VESTNIK UDMURTSKOGO UNIVERSITETA. MATEMATIKA. MEKHANIKA. KOMP'YUTERNYE NAUKI
MATHEMATICS 2024. Vol. 34. Issue 3. Pp. 321–338.

MSC2020: 35K20, 35R09, 35R30

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INVERSE COEFFICIENT PROBLEM FOR A PARTIAL DIFFERENTIAL EQUATION WITH MULTI-TERM ORDERS FRACTIONAL RIEMANN–LIOUVILLE DERIVATIVES

This work studies direct initial boundary value and inverse coefficient determination problems for a one-dimensional partial differential equation with multi-term orders fractional Riemann–Liouville derivatives. The unique solvability of the direct problem is investigated and a priori estimates for its solution are obtained in weighted spaces, which will be used for studying the inverse problem. Then, the inverse problem is equivalently reduced to a nonlinear integral equation. The fixed-point principle is used to prove the unique solvability of this equation.

Keywords: fractional order equation, direct problem, inverse problem, Fourier method, Mittag–Leffler function, Laplace transform, existence, uniqueness.

DOI: 10.35634/vm240302

Introduction and formulation of problem

Fractional partial differential equations are generalizations of partial differential equations to arbitrary (non-integer) orders. These equations have attracted considerable interest in the applied sciences because of their ability to model complex phenomena. It should also be observed that these equations capture nonlocal relations in space and time with power-law memory kernels [1].

In this paper, we focus on the initial-boundary value problem for the partial differential equation with multi-term orders fractional derivatives

$$u_t + \sum_{k=1}^{m-1} \mu_k D_{0+,t}^{\alpha_k} u - \mu_m D_{0+,t}^{\alpha_m} u_{xx} + q(t)u = f(x,t), \quad 0 < x < l, \quad 0 < t \le T, \tag{0.1}$$

with the initial

$$u(x,0) = \varphi(x), \quad 0 \le x \le l, \tag{0.2}$$

and the boundary conditions

$$u(0,t) = u(l,t) = 0, \quad 0 \le t \le T, \tag{0.3}$$

where $0 < \alpha_k < 1$, $\alpha_1 \le \alpha_2 \le \ldots \le \alpha_{m-1}$, $\mu_k \in [\underline{\mu}, \bar{\mu}]$, $0 < \underline{\mu} < \bar{\mu}$, $k = 1, 2, \ldots, m$, $D_{0+,t}^{\gamma}$ is a fractional derivative in the sense of Riemann–Liouville with respect to the variable t, defined by the equalities (see [2, pp. 69–90]):

$$(D_{0+,t}^{\gamma} g)(t) = \frac{d}{dt} I_{0+,t}^{1-\gamma} g(t) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_0^t \frac{g(\tau)}{(t-\tau)^{\gamma}}\, d\tau, \quad 0 < \gamma < 1,$$

$$D_{0+,t}^{\gamma} g(t) = g'(t), \quad \gamma = 1,$$

$$I_{0+,t}^{\gamma} g(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{g(\tau)}{(t-\tau)^{1-\gamma}}\, d\tau, \quad 0 < \gamma < 1,$$

$f(x,t)$, $\varphi(x)$ are given functions, l, T are some positive fixed numbers.

We refer the monograph [2], as reflecting the main approaches to the study of various problems for fractional diffusion and diffusion-wave equations (see also the extensive bibliography there on this issue). In [3], the Cauchy-type problem for the fractional diffusion equation with a discretely distributed differentiation operator was studied. The properties of the fundamental solution were studied using the Wright function. Initial-boundary value problems in bounded domains for diffusion and diffusion-wave equations containing multi-term orders fractional differential operators with Caputo derivatives were solved by the method of separation of variables [4–6], and by methods of integral transformations [7].

The equation (0.1) with $\alpha_m = 0$ and $m = 2$ describes the processes of transfer of immobile solute in highly inhomogeneous porous media [8]. In the works [9,10], the analytical formulas for solution of the Cauchy problem for these equations were constructed. The work [11] is devoted to constructing a fundamental solution of (0.1) and, with its help, solving the Cauchy problem for this equation when $\alpha_m = 0$ and $m = 2$. Equation (0.1) differs from the fractional diffusion-wave equation with the presence of both classical and fractional derivatives of the unknown function.

The problem of determining function $u(x,t)$ satisfying equalities (0.1)–(0.3), which are given by functions $q(t)$, $f(x,t)$, $\varphi(x)$, numbers μ_k and α_k, is called a *direct problem*, and by its *regular solution* we mean a function $u(x,t)$, defined in a closed domain $\overline{\Omega} := \{(x,t)\colon x \in [0,l],\ 0 \le t \le T\}$, subject to conditions:

(1) $\left\{t^\gamma u_t,\ t^\gamma D^{\alpha_k}_{0+,t} u,\ t^\gamma D^{\alpha_m}_{0+,t} u_{xx}\right\} \in C(\overline{\Omega})$, $\gamma = \max\{\alpha_k,\ k = 1,\dots,m\}$;

(2) $u(x,t)$ is twice continuously differentiable with respect to x for each $0 < t \le T$;

(3) for each $x \in (0,l)$ fractional integrals of the function $u(x,t)$: $I^{1-\alpha_k}_{0+,t} u(x,t)$, $I^{1-\alpha_m}_{0+,t} u_{xx}(x,t)$ are continuously differentiable with respect to $t \in (0,T]$;

(4) the equalities (0.1)–(0.3) are satisfied.

The main goal of this work is to study the unique solvability of the *inverse problem* of determining the variable coefficient $q(t)$ of equation (0.1) according to the known condition for solution to the direct problem (0.1)–(0.3):

$$\int_0^l p(x)u(x,t)\,dx = h(t), \quad t \in [0,T], \tag{0.4}$$

where $p(x)$, $h(t)$ are the given functions. In heat propagation in a thin rod in which the law of variation $h(t)$ of the total quantity of heat in the rod is given in [12]. This integral condition in parabolic problems is also called heat moments which are analyzed in [13].

In recent years, numerous studies have been carried out on inverse problems involving various corresponding fractional operators, including the diffusion operator. The book [14, pp. 443–464] provides a review of inverse spatially dependent coefficient problems for fractional diffusion equations (see also the list of references). Basically, the questions of uniqueness and stability were investigated using the maximum principle and the method of the Carleman's estimates. In works [15, 16], inverse coefficient nonlinear problems for fractional diffusion-wave equations in the bounded domain were investigated by the method of the integral equations.

In [17], the inverse problem of determining the time-dependent zero-coefficient in the Cauchy problem for the time-fractional diffusion equation by a single observation at the point $x = 0$ of the diffusion process. Recently, the works [18–20] have been published in which multidimensional inverse problems of identifying the coefficient at the lowest term of the fractional diffusion equation were investigated. Local existence theorems, global uniqueness, and conditional stability estimates for solutions are obtained. The article [21] deals with an inverse potential problem for a semilinear generalized fractional diffusion equation with a time-dependent principal part. The well-posedness of the direct problem is investigated by using the well known Rothe's method. The existence and uniqueness of the inverse problem are obtained by employing the Arzelà–Ascoli theorem, a coerciveness of the fractional derivative and Grönwall's inequality, as well as the regularities of the direct problem. We recommend browsing the reference list of [21] for other inverse coefficient problems.

Throughout this work, regarding the input data, we will assume that the following requirements are met:

(A1) $\varphi(x) \in C^2[0,l]$, $\varphi'''(x) \in L_2[0,l]$, $\varphi(0) = \varphi(l) = \varphi''(0) = \varphi''(l) = 0$;

(A2) $f(x,\cdot) \in C[0,T]$ for all $t \in [0,T]$, $f(\cdot,t) \in C^2[0,l]$, $f^{(3)}(\cdot,t) \in L_2[0,l]$, $f(0,t) = f(l,t) = f_{xx}(0,l) = f_{xx}(l,t) = 0$;

(A3) $h(t) \in C^1[0,T]$, $|h(t)| \geq \frac{1}{h_0} = \text{const} > 0$ for all $t \in [0,T]$, $\int_0^l p(x)\varphi(x)\,dx = h(0)$;

(A4) $p(x) \in C^2[0,l]$, $p(0) = p(l) = 0$.

§ 1. Preliminaries

In this section, we present well known definitions and statements that will be used for the proofs of main results.

In this work, we use the following weighted function spaces (see, for example, [2, pp. 4, 162]):

$$C_\gamma[0,T] := \{g\colon (0,T] \to R\colon t^\gamma g(t) \in C[0,T],\ 0 \leq \gamma < 1\},$$

with the norms

$$\|g\|_\gamma = \|t^\gamma g(t)\|_C = \|t^\gamma g(t)\| = \max_{t\in[0,T]} |t^\gamma g(t)|.$$

The Mittag–Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ are defined by the following series:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

respectively, where $\alpha, \beta \in \mathbb{C}$, $\operatorname{Re}(\alpha) > 0$.

The three-parameter Mittag–Leffler function is defined as follows

$$E^\gamma_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \quad z, \alpha, \beta, \gamma \in \mathbb{C}, \quad \operatorname{Re}(\alpha) > 0,$$

here $(\gamma)_n$ is the Pochhammer symbol:

$$(\gamma)_n = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)} \text{ and } (\gamma)_0 \equiv 1 \quad \big(\operatorname{Re}(\gamma) > -n;\ n \in \mathbb{N};\ \gamma \notin \{0, -1, -2, \ldots\}\big).$$

Next we need the following statements.

Proposition 1.1 ([2, p. 84]). *The following Laplace transformation is correct:*

$$L\big[D^\gamma_{0+,t} g(t)\big](s) = s^\gamma L\big[g(t)\big](s) - \big(I^{1-\gamma}_{0+,t} g\big)(0+), \quad 0 < \gamma \leq 1. \tag{1.1}$$

The following formula also holds for the Laplace transform of the function $t^{\beta-1} E^\gamma_{\alpha,\beta}(\pm\omega t^\alpha)$:

$$L\big[t^{\beta-1} E^\gamma_{\alpha,\beta}(\pm\omega t^\alpha)\big](s) = \int_0^\infty e^{-st} t^{\beta-1} E^\gamma_{\alpha,\beta}(\pm\omega t^\alpha)\,dt = \frac{s^{\alpha\gamma-\beta}}{(s^\alpha \mp \omega)^\gamma}, \tag{1.2}$$

where $\operatorname{Re}(s) > 0$, $\operatorname{Re}(\beta) > 0$, $\omega \in \mathbb{C}$, *and* $|\omega/s^\alpha| < 1$.

Definition 1.1 ([23, p. 2]). The $H(z)$ function is defined by means of a Mellin–Barnes type integral in the following manner:

$$H(z)=H^{m,n}_{p,q}\left[z\middle|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right]=\frac{1}{2\pi i}\int_L \Theta(s)z^{-s}\,ds,$$

where $i=(-1)^{\frac{1}{2}}$, $z\neq 0$, and $z^{-s}=\exp\bigl[-s(\ln|z|+i\arg z)\bigr]$, where $\ln|z|$ represents the natural logarithm of $|z|$ and $\arg z$ is not necessarily the principal value. Here

$$\Theta(s)=\frac{\prod\limits_{j=1}^{m}\Gamma(b_j+B_js)\prod\limits_{j=1}^{n}\Gamma(1-a_j-A_js)}{\prod\limits_{j=m+1}^{q}\Gamma(1-b_j-B_js)\prod\limits_{j=n+1}^{p}\Gamma(a_j+A_js)}.$$

An empty product is always interpreted as unity; $n,m,p,q\in N_0$ with $0\le n\le p$, $1\le m\le q$, $A_l,B_j\in\mathbb{R}_+$, $a_l,b_j\in\mathbb{R}(\mathbb{C})$, $l=1,2,\dots,p$, $j=1,2,\dots,q$. L is a suitable contour separating the poles

$$\zeta_{j\nu}=-\left(\frac{b_j+\nu}{B_j}\right),\quad j=1,2,\dots,m,\quad \nu=0,1,2,\dots,$$

of the gamma-function $\Gamma(b_j+sB_j)$ from the poles

$$\omega_{\lambda k}=\left(\frac{1-a_\alpha+k}{A_\lambda}\right),\quad \lambda=1,2,\dots,n,\quad k=0,1,2,\dots,$$

of the gamma-function $\Gamma(1-a_\lambda-sA_\lambda)$, that is

$$A_\lambda(b_j+\nu)\neq B_j(a_\lambda-k-1),\quad j=1,2,\dots,m,\quad \lambda=1,2,\dots,n,\quad \nu,k=1,2,3,\dots,\tag{1.3}$$

$$\mu=\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j,\tag{1.4}$$

$$\alpha=\sum_{j=1}^{n}A_j-\sum_{j=n+1}^{p}A_j+\sum_{j=1}^{m}B_j-\sum_{j=m+1}^{q}B_j.\tag{1.5}$$

Proposition 1.2 ([23, p. 19]). *Let μ and α be given as in (1.4) and (1.5), and let condition (1.3) be satisfied. Then, if $\mu\ge 0$ or ($\mu<0$, $\alpha>0$, and $|\arg z|<\frac{1}{2}\pi\alpha$), then the H-function has an asymptotic expansion at zero given by*

$$H^{m,n}_{p,q}(z)=O(z^c),\quad |z|\to 0,$$

here

$$c=\min_{1\le j\le m}\left[\frac{\operatorname{Re}(b_j)}{B_j}\right],$$

$\operatorname{Re}(b_j)$ *is the real part of* b_j.

Recall that the function $E^{\gamma}_{\alpha,\beta}(z)$ can be rewritten in terms of the Fox H-function as (see [2, p. 67]):

$$E^{\gamma}_{\alpha,\beta}(z)=\frac{1}{\Gamma(\gamma)}H^{1,1}_{1,2}\left[-z\middle|\begin{matrix}(1-\gamma,1)\\(0,1),(1-\beta,\alpha)\end{matrix}\right].\tag{1.6}$$

Proposition 1.3 ([2, p. 79]). *Let $a \in \mathbb{R}$ and $0 < \alpha < 1$. Also let the functions $f(x)$ and $k(x)$ be defined on $[a, b]$ such that*

$$f(x) \in C[a,b] \ \ \textit{and} \ \ L(x) = \int_0^x \tau^{-\alpha} k(x-\tau)\, d\tau \in C^1[a,b].$$

Then, for any $x \in [a, b]$,

$$D^{\alpha}_{a+}\left[\int_a^t k(t-u) f(u)\, du\right](x) =$$
$$= \int_a^x D^{\alpha}_{a+}\big[k(t-a)\big](u) f(x+a-u)\, du + f(x) \lim_{x\to a+} I^{1-\alpha}_{a+}\big[k(t-a)\big](x).$$

Proposition 1.4 ([24, p. 104]). *Let $\mu,\ \alpha,\ \beta > 0,\ a \in \mathbb{R}$. Then the following formulas for the Riemann–Liouville and the Liouville fractional integration and differentiation of the Prabhakar function are valid:*

$$\Big\{D^{\mu}_{0+,t}\big[t^{\beta-1} E^{\gamma}_{\alpha,\beta}(at^{\alpha})\big]\Big\}(x) = x^{\beta-\mu-1} E^{\gamma}_{\alpha,\beta-\mu}(ax^{\alpha}).$$

§ 2. Existence and uniqueness results for direct problem solution

The following statement is true.

Theorem 2.1. *Let $0 < \alpha_k \le 1$, $k = 1, 2, \ldots, m$, and conditions* (A1)–(A2) *are satisfied. Then there is a unique regular solution to the direct problem* (0.1)–(0.3).

P r o o f. By applying the Fourier method, the solution $u(x, t)$ of the problem (0.1)–(0.3) can be expanded in a uniformly convergent series in term of eigenfunctions of the form

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x) u_n(t), \tag{2.1}$$

where

$$X_n(x) = \sqrt{\frac{2}{l}} \sin\sqrt{\lambda_n} x, \quad \lambda_n = \left(\frac{\pi n}{l}\right)^2, \quad n = 1, 2, 3, \ldots.$$

Taking into account equality (2.1), we obtain from (0.1) the following equation:

$$u_n'(t) + \sum_{k=1}^{m-1} \mu_k D^{\alpha_k}_{0+,t} u_n(t) + \mu_m \lambda_n D^{\alpha_m}_{0+,t} u_n(t) = F_n(t; q, u, f), \tag{2.2}$$

where

$$F_n(t; q, u, f) = f_n(t) - q(t) u_n(t) = F_n(t),$$
$$f_n(t) = \int_0^l f(x,t) X_n(x)\, dx.$$

The following equalities follow from (0.2):

$$u_n(0) = \int_0^l u(x,0) X_n(x)\, dx = \int_0^l \varphi(x) X_n(x)\, dx = \varphi_n, \quad n = 1, 2, 3, \ldots. \tag{2.3}$$

Thus, we obtained ordinary fractional differential equations (2.2) for unknown functions $u_n(t)$ with initial conditions (2.3). To solve them, we will formally apply the Laplace transform. Then, we get the following expression:

$$s\omega_n(s) - \varphi_n + \sum_{k=1}^{m-1} \mu_k s^{\alpha_k}\omega_n(s) + \mu_m\lambda_n s^{\alpha_m}\omega_n(s) = \Phi_n(s), \tag{2.4}$$

where

$$L[u_n(t)] = \omega_n(s), \quad L\big[F_n(t)\big] = \Phi_n(s).$$

Next we show that $u_n(t)$ is continuous on $[0, T]$. Then the second term on the right side of (1.1) will be equal to zero.

Solving equation (2.4), we get

$$\omega_n(s) = \frac{\varphi_n + \Phi_n(s)}{s + \sum\limits_{k=1}^{m-1} \mu_k s^{\alpha_k} + \mu_m\lambda_n s^{\alpha_m}}. \tag{2.5}$$

Next, we calculate the inverse Laplace transform. First, we transform formula (2.5) for

$$\left| \frac{\sum\limits_{k=1}^{m-1} \mu_k s^{\alpha_k}}{s + \mu_m\lambda_n s^{\alpha_m}} \right| < 1,$$

as follows:

$$\frac{\varphi_n + \Phi_n(s)}{s + \sum\limits_{k=1}^{m-1} \mu_k s^{\alpha_k} + \mu_m\lambda_n s^{\alpha_m}} = \frac{\varphi_n + \Phi_n(s)}{s + \mu_m\lambda_n s^{\alpha_m}} \frac{1}{1 + \frac{\sum_{k=1}^{m-1} \mu_k s^{\alpha_k}}{s+\mu_m\lambda_n s^{\alpha_m}}} =$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\big(\varphi_n + \Phi_n(s)\big)}{s + \mu_m\lambda_n s^{\alpha_m}} \left(\frac{\sum\limits_{k=1}^{m-1} \mu_k s^{\alpha_k}}{s + \mu_m\lambda_n s^{\alpha_m}} \right)^r = \sum_{r=0}^{\infty} \frac{(-1)^r\big(\varphi_n + \Phi_n(s)\big)\left(\sum\limits_{k=1}^{m-1} \mu_k s^{\alpha_k}\right)^r}{(s + \mu_m\lambda_n s^{\alpha_m})^{r+1}} =$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\big(\varphi_n + \Phi_n(s)\big)}{(s + \mu_m\lambda_n s^{\alpha_m})^{r+1}} \left(\sum_{k_1+k_2+\ldots+k_{m-1}=r} \frac{r!}{k_1!k_2!\ldots k_{m-1}!} \right) \left[\prod_{\nu=1}^{m-1} (\mu_\nu)^{k_\nu} \right] s^{\sum\limits_{\nu=1}^{m-1} \alpha_\nu k_\nu},$$

if we also take into account the following relation:

$$(x_1 + x_2 + \ldots + x_{m-1})^r = \left(\sum_{k_1+k_2+\ldots+k_{m-1}=r} \frac{r!}{k_1!k_2!\ldots k_{m-1}!} \right) \left[\prod_{\nu=1}^{m-1} (x_\nu)^{k_\nu} \right],$$

where the summation is taken over all $k_1, k_2, \ldots, k_{m-1} \in N_0$ such that $k_1 + k_2 + \ldots k_{m-1} = r$ (see, for example, [22, p. 823]).

According to (1.2) and (1.6), for $s \in C$ and $|\mu_m\lambda_n s^{\alpha_m - 1}| < 1$, we have

$$\frac{s^{\sum\limits_{\nu=1}^{m-1} \alpha_\nu k_\nu - \alpha_m(r+1)}}{(s^{1-\alpha_m} + \mu_m\lambda_n)^{r+1}} = L\left[t^{r - \sum\limits_{\nu=1}^{m-1} \alpha_\nu k_\nu} E^{r+1}_{1-\alpha_m, r+1-\sum\limits_{\nu=1}^{m-1} \alpha_\nu k_\nu} \left(-\mu_m\lambda_n t^{1-\alpha_m} \right) \right] =$$

$$= L\left[t^{r - \sum\limits_{\nu=1}^{m-1} \alpha_\nu k_\nu} \frac{1}{r!} H^{1,1}_{1,2}\left[\mu_m\lambda_n t^{1-\alpha_m} \middle| \begin{matrix} (-r, 1) \\ (0,1), \left(\sum\limits_{\nu=1}^{m-1} \alpha_\nu k_\nu - r, 1 - \alpha_m \right) \end{matrix} \right] \right].$$

Using the well-known theorem on the Laplace transform of the convolution of two functions, from (2.4) taking into account the above calculations, we obtain the following formal solution to problems (2.2) and (2.3):

$$u_n(t) = \varphi_n G_n(t) + \int_0^t G_n(t-\tau) F_n(\tau)\, d\tau, \tag{2.6}$$

where

$$\begin{aligned} G_n(t) = \sum_{r=0}^{\infty} \left(\sum_{k_1+\ldots+k_{m-1}=r} \frac{(-1)^r}{k_1!\ldots k_{m-1}!} \right) \left[\prod_{\nu=1}^{m-1} (\mu_\nu)^{k_\nu} \right] \times \\ \times\, t^{\beta-1} H_{1,2}^{1,1}\left[\mu_m \lambda_n t^{1-\alpha_m} \middle| \begin{matrix} (-r,1) \\ (0,1),(1-\beta,1-\alpha_m) \end{matrix} \right], \quad \beta = r - \sum_{\nu=1}^{m-1} \alpha_\nu k_\nu + 1. \quad \square \end{aligned} \tag{2.7}$$

We have the following statement.

Lemma 2.1. *The following estimates are valid:*

$$\begin{gathered} \left|G_n(t)\right| \le C_T, \\ t^\gamma \left| D_{0+,t}^{\alpha_k} G_n(t) \right| \le C_T t^{\gamma-\alpha_k}, \\ \left|u_n(t)\right| \le \|\Psi_n^0\| e^{C_T \|q\|_\gamma \frac{T^{1-\gamma}}{1-\gamma}}, \\ t^\gamma \left| D_{0+,t}^{\alpha_k} u_n(t) \right| \le C_T t^\gamma \left(|\varphi_n| t^{-\alpha_k} + \frac{\|f_n\|}{1-\alpha_k} t^{1-\alpha_k} + \|u_n\| \|q\|_\gamma B(1-\alpha_k, 1-\gamma) \right), \\ t^\gamma \left|u_n'(t)\right| \le t^\gamma \left|F_n(t;q,u,f)\right| + \sum_{k=1}^{m-1} \mu_k \left| t^\gamma D_{0+,t}^{\alpha_k} u_n(t) \right| + \mu_m \lambda_n \left| t^\gamma D_{0+,t}^{\alpha_m} u_n(t) \right|, \end{gathered}$$

where C_T is some constant depending on μ_ν, α_1, T,

$$\Psi_n^0(t) = \left|\varphi_n G_n(t)\right| + \left| \int_0^t G_n(t-\tau) f_n(\tau)\, d\tau \right|,$$

$B(\cdot,\cdot)$ is the Euler's beta function.

P r o o f. First, we will prove that the series in $G_n(t)$ converges. To do this, consider the H-Fox's function. Possible singularities for the Fox H function can only exist for $t \to 0$. Taking into account Proposition 1.2, we have

$$H_{1,2}^{1,1}\left[\mu_m \lambda_n t^{1-\alpha_m} \middle| \begin{matrix} (-r,1) \\ (0,1),(1-\beta,1-\alpha_m) \end{matrix} \right] \le C, \quad t \to 0.$$

As a result, taking into account (2.7), we arrive at the following asymptotic estimates for the function $G_n(t)$:

$$\begin{aligned} |G_n(t)| &\le C \sum_{r=0}^{\infty} \left(\sum_{k_1+k_2+\ldots+k_{m-1}=r} \frac{r!}{k_1!k_2!\ldots k_{m-1}!} \right) \left[\prod_{\nu=1}^{m-1} (\mu_\nu)^{k_\nu} \right] \frac{t^{(1-\alpha_1)r}}{r!} = \\ &= C \sum_{r=0}^{\infty} \left(\sum_{\nu=1}^{m-1} \mu_\nu \right)^r \frac{t^{(1-\alpha_1)r}}{r!} = C e^{M t^{1-\alpha_1}} \le C e^{M T^{1-\alpha_1}} =: C_T, \end{aligned}$$

where $M = \sum\limits_{\nu=1}^{m-1} \mu_\nu$.

Based on above, we have the fact that the series in the formula (2.7) is bounded. Besides, the function $G_n(t)$ is continuous in $[0,T]$ for all $n = 1,2,3,\dots$.

We will prove the existence of a solution to the integral equation (2.6) in the space $C[0,T]$ by the method of successive approximations. For this equation, we consider a sequence of functions in $[0,T]$:

$$\left(u_n\right)_m(t) = -\int_0^t G_n(t-\tau)q(\tau)\left(u_n\right)_{m-1}(\tau)\,d\tau,$$

where

$$\left|\left(u_n\right)_0(t)\right| \le \left|\varphi_n G_n(t)\right| + \left|\int_0^t G_n(t-\tau)f_n(\tau)\,d\tau\right| := \Psi_n^0(t) \le \|\Psi_n^0\|,$$
$$\left(u_n\right)_1(t) = -\int_0^t G_n(t-\tau)q(\tau)\left(u_n\right)_0(\tau)\,d\tau.$$

We get the following estimates:

$$\left|\left(u_n\right)_1(t)\right| \le C_T\left|\int_0^t q(\tau)\Psi_n^0(\tau)\,d\tau\right| \le C_T\|\Psi_n^0\|\|q\|_\gamma\left|\int_0^t \tau^{-\gamma}\,d\tau\right| \le C_T\|\Psi_n^0\|\|q\|_\gamma\frac{t^{1-\gamma}}{1-\gamma}.$$

Similarly

$$\left|\left(u_n\right)_2(t)\right| = \left|\int_0^t G_n(t-\tau)q(\tau)\left(u_n\right)_1(\tau)\,d\tau\right| \le$$
$$\le C_T\left|\int_0^t q(\tau)\left(u_n\right)_1(\tau)\,d\tau\right| \le \frac{\|\Psi_n^0\|}{2!}\left(C_T\|q\|_\gamma\frac{t^{1-\gamma}}{1-\gamma}\right)^2.$$

For arbitrary $m = 1,2,\dots,$ we have:

$$\left|\left(u_n\right)_m(t)\right| \le C_T\left|\int_0^t q(\tau)\left(u_n\right)_{m-1}(\tau)\,d\tau\right| \le \frac{\|\Psi_n^0\|}{m!}\left(C_T\|q\|_\gamma\frac{t^{1-\gamma}}{1-\gamma}\right)^m.$$

From the above estimates, it follows that a sequence of functions

$$u_n(t) = \sum_{m=0}^{\infty}\left(u_n\right)_m(t) \tag{2.8}$$

converges uniformly in $[0,T]$. Indeed, (2.8) can be majorized on $[0,T]$ by a convergent number series as follows:

$$\left|u_n(t)\right| = \left|\sum_{m=0}^{\infty}\left(u_n\right)_m(t)\right| \le \sum_{m=0}^{\infty}\frac{\|\Psi_n^0\|}{m!}\left(C_T\|q\|_\gamma\frac{t^{1-\gamma}}{1-\gamma}\right)^m = \|\Psi_n^0\|e^{C_T\|q\|_\gamma\frac{T^{1-\gamma}}{1-\gamma}}.$$

Now let's calculate the fractional derivative $D_{0+,t}^{\alpha_k}u_n(t)$ for $k = 1,2,\dots,m$:

$$D_{0+,t}^{\alpha_k}u_n(t) = \varphi_n D_{0+,t}^{\alpha_k}G_n(t) + D_{0+,t}^{\alpha_k}\int_0^t G_n(t-\tau)F_n(\tau)\,d\tau. \tag{2.9}$$

In this case given Proposition 1.4, we have

$$D_{0+,t}^{\alpha_k}G_n(t)=\sum_{r=0}^{\infty}\left(\sum_{k_1+\ldots+k_{m-1}=r}\frac{(-1)^r r!}{k_1!\ldots k_{m-1}!}\right)\left[\prod_{\nu=1}^{m-1}(\mu_\nu)^{k_\nu}\right]\times$$
$$\times D_{0+,t}^{\alpha_k}\{t^{\beta-1}E_{1-\alpha_m,\beta}^{r+1}(-\lambda_n\mu_m t^{1-\alpha_m})\}=$$
$$=\sum_{r=0}^{\infty}\left(\sum_{k_1+\ldots+k_{m-1}=r}\frac{(-1)^r r!}{k_1!\ldots k_{m-1}!}\right)\left[\prod_{\nu=1}^{m-1}(\mu_\nu)^{k_\nu}\right]\times$$
$$\times t^{\beta-1-\alpha_k}E_{1-\alpha_m,\beta-\alpha_k}^{r+1}(-\lambda_n\mu_m t^{1-\alpha_m}).$$

Next, taking into account Proposition 1.3, we calculate the fractional derivative of the second term on the right side of the formula (2.9):

$$D_{0+,t}^{\alpha_k}\int_0^t G_n(t-\tau)F_n(\tau)\,d\tau=\int_0^t D_{0+,\tau}^{\alpha_k}G_n(\tau)F_n(t-\tau)\,d\tau.$$

Thus, we have the final form of the function $D_{0+,t}^{\alpha_k}u_n(t)$:

$$D_{0+,t}^{\alpha_k}u_n(t)=\varphi_n D_{0+,t}^{\alpha_k}G_n(t)+\int_0^t D_{0+,\tau}^{\alpha_k}G_n(\tau)F_n(t-\tau)\,d\tau.$$

Just as above, we obtain estimates for the function $t^\gamma D_{0+,t}^{\alpha_k}u_n(t)$:

$$\left|t^\gamma D_{0+,t}^{\alpha_k}u_n(t)\right|\le\left|\varphi_n t^\gamma D_{0+,t}^{\alpha_k}G_n(t)\right|+\left|t^\gamma\int_0^t D_{0+,t}^{\alpha_k}G_n(t-\tau)f_n(\tau)\,d\tau\right|+$$
$$+\left|t^\gamma\int_0^t D_{0+,t}^{\alpha_k}G_n(t-\tau)q(\tau)u_n(\tau)\,d\tau\right|\le C_T|\varphi_n|t^{\gamma-\alpha_k}+C_T t^\gamma\|f_n\|\int_0^t(t-\tau)^{-\alpha_k}\,d\tau+$$
$$+C_T\|u_n\|\|q\|_\gamma t^\gamma\int_0^t(t-\tau)^{-\alpha_k}t^\gamma\,d\tau=$$
$$=C_T t^\gamma\left(|\varphi_n|t^{-\alpha_k}+\frac{\|f_n\|t^{1-\alpha_k}}{1-\alpha_k}+\|u_n\|\|q\|_\gamma B(1-\alpha_k,1-\gamma)\right).$$

Evaluate the function $\left|D_{0+,t}^{\alpha_k}G_n(t)\right|$ as follows:

$$\left|D_{0+,t}^{\alpha_k}G_n(t)\right|\le\sum_{r=0}^{\infty}\left(\sum_{k_1+k_2+\ldots+k_{m-1}=r}\frac{r!}{k_1!k_2!\ldots k_{m-1}!}\right)\left[\prod_{\nu=1}^{m-1}(\mu_\nu)^{k_\nu}\right]\times$$
$$\times\left|t^{r-\sum\limits_{\nu=1}^{m-1}\alpha_\nu k_\nu-\alpha_k}E^{r+1}_{1-\alpha_m,r-\sum\limits_{\nu=1}^{m-1}\alpha_\nu k_\nu-\alpha_k}(-\mu_m\lambda_n t^{1-\alpha_m})\right|=$$
$$=\sum_{r=0}^{\infty}\left(\sum_{k_1+k_2+\ldots+k_{m-1}=r}\frac{1}{k_1!k_2!\ldots k_{m-1}!}\right)\left[\prod_{\nu=1}^{m-1}(\mu_\nu)^{k_\nu}\right]\times$$
$$\times\left|t^{r-\sum\limits_{\nu=1}^{m-1}\alpha_\nu k_\nu-\alpha_k}H^{1,1}_{1,2}\left[\mu_m\lambda_n t^{1-\alpha_m}\middle|\begin{matrix}(-r,1)\\(0,1),\left(\sum\limits_{\nu=1}^{m-1}\alpha_\nu k_\nu-r,1-\alpha_m\right)\end{matrix}\right]\right|\le$$
$$=\sum_{r=0}^{\infty}\left(\sum_{k_1+k_2+\ldots+k_{m-1}=r}\frac{1}{k_1!k_2!\ldots k_{m-1}!}\right)\left[\prod_{\nu=1}^{m-1}(\mu_\nu)^{k_\nu}\right]t^{r-\sum\limits_{\nu=1}^{m-1}\alpha_\nu k_\nu-\alpha_k}C_T\le$$
$$\le C_T\sum_{r=0}^{\infty}\left(\sum_{\nu=1}^{m-1}\mu_\nu\right)^r t^{(1-\alpha_1)r-\alpha_k},$$

$$t^{\gamma}\left|D_{0+,t}^{\alpha_k}G_n(t)\right| \le Ct^{\gamma-\alpha_k}e^{Mt^{1-\alpha_1}} \le C_T t^{\gamma-\alpha_k}.$$

Estimates for the function $u_n'(t)$ can be obtained from equation (2.2). Lemma 2.1 is proven. □

Consider the convergence of the following series:

$$\sum_{n=1}^{\infty}\lambda_n\|\Psi_n^0\| \le C_T\sum_{n=1}^{\infty}\lambda_n\big(|\varphi_n| + T\|f_n\|\big) =$$
$$= C_T\sum_{n=1}^{\infty}\lambda_n\left(\left|\int_0^l \varphi(x)X_n(x)\,dx\right| + T\left\|\int_0^l f(x,t)X_n(x)\,dx\right\|\right). \tag{2.10}$$

Integrating the Fourier coefficients of functions $\varphi(x)$ and $f(x,t)$ by parts, and taking into account conditions (A1) and (A2), we obtain

$$\int_0^l \sqrt{\frac{2}{l}}\varphi(x)\sin\frac{\pi nx}{l}\,dx = -\int_0^l \frac{l^3}{\pi^3n^3}\sqrt{\frac{2}{l}}\varphi^{(3)}(x)\cos\frac{\pi nx}{l}\,dx =: \frac{l^3}{\pi^3n^3}\widehat{\varphi}_n,$$
$$\int_0^l \sqrt{\frac{2}{l}}f(x,t)\sin\frac{\pi nx}{l}\,dx = -\int_0^l \frac{l^3}{\pi^3n^3}\sqrt{\frac{2}{l}}f_{xxx}(x,t)\cos\frac{\pi nx}{l}\,dx =: \frac{l^3}{\pi^3n^3}\widehat{f}_n(t).$$

Let's write the series (2.10) using the last equality and apply the Cauchy–Bunyakovsky's inequality to this series

$$\sum_{n=1}^{\infty}\lambda_n\big(|\varphi_n| + T\|f_n\|\big) = \sum_{n=1}^{\infty}\frac{l}{\pi n}\big(|\widehat{\varphi}_n| + T\|\widehat{f}_n\|\big) \le \tag{2.11}$$
$$\le \frac{l}{\pi}\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^2}}\sqrt{\sum_{n=1}^{\infty}|\widehat{\varphi}_n|^2} + T\frac{l}{\pi}\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^2}}\sqrt{\sum_{n=1}^{\infty}\|\widehat{f}_n\|^2} \le C(\|\varphi'''\|_{L_2[0,l]} + T\|f_{xxx}\|_{L_2([0,l],[0,T])}).$$

Next, we examine the continuity of functions $u(x,t)$, $D_{0+,t}^{\alpha_k}u(x,t)$, and $D_{0+,t}^{\alpha_m}u_{xx}(x,t)$:

$$\big|u(x,t)\big| = \sum_{n=1}^{\infty}\big|u_n(t)X_n(x)\big| \le \sum_{n=1}^{\infty}\big|u_n(t)\big| \le C_T\sum_{n=1}^{\infty}\big(|\varphi_n| + T\|f_n\|\big)e^{C_T\|q\|_{\gamma}\frac{T^{1-\gamma}}{1-\gamma}},$$

$$\big|t^{\gamma}D_{0+,t}^{\alpha_k}u(x,t)\big| \le \sum_{n=1}^{\infty}\big|t^{\gamma}D_{0+,t}^{\alpha_k}u_n(t)X_n(x)\big| \le$$
$$\le C_T t^{\gamma}\sum_{n=1}^{\infty}\left(|\varphi_n|t^{-\alpha_k} + \frac{\|f_n\|}{1-\alpha_k}t^{1-\alpha_k} + \|u_n\|\|q\|_{\gamma}B(1-\alpha_k, 1-\gamma)\right),$$

$$\big|t^{\gamma}D_{0+,t}^{\alpha_m}u_{xx}(x,t)\big| \le \sum_{n=1}^{\infty}\big|t^{\gamma}D_{0+,t}^{\alpha_m}u_n(t)X_n''(x)\big| \le$$
$$\le C_T t^{\gamma}\sum_{n=1}^{\infty}\lambda_n\left(|\varphi_n|t^{-\alpha_m} + \frac{\|f_n\|}{1-\alpha_m}t^{1-\alpha_m} + \|u_n\|\|q\|_{\gamma}B(1-\alpha_m, 1-\gamma)\right).$$

The uniform convergence of the above series follows from formula (2.11), so function (2.1) is a classical solution to the problem (0.1)–(0.3). Theorem 2.1 is proven. □.

Let us derive an estimate for the norm of the difference between the solution of the original integral equation (2.6) and the solution of this equation with the perturbed functions $\widetilde{\varphi}_n$, $\widetilde{f}_n$ and $\widetilde{q}$. Let $\widetilde{u}_n$ be the solution to the integral equation (2.6) corresponding to the functions $\widetilde{\varphi}_n$, $\widetilde{f}_n$ and $\widetilde{q}$:

$$\widetilde{u}_n(t) = \widetilde{\varphi}_nG_n(t) + \int_0^t J_n(t-\tau)(\widetilde{f}_n(\tau) - \widetilde{q}(\tau)\widetilde{u}_n(\tau))\,d\tau, \quad n = 1, 2, \ldots. \tag{2.12}$$

Subtract the equations (2.6) and (2.12) one from the other, by introducing the notations $u_n - \widetilde{u}_n = \overline{u}_n$, $f_n - \widetilde{f}_n = \overline{f}_n$, $q - \widetilde{q} = \overline{q}$ and $\varphi_n - \widetilde{\varphi}_n = \overline{\varphi}_n$, we obtain the integral equation

$$\overline{u}_n(t) = \overline{\varphi}_n G_n(t) + \int_0^t G_n(t-\tau)\overline{f}_n(\tau)\,d\tau - \int_0^t G_n(t-\tau)\big(\overline{q}(\tau)u_n(\tau) + \widetilde{q}(\tau)\overline{u}_n(\tau)\big)\,d\tau. \quad (2.13)$$

Applying the method of successive approximations to the linear equation (2.13) with respect to $\overline{u}_n(t)$ as follows

$$\Big|\big(\overline{u}\big)_0(t)\Big| \le C_T|\overline{\varphi}_n| + C_T T\|\overline{f}_n\| + C_T T\|\overline{q}\|_\gamma \|\Psi_n^o\| e^{C_T\|q\|_\gamma \frac{T^{1-\gamma}}{1-\gamma}},$$

$$\Big|\big(\overline{u}\big)_m(t)\Big| \le \int_0^t G_n(t-\tau)\widetilde{q}(\tau)\Big|\big(u_n\big)_{m-1}(\tau)\Big|\,d\tau,$$

we have the estimates for all $t \in [0,T]$ and $n = 1, 2, 3, \dots$:

$$\big|\overline{u}_n(t)\big| \le C_T T\Big(|\overline{\varphi}_n| + \|\overline{f}_n\| + \|\overline{q}\|_\gamma \|\Psi_n^o\| e^{C_T\|q\|_\gamma \frac{T^{1-\gamma}}{1-\gamma}}\Big) e^{C_T\|\widetilde{q}\|_\gamma \frac{T^{1-\gamma}}{1-\gamma}}.$$

These relations will be used in the next Section.

§ 3. Inverse problem

First, we multiply both sides of the equation (0.1) by $p(x)$ and integrate over the variable x on the interval $[0, l]$, and we obtain

$$\int_0^l p(x)u_t(x,t)\,dx + \int_0^l \sum_{k=1}^{m-1} \mu_k p(x) D_{0+,t}^{\alpha_k} u(x,t)\,dx - \mu_m \int_0^l p(x) D_{0+,t}^{\alpha_m} u_{xx}(x,t)\,dx + \\ + \int_0^l p(x)q(t)u(x,t)\,dx = \int_0^l p(x)f(x,t)\,dx. \quad (3.1)$$

If we use an additional condition (0.4), then the equation (3.1) takes the following form:

$$h'(t) + \sum_{k=1}^{m-1} \mu_k D_{0+,t}^{\alpha_k} h(t) - \mu_m D_{0+,t}^{\alpha_m} \int_0^l p(x)u_{xx}(x,t)\,dx + q(t)h(t) = \phi(t), \quad (3.2)$$

where $\phi(t) = \int_0^l p(x)f(x,t)\,dx$.

We integrate by parts twice in the third term on the left side of (3.2) and obtain

$$h'(t) + \sum_{k=1}^{m-1} \mu_k D_{0+,t}^{\alpha_k} h(t) - \mu_m D_{0+,t}^{\alpha_m} \int_0^l p''(x)u(x,t)\,dx + q(t)h(t) = \phi(t).$$

Taking into account the expansion of the function $u(x,t)$ in the system $X_n(x)$ into a Fourier series and obtain the following equality:

$$h'(t) + \sum_{k=1}^{m-1} \mu_k D_{0+,t}^{\alpha_k} h(t) - \mu_m D_{0+,t}^{\alpha_m} \int_0^l p''(x) \sum_{n=1}^{\infty} X_n(x)u_n(t)\,dx + q(t)h(t) = \phi(t),$$

or

$$h'(t) + \sum_{k=1}^{m-1} \mu_k D_{0+,t}^{\alpha_k} h(t) - \mu_m \sum_{n=1}^{\infty} p_n'' D_{0+,t}^{\alpha_m} u_n(t) + q(t)h(t) = \phi(t), \quad (3.3)$$

where

$$p_n'' = \int_0^1 X_n(x)p''(x)\,dx.$$

Let us apply the fractional derivative $D_{0+,t}^{\alpha_m}$ to both sides of formula (2.6), and we have:

$$D_{0+,t}^{\alpha_m}u_n(t) = \varphi_n D_{0+,t}^{\alpha_m}G_n(t) + D_{0+,t}^{\alpha_m}\int_0^t G_n(t-\tau)f_n(\tau)\,d\tau - D_{0+,t}^{\alpha_m}\int_0^t G_n(t-\tau)u_n(\tau)q(\tau)\,d\tau.$$

Now we use Proposition 1.3, then, we have

$$D_{0+,t}^{\alpha_m}\int_0^t G_n(t-\tau)f_n(\tau)\,d\tau = \int_0^t G_{n,\alpha_m}(t-\tau)f_n(\tau)\,d\tau,$$
$$D_{0+,t}^{\alpha_m}\int_0^t G_n(t-\tau)u_n(\tau)q(\tau)\,d\tau = \int_0^t G_{n,\alpha_m}(t-\tau)u_n(\tau)q(\tau)\,d\tau,$$

where

$$D_{0+,t}^{\alpha_m}G_n(t) = G_{n,\alpha_m}(t).$$

In view of this, equation (3.3) will have the following form:

$$h'(t) + \sum_{k=1}^{m-1}\mu_k D_{0+,t}^{\alpha_k}h(t) - \mu_m\sum_{n=1}^{\infty}p_n''\varphi_n G_{n,\alpha_m}(t) -$$
$$-\mu_m\sum_{n=1}^{\infty}p_n''\int_0^t G_{n,\alpha_m}(t-\tau)\big(f_n(\tau) - q(\tau)u_n(\tau)\big)\,d\tau + q(t)h(t) = \phi(t).$$

As a result, we find an integral equation for an unknown function $q(t)$:

$$q(t) = H(t) - \frac{\mu_m}{h(t)}\sum_{n=1}^{\infty}p_n''\int_0^t G_{n,\alpha_m}(t-\tau)q(\tau)u_n(\tau)\,d\tau, \tag{3.4}$$

where

$$H(t) = -\frac{1}{h(t)}\Bigg(h'(t) + \sum_{k=1}^{m-1}\mu_k D_{0+,t}^{\alpha_k}h(t) - \mu_m\varphi_n\sum_{n=1}^{\infty}p_n''G_{n,\alpha_m}(t) -$$
$$-\mu_m\sum_{n=1}^{\infty}p_n''\int_0^t G_{n,\alpha_m}(t-\tau)f_n(\tau)\,d\tau - \phi(t)\Bigg).$$

Rewrite (3.4) in a more convenient form

$$q(t) = A[q](t). \tag{3.5}$$

Theorem 3.1. *Suppose that conditions* (A1)–(A5) *are satisfied. Then there exists a number* $T^* \in (0,T)$ *such that, for all* $0 < \alpha_k \le \gamma < 1$, *inverse problem* (0.1)–(0.4) *has a unique solution, belonging to the class* $C_\gamma[0,T]$.

P r o o f. Consider a ball

$$B\big(H(t),\rho\big) = \big\{q(t)\colon q(t)\in C_\gamma[0,T],\ \|q(t)-H(t)\|_\gamma \le \rho\big\}.$$

Obviously, for functions $q(t)\in B\big(H(t),\rho\big)$, the estimate $\|q\|_\gamma \le 2\rho$ is valid.

Let a function $q \in B\big(H(t), \rho\big)$. Let us prove that, with an appropriate choice of $T > 0$, the operator A on the set $B\big(H(t), \rho\big)$ is contractive. We first show that if we choose $T > 0$ appropriately, then the operator A maps a ball to the same ball, that is, $Aq \in B\big(H(t), \rho\big)$:

$$\begin{aligned}\big\|A[q] - H(t)\big\|_\gamma &= \left\|\frac{\mu_m}{h(t)}\sum_{n=1}^{\infty} p_n'' \int_0^t G_{n,\alpha_m}(t-\tau)q(\tau)u_n(\tau)\,d\tau\right\|_\gamma \le \\ &\le h_0\|q\|_\gamma \mu_m C_T \left\|\sum_{n=1}^{\infty} t^\gamma p_n'' \int_0^t (t-\tau)^{-\alpha_m}\tau^{-\gamma}u_n(\tau)\,d\tau\right\| \le \\ &\le h_0\|q\|_\gamma \mu_m C_T e^{C_T\|q\|_\gamma \frac{T^{1-\gamma}}{1-\gamma}} T^{1-\alpha_m}\left\|\sum_{n=1}^{\infty} p_n''\|\Psi_n\| B(1-\alpha_m, 1-\gamma)\right\| \le \\ &\le 2\rho h_0 \mu_m C_T e^{2C_T\rho\frac{T^{1-\gamma}}{1-\gamma}} T^{1-\alpha_m} B(1-\alpha_m, 1-\gamma)\|p''\|_{L_2[0,l]}\|\Psi^0\|_{C([0,T],L_2[0,l])}.\end{aligned}$$

Let T_1 be the positive root of the equation

$$W_1(T) = 2\rho h_0 \mu_m C_T e^{2C_T\rho\frac{T^{1-\gamma}}{1-\gamma}} T^{1-\alpha_m} B(1-\alpha_m, 1-\gamma)\|p''\|_{L_2[0,l]}\|\Psi^0\|_{C([0,T],L_2[0,l])} = \rho.$$

Now consider two functions $q(t)$ and $\widetilde{q}(t)$ belonging to the ball $B\big(H(t), \rho\big)$, and estimate the distance between their images $A[q](t)$ and $A[\widetilde{q}](t)$ in the space $C_\gamma[0, T]$. The function $\widetilde{u}_n(t)$ corresponding to $\widetilde{q}(t)$ satisfies the integral equation (2.12) with the functions $\varphi_n = \widetilde{\varphi}_n$ and $f_n(t) = \widetilde{f}_n(t)$. Compiling the difference $A[q](t) - A[\widetilde{q}](t)$ using the equations (2.6), (2.12) and then estimating its norm, we obtain

$$\begin{aligned}\big\|A[q] - A[\widetilde{q}]\big\|_\gamma &= \left\|\frac{t^\gamma}{h(t)}\sum_{n=1}^{\infty} p_n'' \int_0^t G_{n.\beta}(t-\tau)\big(u_n(\tau)\overline{q}(\tau) + \widetilde{q}(\tau)\overline{u}_n(\tau)\big)\,d\tau\right\| \le \\ &\le h_0 C_T e^{2\rho C_T\frac{T^{1-\gamma}}{1-\gamma}}\|\overline{q}\|_\gamma \left\|t^\gamma \sum_{n=1}^{\infty} p_n''\|\Psi_n^0\| \int_0^t (t-\tau)^{-\beta}\tau^{-\gamma}d\tau\right\|\left(1 + 2\rho e^{2\rho C_T\frac{T^{1-\gamma}}{1-\gamma}}\right) \le \\ &\le h_0 C_T e^{2\rho C_T\frac{T^{1-\gamma}}{1-\gamma}}\left(1 + 2\rho e^{2\rho C_T\frac{T^{1-\gamma}}{1-\gamma}}\right)T^{1-\beta} \times \\ &\times B(1-\gamma, 1-\beta)\|p''\|_{L_2[0,T]}\|\Psi^0\|_{C([0,T],L_2[0,l])}\|\overline{q}\|_\gamma.\end{aligned}$$

Let T_2 be the positive root of the equation

$$\begin{aligned}W_2(T) = h_0 C_T e^{2\rho C_T\frac{T^{1-\gamma}}{1-\gamma}}\left(1 + 2\rho e^{2\rho C_T\frac{T^{1-\gamma}}{1-\gamma}}\right)T^{1-\beta} \times \\ \times B(1-\gamma, 1-\beta)\|p''\|_{L_2[0,T]}\|\Psi^0\|_{C([0,T],L_2[0,l])} = 1.\end{aligned}$$

Then, for $T \in (0, T^*)$ operator A compresses the distance between elements $q(t), \widetilde{q}(t) \in \in B\big(H(t), \rho\big)$. Consequently, if we choose $T^* < \min(T_1, T_2)$, then the operator A is contractive in the ball $B\big(H(t), \rho\big)$. Then, in accordance with Banach's theorem, the operator A has a unique fixed point in the ball $B\big(H(t), \rho\big)$, i. e., there is a unique solution to the equation (3.5). Theorem 3.1 is proven. □

Conclusion

In this work, we considered the inverse problem of determining the time-dependent coefficient in a one-dimensional fractional order equation with initial boundary conditions and an overdetermination condition. First, the Fourier method reduces the problem to equivalent integral equations. After this, using estimates of the Mittag–Leffler function and the method of successive approximations, an estimate for the solution of the direct problem was obtained in terms of the norm of the unknown coefficient, which was used in the study of the inverse problem. The inverse problem is reduced to an equivalent integral equation of Volterra type. To solve this equation, the principle of compressed mapping is applied. The results of local existence and uniqueness are proven.

Funding. The work of the first author was supported by the Ministry of Science and Higher Education of the Russian Federation, grant 075-02-2024-1447.

REFERENCES

1. Podlubny I. *Fractional differential equations*, San Diego: Academic Press, 1999.
2. Kilbas A. A., Srivastava H. M., Trujillo J. J. *Theory and Applications of Fractional Differential Equations*, Amsterdam: Elsevier, 2006.
3. Pskhu A. V. Fractional diffusion equation with discretely distributed differentiation operator, *Sibirskie Èlektronnye Matematicheskie Izvestiya*, 2016, vol. 13, pp. 1078–1098 (in Russian). https://doi.org/10.17377/semi.2016.13.086
4. Luchko Yu. Initial-boundary-value problems for the generalized multi-term time-fractional diffusion equation, *Journal of Mathematical Analysis and Applications*, 2011, vol. 374, issue 2, pp. 538–548. https://doi.org/10.1016/j.jmaa.2010.08.048
5. Daftardar–Gejji V., Bhalekar S. Boundary value problems for multi-term fractional differential equations, *Journal of Mathematical Analysis and Applications*, 2008, vol. 345, issue 2, pp. 754–765. https://doi.org/10.1016/j.jmaa.2008.04.065
6. Jiang H., Liu Fawang, Turner I. W., Burrage K. Analytical solutions for the multi-term time-fractional diffusion-wave/diffusion equations in a finite domain, *Computers and Mathematics with Applications*, 2012, vol. 64, issue 10, pp. 3377–3388. https://doi.org/10.1016/j.camwa.2012.02.042
7. Liu Xiao-jing, Wang Ji-zeng, Wang Xiao-min, Zhou You-he. Exact solutions of multi-term fractional diffusion-wave equations with Robin type boundary conditions, *Applied Mathematics and Mechanics*, 2014, vol. 35, issue 1, pp. 49–62. https://doi.org/10.1007/s10483-014-1771-6
8. Schumer R., Benson D. A., Meerschaert M. M., Baeumer B. Fractal mobile/immobile solute transport, *Water Resources Research*, 2003, vol. 39, issue 10. https://doi.org/10.1029/2003WR002141
9. Durdiev D. K., Shishkina E. L., Sitnik S. M. The explicit formula for solution of anomalous diffusion equation in the multi-dimensional space, *Lobachevskii Journal of Mathematics*, 2021, vol. 42, issue 6, pp. 1264–1273. https://doi.org/10.1134/S199508022106007X
10. Sultanov M. A., Durdiev D. K., Rahmonov A. A. Construction of an explicit solution of a time-fractional multidimensional differential equation, *Mathematics*, 2021, vol. 9, issue 17, 2052. https://doi.org/10.3390/math9172052
11. Hasanov I. I., Akramova D. I., Rakhmonov A. A. Investigation of the Cauchy problem for one fractional order equation with the Riemann–Liouville operator, *Journal of Samara State Technical University, Ser. Physical and Mathematical Sciences*, 2023, vol. 27, no. 1, pp. 64–80 (in Russian). https://doi.org/10.14498/vsgtu1952
12. Ionkin N. I. Solution of a boundary-value problem in heat conduction with a nonclassical boundary condition, *Differential Equations*, 1977, vol. 13, pp. 204–211. https://zbmath.org/0403.35043
13. Ivanchov M. I., Pabyrivs'ka N. V. Simultaneous determination of two coefficients of a parabolic equation in the case of nonlocal and integral conditions, *Ukrainian Mathematical Journal*, 2001, vol. 53, no. 5, pp. 674–684. https://doi.org/10.1023/A:1012570031242
14. Kochubei A., Luchko Yu. *Handbook of fractional calculus with applications. Vol. 2. Fractional differential equations*, De Gruyter, 2019. https://doi.org/10.1515/9783110571660
15. Durdiev D. K., Turdiev H. H. Inverse coefficient problem for a time-fractional wave equation with initial-boundary and integral type overdetermination conditions, *Mathematical Methods in the Applied Sciences*, 2024, vol. 47, issue 6, pp. 5329–5340. https://doi.org/10.1002/mma.9867

16. Turdiev H. H. Inverse coefficient problems for a time-fractional wave equation with the generalized Riemann–Liouville time derivative, *Russian Mathematics*, 2023, vol. 67, issue 10, pp. 14–29. https://doi.org/10.3103/S1066369X23100092
17. Durdiev D. K. Inverse coefficient problem for the time-fractional diffusion equation, *Eurasian Journal of Mathematical and Computer Applications*, 2021, vol. 9, issue 1, pp. 44–54. https://doi.org/10.32523/2306-6172-2021-9-1-44-54
18. Durdiev U. D. Problem of determining the reaction coefficient in a fractional diffusion equation, *Differential Equations*, 2021, vol. 57, issue 9, pp. 1195–1204. https://doi.org/10.1134/S0012266121090081
19. Durdiev D. K. Convolution kernel determination problem for the time-fractional diffusion equation, *Physica D: Nonlinear Phenomena*, 2024, vol. 457, 133959. https://doi.org/10.1016/j.physd.2023.133959
20. Durdiev D. K., Rahmonov A. A., Bozorov Z. R. A two-dimensional diffusion coefficient determination problem for the time-fractional equation, *Mathematical Methods in the Applied Sciences*, 2021, vol. 44, issue 13, pp. 10753–10761. https://doi.org/10.1002/mma.7442
21. Ma Wenjun, Sun Liangliang. Inverse potential problem for a semilinear generalized fractional diffusion equation with spatio-temporal dependent coefficients, *Inverse Problems*, 2023, vol. 39, no. 1, 015005. https://doi.org/10.1088/1361-6420/aca49e
22. Abramowitz M., Stegun I. A. *Handbook of mathematical functions with formulas, graphics and mathematical tables*, New York: Dover, 1972.
23. Mathai A. M., Saxena R. K., Haubold H. J. *The H-function. Theory and application*, New York: Springer, 2010. https://doi.org/10.1007/978-1-4419-0916-9
24. Gorenflo R., Kilbas A. A., Mainardi F., Rogosin S. V. *Mittag–Leffler functions, related topics and applications*, Berlin–Heidelberg: Springer, 2014. https://doi.org/10.1007/978-3-662-43930-2

Received 03.04.2024
Accepted 21.08.2024

Durdimurod Qalandarovich Durdiev, Doctor of Physics and Mathematics, Professor, Bukhara Branch of the Institute of Mathematics named after V. I. Romanovskiy at the Academy of Sciences of the Republic of Uzbekistan, ul. M. Iqbol, 11, Bukhara, 200118, Uzbekistan.
ORCID: https://orcid.org/0000-0002-6054-2827
E-mail: d.durdiev@mathinst.ru

Ibrohim Ixtiyorovich Hasanov, Lecturer, Bukhara State University, ul. M. Iqbol, 11, Bukhara, 200118, Uzbekistan.
ORCID: https://orcid.org/0009-0005-6294-5768
E-mail: i.i.hasanov@buxdu.uz, ihasanov998@gamail.com

Citation: D. K. Durdiev, I. I. Hasanov. Inverse coefficient problem for a partial differential equation with multi-term orders fractional Riemann–Liouville derivatives, *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki*, 2024, vol. 34, issue 3, pp. 321–338.

Д. К. Дурдиев, И. И. Хасанов

Обратная коэффициентная задача для уравнения в частных производных со многими производными дробных порядков Римана–Лиувилля

Ключевые слова: уравнение дробного порядка, прямая задача, обратная задача, метод Фурье, функция Миттаг–Леффлера, преобразование Лапласа, существование, единственность.

УДК 517.958

DOI: 10.35634/vm240302

В данной работе изучаются прямая начально-краевая задача и обратная задача определения коэффициента одномерного уравнения в частных производных со многими дробными производными Римана–Лиувилля. Исследована однозначная разрешимость прямой задачи и получены априорные оценки ее решения в весовых пространствах, которые будут использованы при изучении обратной задачи. Далее обратная задача эквивалентно сводится к нелинейному интегральному уравнению. Для доказательства однозначной разрешимости этого уравнения используется принцип неподвижной точки.

Финансирование. Работа первого автора выполнена при финансовой поддержке Министерства науки и высшего образования Российской Федерации, соглашение № 075-02-2024-1447.

СПИСОК ЛИТЕРАТУРЫ

1. Podlubny I. Fractional differential equations. San Diego: Academic Press, 1999.
2. Kilbas A. A., Srivastava H. M., Trujillo J. J. Theory and applications of fractional differential equations. Amsterdam: Elsevier, 2006.
3. Псху А. В. Уравнение дробной диффузии с оператором дискретно распределенного дифференцирования // Сибирские электронные математические известия. 2016. Т. 13. С. 1078–1098. https://doi.org/10.17377/semi.2016.13.086
4. Luchko Yu. Initial-boundary-value problems for the generalized multi-term time-fractional diffusion equation // Journal of Mathematical Analysis and Applications. 2011. Vol. 374. Issue 2. P. 538–548. https://doi.org/10.1016/j.jmaa.2010.08.048
5. Daftardar–Gejji V., Bhalekar S. Boundary value problems for multi-term fractional differential equations // Journal of Mathematical Analysis and Applications. 2008. Vol. 345. Issue 2. P. 754–765. https://doi.org/10.1016/j.jmaa.2008.04.065
6. Jiang H., Liu Fawang, Turner I. W., Burrage K. Analytical solutions for the multi-term time-fractional diffusion-wave/diffusion equations in a finite domain // Computers and Mathematics with Applications. 2012. Vol. 64. Issue 10. P. 3377–3388. https://doi.org/10.1016/j.camwa.2012.02.042
7. Liu Xiao-jing, Wang Ji-zeng, Wang Xiao-min, Zhou You-he. Exact solutions of multi-term fractional diffusion-wave equations with Robin type boundary conditions // Applied Mathematics and Mechanics. 2014. Vol. 35. Issue 1. P. 49–62. https://doi.org/10.1007/s10483-014-1771-6
8. Schumer R., Benson D. A., Meerschaert M. M., Baeumer B. Fractal mobile/immobile solute transport // Water Resources Research. 2003. Vol. 39. Issue 10. https://doi.org/10.1029/2003WR002141
9. Durdiev D. K., Shishkina E. L., Sitnik S. M. The explicit formula for solution of anomalous diffusion equation in the multi-dimensional space // Lobachevskii Journal of Mathematics. 2021. Vol. 42. Issue 6. P. 1264–1273. https://doi.org/10.1134/S199508022106007X
10. Sultanov M. A., Durdiev D. K., Rahmonov A. A. Construction of an explicit solution of a time-fractional multidimensional differential equation // Mathematics. 2021. Vol. 9. Issue 17. 2052. https://doi.org/10.3390/math9172052
11. Хасанов И. И., Акрамова Д. И., Рахмонов А. А. Исследование задачи Коши для одного уравнения дробного порядка с оператором Римана–Лиувилля // Вестник Самарского государственного технического университета. Серия «Физико-математические науки». 2023. Т. 27. № 1. С. 64–80. https://doi.org/10.14498/vsgtu1952

12. Ионкин Н. И. Решение одной краевой задачи теории теплопроводности с неклассическим краевым условием // Дифференциальные уравнения. 1977. Т. 13. № 2. С. 294–304. https://www.mathnet.ru/rus/de2993
13. Ivanchov M. I., Pabyrivs'ka N. V. Simultaneous determination of two coefficients of a parabolic equation in the case of nonlocal and integral conditions // Ukrainian Mathematical Journal. 2001. Vol. 53. No. 5. P. 674–684. https://doi.org/10.1023/A:1012570031242
14. Kochubei A., Luchko Yu. Handbook of fractional calculus with applications. Vol. 2. Fractional differential equations. De Gruyter, 2019. https://doi.org/10.1515/9783110571660
15. Durdiev D. K., Turdiev H. H. Inverse coefficient problem for a time-fractional wave equation with initial-boundary and integral type overdetermination conditions // Mathematical Methods in the Applied Sciences. 2024. Vol. 47. Issue 6. P. 5329–5340. https://doi.org/10.1002/mma.9867
16. Турдиев Х. Х. Обратные коэффициентные задачи для временно-дробного волнового уравнения с обобщенной производной Римана–Лиувилля по времени // Известия высших учебных заведений. Математика. 2023. № 10. С. 46–59. https://doi.org/10.26907/0021-3446-2023-10-46-59
17. Durdiev D. K. Inverse coefficient problem for the time-fractional diffusion equation // Eurasian Journal of Mathematical and Computer Applications. 2021. Vol. 9. Issue 1. P. 44–54. https://doi.org/10.32523/2306-6172-2021-9-1-44-54
18. Дурдиев У. Д. Задача об определении коэффициента реакции в дробном уравнении диффузии // Дифференциальные уравнения. 2021. Т. 57. № 9. С. 1220–1229. https://doi.org/10.31857/S0374064121090089
19. Durdiev D. K. Convolution kernel determination problem for the time-fractional diffusion equation // Physica D: Nonlinear Phenomena. 2024. Vol. 457. 133959. https://doi.org/10.1016/j.physd.2023.133959
20. Durdiev D. K., Rahmonov A. A., Bozorov Z. R. A two-dimensional diffusion coefficient determination problem for the time-fractional equation // Mathematical Methods in the Applied Sciences. 2021. Vol. 44. Issue 13. P. 10753–10761. https://doi.org/10.1002/mma.7442
21. Ma Wenjun, Sun Liangliang. Inverse potential problem for a semilinear generalized fractional diffusion equation with spatio-temporal dependent coefficients // Inverse Problems. 2023. Vol. 39. No. 1. 015005. https://doi.org/10.1088/1361-6420/aca49e
22. Abramowitz M., Stegun I. A. Handbook of mathematical functions with formulas, graphics and mathematical tables. New York: Dover, 1972.
23. Mathai A. M., Saxena R. K., Haubold H. J. The H-function. Theory and application. New York: Springer, 2010. https://doi.org/10.1007/978-1-4419-0916-9
24. Gorenflo R., Kilbas A. A., Mainardi F., Rogosin S. V. Mittag–Leffler functions, related topics and applications. Berlin–Heidelberg: Springer, 2014. https://doi.org/10.1007/978-3-662-43930-2

Поступила в редакцию 03.04.2024

Принята к публикации 21.08.2024

Дурдиев Дурдимурод Каландарович, д. ф.-м. н., профессор, заведующий, Бухарское отделением Института математики им. В. И. Романовского АН Республики Узбекистан, 200118, Узбекистан, г. Бухара, ул. Мухаммад Икбол, 11.
ORCID: https://orcid.org/0000-0002-6054-2827
E-mail: d.durdiev@mathinst.ru

Хасанов Иброхим Ихтиёрович, преподаватель, Бухарский государственный университет, 200118, Узбекистан, г. Бухара, ул. Мухаммад Икбол, 11.
ORCID: https://orcid.org/0009-0005-6294-5768
E-mail: i.i.hasanov@buxdu.uz, ihasanov998@gamail.com

Цитирование: Д. К. Дурдиев, И. И. Хасанов. Обратная коэффициентная задача для уравнения в частных производных со многими производными дробных порядков Римана–Лиувилля // Вестник Удмуртского университета. Математика. Механика. Компьютерные науки. 2024. Т. 34. Вып. 3. С. 321–338.