it follows that

$$\omega_{2n-1,2n-1} = -\omega_{2n,2n}. \tag{54}$$

From (52) and (54), it follows that

$$\chi(\Omega, \lambda) = \lambda^{2n} - \omega_{2n,2n}^2 \lambda^{2n-2}. \tag{55}$$

From (45), (48), and (55), it follows that $\omega_{2n,2n}^2 = -1$. This contradicts the fact that $\omega_{2n,2n}$ is real.

Case 5. Let $s > 2$ and $n \in \mathbb{N}$. Set $\mathcal{N} := \operatorname{diag}\{\underbrace{0, \dots, 0}_{s-2}, N\} \in M_s(\mathbb{R})$ where N is defined by (26), and set

$$\Theta := \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I \\ 0 & 0 & 0 & \dots & \mathcal{N} \end{bmatrix} \in M_{ns}(\mathbb{R}), \quad 0, I, \mathcal{N} \in M_s(\mathbb{R}). \tag{56}$$

Then, carrying out the proof similarly to Case 4, it can be shown that there is no matrix Ω of the form (19), where $0, I, U_i \in UT_s(\mathbb{K})$, $i = \overline{1, n}$, such that Ω is similar to (56). □

§ 6. Proof of Theorem 6

Let $n = 1$. We set $\Omega := U_1 := aI + J \in UT_s(\mathbb{K})$ where $I, J \in M_s(\mathbb{K})$ and $a \in \mathbb{K}$. It is well known that there is no matrix $R_1 \in D_s(\mathbb{K})$ such that $R_1 \sim U_1$. In fact, if $R_1 \sim U_1$, then $\chi(R_1, \lambda) = \chi(U_1, \lambda) = (\lambda - a)^s$, hence, $R_1 = aI$, but $aI \not\sim U_1$ since $s > 1$. We will use the same idea for arbitrary $n \in \mathbb{N}$.

Let $n > 1$ and $s > 1$. Let us construct the matrix Ω of the form (19) where

$$\begin{gathered} U_n = naI, \quad U_{n-1} = -C_n^{n-2} a^2 I, \quad U_{n-2} = C_n^{n-3} a^3 I, \quad \dots, \quad U_2 = (-1)^{n-2} C_n^1 a^{n-1} I, \\ U_1 = (-1)^{n-1} a^n I + J, \quad I, J \in M_s(\mathbb{K}), \quad a \in \mathbb{K}. \end{gathered} \tag{57}$$

Then $U_i \in UT_s(\mathbb{K})$. Let us calculate $\chi(\Omega, \lambda)$. By [12, Theorem 1.1], we have

$$\det(\lambda I_{ns} - \Omega) = \det[\lambda^n I_s - U_n \lambda^{n-1} - U_{n-1} \lambda^{n-2} - \dots - U_1]. \tag{58}$$

By (57), we obtain that

$$[\lambda^n I_s - U_n \lambda^{n-1} - U_{n-1} \lambda^{n-2} - \dots - U_1] = \begin{bmatrix} (\lambda - a)^n & -1 & \dots & 0 & 0 \\ 0 & (\lambda - a)^n & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & (\lambda - a)^n & -1 \\ 0 & 0 & \dots & 0 & (\lambda - a)^n \end{bmatrix}. \tag{59}$$

So, from (58) and (59), it follows that $\chi(\Omega, \lambda) = (\lambda - a)^{ns}$. Thus, all the eigenvalues of the matrix Ω are equal to a. Construct the matrix $\Omega - aI_{ns}$. It can be seen that the minor consisting of the first $ns - 1$ rows and last $ns - 1$ columns of the matrix $\Omega - aI_{ns}$ is not equal to zero, i. e., $\operatorname{rank}(\Omega - aI_{ns}) = ns - 1$. This means that the matrix Ω has only one elementary divisor $(\lambda - a)^{ns}$ and

$$\Omega \sim aI + J, \quad I, J \in M_{ns}(\mathbb{K}).$$

Let us show that there is no matrix Δ of the form (22) where $R_i \in D_s(\mathbb{K})$, $i = \overline{1,n}$, such that $\Delta \sim \Omega$. Let us prove it by contradiction. Suppose that there exists a matrix Δ of the form (22) where

$$\begin{gathered} R_1 = \operatorname{diag}\{r_{11}, \dots, r_{1s}\}, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ R_n = \operatorname{diag}\{r_{n1}, \dots, r_{ns}\}, \end{gathered} \tag{60}$$

such that $\Delta \sim \Omega$. Taking into account (60), we get

$$\det(\lambda I_{ns} - \Delta) = \det[\lambda^n I_s - R_n \lambda^{n-1} - R_{n-1}\lambda^{n-2} - \dots - R_1] = \det[\operatorname{diag}\{q_1(\lambda), \dots, q_s(\lambda)\}] \tag{61}$$

where

$$\begin{gathered} q_1(\lambda) = \lambda^n - r_{n1}\lambda^{n-1} - \dots - r_{21}\lambda - r_{11}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ q_s(\lambda) = \lambda^n - r_{ns}\lambda^{n-1} - \dots - r_{2s}\lambda - r_{1s}. \end{gathered} \tag{62}$$

Since $\Delta \sim \Omega$, we get $\chi(\Delta,\lambda) = \chi(\Omega,\lambda) = (\lambda - a)^{ns}$. Hence, from (61), it follows that $\prod\limits_{j=1}^{s} q_j(\lambda) = (\lambda - a)^{ns}$. Hence, $q_j(\lambda) = (\lambda - a)^n$, $j = \overline{1,s}$. From this, by (62), it follows that

$$r_{ij} = (-1)^{n-i} C_n^{n+1-i} a^{n+1-i}, \quad i = \overline{1,n},$$

for all $j = \overline{1,s}$. Thus, the matrices (60) are scalar matrices and

$$\begin{aligned} &R_n = naI, \quad R_{n-1} = -C_n^2 a^2 I, \quad R_{n-2} = C_n^3 a^3 I, \quad \dots, \\ &R_2 = (-1)^{n-2} C_n^{n-1} a^{n-1} I, \quad R_1 = (-1)^{n-1} a^n I, \quad I, J \in M_s(\mathbb{K}). \end{aligned} \tag{63}$$

In particular, $R_i = U_i$, $i = \overline{2,n}$. From (63), it follows that $\Delta = W \otimes I$ where $I \in M_s(\mathbb{K})$ and

$$W = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -(-a)^n & -(-a)^{n-1}n & -(-a)^{n-2}n(n-1)/2 & \dots & na \end{bmatrix}.$$

The matrix W has the elementary divisor $(\lambda - a)^n$ and $W \sim aI + J$, $I, J \in M_n(\mathbb{K})$. Hence, $\Delta = W \otimes I_s \sim (aI_n + J) \otimes I_s$, and, therefore, Δ has s elementary divisors $(\lambda - a)^n, \dots, (\lambda - a)^n$. We get that the set of elementary divisors of the matrix Δ does not coincide with the set of elementary divisors of the matrix Ω (since $s > 1$). This contradicts the fact that these matrices are similar. We have come to a contradiction. Thus, the theorem is proven. □

§ 7. On simultaneous assignment of spectrum eigenvalues and eigenvectors

In conclusion, we present one property that systems with AMCA have. The property of AMCA was studied in [9] partially (there it was called as AMESA: this name is not completely accurate, see [ZK–2024]). Suppose that, for system (9), (10), the problem of AMCA for CMP by LSOF (or by LSSF) is resolvable. Then, for any $\Gamma_i \in M_s(\mathbb{K})$, $i = \overline{1,n}$, there exists a gain matrix $Q \in M_{ms,ks}(\mathbb{K})$ such that the closed-loop system (12) is reducible by some change of variables $z = Sx$ to the system (17) with the matrix (16). The system (17) with the matrix (16) is equivalent to the differential equation of nth order in the space $\mathbb{K}^s$:

$$\mathcal{X}^{(n)} + \Gamma_1 \mathcal{X}^{(n-1)} + \dots + \Gamma_n \mathcal{X} = 0, \quad \mathcal{X} \in \mathbb{K}^s. \tag{64}$$

The AMCA property allows us to assign coefficients of the matrices Γ_i, $i = \overline{1,n}$. In particular, this allows us to assign arbitrary modes (corresponding to eigenvalues) to solutions of differential equation (64) and simultaneously assign eigenvectors with a high degree of freedom. The following theorem was proven [9, Theorem 11].

Theorem 7. *For any different $\lambda_\xi \in \mathbb{R}$, $\xi = \overline{1,ns}$, and for any linear independent vectors $h_1,\dots,h_s \in \mathbb{R}^s$ there exist matrices $\Gamma_j \in M_s(\mathbb{R})$, $j = \overline{1,n}$, such that the general solution of system* (64) *has the form*

$$\begin{aligned}\mathcal{X}(t) = C_1 h_1 \exp(\lambda_1 t) + C_2 h_2 \exp(\lambda_2 t) + \ldots + C_s h_s \exp(\lambda_s t)\\ + C_{s+1} h_1 \exp(\lambda_{s+1} t) + \ldots + C_{2s} h_s \exp(\lambda_{2s} t) + \ldots\\ + C_{(n-1)s+1} h_1 \exp(\lambda_{(n-1)s+1} t) + \ldots + C_{ns} h_s \exp(\lambda_{ns} t).\end{aligned}$$

In [9, Remark 16], it was noted that the condition of [9, Theorem 11] that all λ_ξ are different can be weakened. We will prove this statement here. Moreover, we assume here that $\lambda_\xi \in \mathbb{K}$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ (and not only $\mathbb{K} = \mathbb{R}$).

Lemma 2. *Let $\Omega = (\lambda_1, \lambda_2, \dots, \lambda_{ns})$ be an ordered list of ns numbers $\lambda_\xi \in \mathbb{K}$ such that not all λ_ξ are necessarily different, but the multiplicity* (*that is, the number of repetitions*) *of each number in the list Ω does not exceed s. Then these numbers can be placed in the matrix $\Phi = \{\phi_{ij}\}$, $i = 1,\dots,n$, $j = 1,\dots,s$, in such a way that no column contains two identical numbers, that is, $\phi_{ij} = \phi_{\nu\mu} \Rightarrow j \neq \mu$.*

P r o o f. We renumber the elements λ_ξ of Ω as follows. Among the numbers of the list Ω, we select the number with the greatest multiplicity $i_1 \le s$. Denote these numbers as $\lambda'_1 = \lambda'_2 = \ldots = \lambda'_{i_1}$ with the indices from 1 to i_1. Next, among the remaining numbers of the list Ω, we select the number with the greatest multiplicity i_2 ($i_2 \le i_1 \le s$). Denote these numbers as $\lambda'_{i_1+1} = \ldots = \lambda'_{i_1+i_2}$. And so on. We obtain the ordered list

$$\begin{aligned}\Omega' = (\lambda'_1, \dots, \lambda'_{i_1}, \lambda'_{i_1+1}, \dots, \lambda'_{i_1+i_2}, \dots, \lambda'_{i_1+i_2+\ldots+i_k}),\\ i_k \le \ldots \le i_2 \le i_1 \le s.\end{aligned} \tag{65}$$

Let's place the list Ω' into the matrix Φ as follows: we place the elements of the list Ω' in order into the first row of the matrix Φ; then, when we reach the end of the first row of the matrix Φ, we move to the second row of the matrix Φ, and place the subsequent elements of the list Ω' in order into the second row of the matrix Φ; and so on, reaching the end of the jth row of the matrix Φ, we move to the next row. By doing this, we will place all the elements of the list Ω' into the matrix Φ in such a way that

$$\Omega' = (\phi_{11}, \phi_{12}, \dots, \phi_{1s}, \phi_{21}, \dots, \phi_{2s}, \dots, \phi_{n1}, \dots, \phi_{ns}).$$

This placement method will ensure that the required condition is met. Indeed, the equality of two elements of one column would mean that the multiplicity of this number is greater than s, and this contradicts with (65). □

Let an arbitrary set of linear independent vectors $h_1,\dots,h_s \in \mathbb{K}^s$ be given. Let an arbitrary ordered list $\Omega = (\lambda_1, \lambda_2, \dots, \lambda_{ns})$ of ns numbers $\lambda_\xi \in \mathbb{K}$ be given such that the multiplicity of each number in the list Ω does not exceed s. Let us renumber the list Ω according to Lemma 2 into the list Ω' and denote it again by Ω (in this case we will say that the list Ω is ordered according to Lemma 2). Let us construct the following vector functions:

$$\begin{array}{lll}\psi_{1,1}(t) = h_1 \exp(\lambda_1 t), & \psi_{1,2}(t) = h_2 \exp(\lambda_2 t), & \psi_{1,s}(t) = h_s \exp(\lambda_s t),\\ \psi_{2,1}(t) = h_1 \exp(\lambda_{s+1} t), & \psi_{2,2}(t) = h_2 \exp(\lambda_{s+2} t), & \psi_{2,s}(t) = h_s \exp(\lambda_{2s} t),\\ \dots\dots\dots & \dots\dots\dots & \dots\dots\dots,\\ \psi_{n,1}(t) = h_1 \exp(\lambda_{(n-1)s+1} t), & \psi_{n,2}(t) = h_2 \exp(\lambda_{(n-1)s+2} t), & \psi_{n,s}(t) = h_s \exp(\lambda_{ns} t).\end{array} \tag{66}$$

From the construction, it follows obviously that the vector functions (66) are linearly independent.

The following theorem takes place.

Theorem 8. *For any linear independent vectors* $h_1, \ldots, h_s \in \mathbb{K}^s$, *and for any* $\lambda_\xi \in \mathbb{K}$, $\xi = \overline{1, ns}$, *such that the vector functions* (66) *are linearly independent, there exist matrices* $\Gamma_j \in M_s(\mathbb{R})$, $j = \overline{1, n}$, *such that the general solution of system* (64) *has the form*

$$\mathcal{X}(t) = \sum_{j=1}^{n} \sum_{i=1}^{s} C_{j,i} \psi_{j,i}(t) \tag{67}$$

where $\psi_{j,i}(t)$ *are defined by* (66).

The proof of Theorem 8 repeats the proof of Theorem 7 (see [9, Theorem 11]) up to the last paragraph of the proof. Further, distinctness of λ_ξ is not assumed, but linear independence of (67) is assumed.

Theorem 9. *For any linear independent vectors* $h_1, \ldots, h_s \in \mathbb{K}^s$, *and for an arbitrary list* $\Omega = (\lambda_1, \lambda_2, \ldots, \lambda_{ns})$ *of* ns *numbers* $\lambda_\xi \in \mathbb{K}$ *(such that the multiplicity of each number in the list* Ω *does not exceed* s*) that is ordered according to Lemma 2, there exist matrices* $\Gamma_j \in M_s(\mathbb{R})$, $j = \overline{1, n}$, *such that the general solution of system* (64) *has the form* (67) *where* $\psi_{j,i}(t)$ *are defined by* (66).

The proof of Theorem 9 follows from Theorem 8 and the fact that if the list Ω is ordered according to Lemma 2, then the functions (66) are linearly independent.

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О назначении произвольных матричных коэффициентов для характеристического матричного многочлена блочных матричных линейных систем управления

Ключевые слова: линейная стационарная система управления, назначение спектра собственных значений, линейная статическая обратная связь, блочная матричная система.

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Для блочных матричных линейных систем управления изучается свойство, обеспечивающее назначение произвольных матричных коэффициентов для характеристического матричного полинома. Это свойство является обобщением свойства назначаемости спектра собственных значений или назначаемости произвольных коэффициентов характеристического полинома, от систем с блочными матрицами со скалярными блоками $(s = 1)$ на системы с блочными матрицами с блоками более высоких размерностей $(s > 1)$. По сравнению со скалярным случаем $(s = 1)$ в блочных случаях более высоких размерностей $(s > 1)$ появляются новые особенности, отсутствующие в скалярном случае. Вводятся новые свойства, обеспечивающие назначение произвольных (верхнетреугольных, нижнетреугольных, диагональных) матричных коэффициентов для характеристического матричного полинома. В скалярном случае все описанные свойства эквивалентны друг другу, однако в блочных случаях более высоких размерностей это не так. Устанавливаются импликации между этими свойствами.

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