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INVERSE PROBLEMS FOR THE BEAM VIBRATION EQUATION WITH INVOLUTION

This article considers inverse problems for a fourth-order hyperbolic equation with involution. The existence and uniqueness of a solution of the studied inverse problems is established by the method of separation of variables. To apply the method of separation of variables, we prove the Riesz basis property of the eigenfunctions for a fourth-order differential operator with involution in the space $L_2(-1,1)$. For proving theorems on the existence and uniqueness of a solution, we widely use the Bessel inequality for the coefficients of expansions into a Fourier series in the space $L_2(-1,1)$. A significant dependence of the existence of a solution on the equation coefficient α is shown. In each of the cases $\alpha < -1$, $\alpha > 1$, $-1 < \alpha < 1$ representations of solutions in the form of Fourier series in terms of eigenfunctions of boundary value problems for a fourth-order equation with involution are written out.

Keywords: differential equations with involution, inverse problem, eigenvalue, eigenfunction, Fourier method.

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Introduction

The work is devoted to the study of inverse problems for a differential equation of the type

$$u_{tt}(x,t) + u_{xxxx}(x,t) + \alpha \cdot u_{xxxx}(-x,t) = f(x), \quad (x,t) \in E, \tag{0.1}$$

in the rectangular area $E = \{-1 < x < 1,\ 0 < t < T\}$, where the real number $\alpha \neq \pm 1$. Equation (0.1) is nonlocal, since there is a dependence on the derivatives at the points x and $-x$.

Transformation S of the function $q(x)$, $-1 \le x \le 1$, is called an involution if it satisfies the condition $S^2 q(x) = q(x)$.

For $\alpha = 0$, Equation (0.1) is a well-known classical equation that describes the vibration of a beam (see, for example [1]). Among the papers devoted to inverse problems for equations of this type, we can note the recent paper [2, 3] (see also references therein). The papers [4, 5] are devoted to inverse problems for second-order equations. Differential equations with involution are studied in many works (see [6–8]). Recently, many papers have appeared (see [9–26]) devoted to the solvability of direct and inverse problems for partial differential equations with involution. However, inverse problems for differential equations of the fourth order with involution were not considered. The purpose of this work is to solve the following two problems.

IPD: Find a pair of functions $u(x,t)$ and $f(x)$ in the domain E satisfying Equation (0.1), initial conditions

$$u(x,0) = \varphi(x), \quad u(x,T) = \psi(x), \quad u_t(x,0) = 0, \quad x \in [-1,1], \tag{0.2}$$

and Dirichlet boundary conditions

$$u(-1,t) = 0, \quad u(1,t) = 0, \quad u_{xx}(-1,t) = 0, \quad u_{xx}(1,t) = 0, \quad t \in [0,T], \tag{0.3}$$

where $\varphi(x)$ and $\psi(x)$ are given sufficiently smooth functions.

IPN: Find a pair of functions $u(x,t)$ and $f(x)$ in the domain E satisfying Equation (0.1), initial conditions (0.2) and Neumann boundary conditions

$$u_x(-1,t) = 0, \quad u_x(1,t) = 0, \quad u_{xxx}(-1,t) = 0, \quad u_{xxx}(1,t) = 0, \quad t \in [0,T]. \tag{0.4}$$

We call the solution of problem (0.1), (0.2), (0.3) the pair of functions $u(x,t)$ and $f(x)$ satisfying Equation (0.1), conditions (0.2) and (0.3) such that $u(x,t) \in C^{2,1}_{x,t}(\overline{E}) \cap C^{4,2}_{x,t}(E)$, $f(x) \in C[-1,1]$.

We call the solution of problem (0.1), (0.2), (0.4) the pair of functions $u(x,t)$ and $f(x)$ satisfying Equation (0.1), conditions (0.2) and (0.4) such that $u(x,t) \in C^{3,1}_{x,t}(\overline{E}) \cap C^{4,2}_{x,t}(E)$, $f(x) \in C[-1,1]$.

§ 1. Spectral problems for a fourth-order equation with involution

Let us consider a homogeneous equation corresponding to Equation (0.1). Using the method of separation of variables, we obtain the following, previously unstudied, spectral problems:

$$X^{IV}(x) + \alpha \cdot X^{IV}(-x) = \lambda X(x), \tag{1.1}$$

$$X(-1) = 0, \quad X(1) = 0, \quad X''(-1) = 0, \quad X''(1) = 0, \tag{1.2}$$

$$X'(-1) = 0, \quad X'(1) = 0, \quad X'''(-1) = 0, \quad X'''(1) = 0. \tag{1.3}$$

Boundary conditions (1.2), (1.3) follow from conditions (0.3), (0.4), respectively. Denote $\alpha_0 = \sqrt[4]{\frac{1}{1+\alpha}}$, $\alpha_1 = \sqrt[4]{\frac{1}{1-\alpha}}$, $\rho = \sqrt[4]{\lambda}$. The function

$$X(x) = c_1\big(e^{\alpha_0 \rho x} + e^{-\alpha_0 \rho x}\big) + c_2\big(e^{i\alpha_0 \rho x} + e^{-i\alpha_0 \rho x}\big) + c_3\big(e^{\alpha_1 \rho x} - e^{-\alpha_1 \rho x}\big) + c_4\big(e^{i\alpha_1 \rho x} - e^{-i\alpha_1 \rho x}\big)$$

is a solution to Equation (1.1). Now it is not difficult to calculate the eigenvalues of the spectral problems. Eigenvalues

$$\lambda_{k1} = (1+\alpha)(\pi k - \pi/2)^4, \quad \lambda_{k2} = (1-\alpha)(\pi k)^4, \quad k \in N, \tag{1.4}$$

of spectral problem (1.1), (1.2) correspond to the eigenfunctions

$$X_{k1} = \cos(k - 1/2)\pi x, \quad X_{k2} = \sin(\pi k x), \quad k \in N. \tag{1.5}$$

Eigenvalues

$$\lambda_{k1} = (1+\alpha)(\pi k)^4, \quad \lambda_{k2} = (1-\alpha)(\pi k - \pi/2)^4, \quad k \in N, \tag{1.6}$$

of spectral problem (1.1), (1.3) correspond to the eigenfunctions

$$X_0 = 1, \quad X_{k1} = \cos(\pi k x), \quad X_{k2} = \sin(k - 1/2)\pi x, \quad k \in N. \tag{1.7}$$

Lemma 1.1. *Each of the systems of functions* (1.5) *and* (1.7) *is complete and orthonormal in the space* $L_2(-1,1)$.

P r o o f. The orthonormality of the systems is checked by direct calculations. Therefore, we will only prove the completeness of systems (1.5) and (1.7). Consider system (1.5). Let for any function $g(x)$ from the class $L_2(-1,1)$ the following conditions be satisfied:

$$\int_{-1}^{1} g(x) \cos(k - 1/2)\pi x \, dx = 0, \quad k \in N, \tag{1.8}$$

and

$$\int_{-1}^{1} g(x)\sin \pi kx\, dx = 0, \quad k \in N. \tag{1.9}$$

Let us show that $g(x) \equiv 0$ in $(-1, 1)$. To do this, we transform equality (1.8)

$$0 = \int_{-1}^{1} g(x)\cos(k - 1/2)\pi x\, dx = \int_{-1}^{0} g(x)\cos(k - 1/2)\pi x\, dx +$$
$$+ \int_{0}^{1} g(x)\cos(k - 1/2)\pi x\, dx = \int_{0}^{1} \big(g(x) + g(-x)\big)\cos(k - 1/2)\pi x\, dx.$$

Since the system $\{\cos(k-1/2)\pi x\}_{k\in N}$ is complete [27] in $L_2(0, 1)$, it follows that the function $g(x)$, $-1 < x < 1$, is odd, i. e., $g(-x) = -g(x)$. Further, we similarly transform equality (1.9):

$$0 = \int_{-1}^{1} g(x)\sin \pi kx\, dx = \int_{-1}^{0} g(x)\sin \pi kx\, dx + \int_{0}^{1} g(x)\sin \pi kx\, dx =$$
$$= \int_{0}^{1} \big(g(x) - g(-x)\big)\sin \pi kx\, dx.$$

From this equality and the completeness of the trigonometric system $\{\sin \pi kx\}_{k\in N}$ in the space $L_2(0, 1)$, we get the equality $g(-x) = g(x)$, $-1 < x < 1$. Therefore, $g(x) \equiv 0$ in $(-1, 1)$. The completeness of system (1.5) is proved. The completeness of the system of functions (1.7) is proved similarly. The Lemma is proved. □

It should be noted that spectral problems for differential equations with involution have been studied in recent decades. Regarding the papers devoted to spectral problems for differential equations of the first and second orders, we refer readers to [28] (see also references therein). For results concerning nonclassical spectral problems, we refer readers to [29, 30].

Remark 1.1. The explicit form of eigenvalues (1.4), (1.6) shows that for $\alpha > 1$ or $\alpha < -1$ direct problems for equation (0.1) may be incorrect.

§ 2. Formal solution to the problem

The unknown functions $u(x, t)$ and $f(x)$ can be represented as a Fourier series in terms of spectral eigenfunctions of problem (1.1), (1.2). The function $u(x, t)$ as a function of a variable x is represented as

$$u(x, t) = \sum_{k=1}^{\infty} \big[u_{k1}(t)\cos(k - 1/2)\pi x + u_{k2}(t)\sin \pi kx\big] \tag{2.1}$$

and the function $f(x)$ is represented as

$$f(x) = \sum_{k=1}^{\infty} \big[f_{k1}\cos(k - 1/2)\pi x + f_{k2}\sin \pi kx\big], \tag{2.2}$$

where the coefficients $u_{k1}(t)$, $u_{k2}(t)$, f_{k1}, f_{k2} are unknown. Substituting (2.1) and (2.2) into Equation (0.1), we obtain the following equations connecting the functions $u_{k1}(t)$, $u_{k2}(t)$ and constants f_{1k}, f_{2k}:

$$u''_{k1}(t) + (1 + \alpha)(\pi k - \pi/2)^4 u_{k1}(t) = f_{k1}, \tag{2.3}$$
$$u''_{k2}(t) + (1 - \alpha)(\pi k)^4 u_{k2}(t) = f_{k2}. \tag{2.4}$$

Equations (2.3), (2.4) are inhomogeneous second-order linear differential equations with constant coefficients. Let us write out their general solutions

$$u_{k1}(t) = C_{k1}\cos\sqrt{1+\alpha}(\pi k-\pi/2)^2 t + C_{k2}\sin\sqrt{1+\alpha}(\pi k-\pi/2)^2 t + \frac{f_{k1}}{(1+\alpha)(\pi k-\pi/2)^4},$$
$$u_{k2}(t) = D_{k1}\cos\sqrt{1-\alpha}(\pi k)^2 t + D_{k2}\sin\sqrt{1-\alpha}(\pi k)^2 t + \frac{f_{k2}}{(1-\alpha)(\pi k)^4},$$

where unknown constants C_{k1}, C_{k2}, D_{k1}, D_{k2}, f_{k1}, f_{k2} can be determined using the initial conditions (0.2). Let φ_{ki}, ψ_{ki}, $i=1,2$, be the coefficients of the expansion into the Fourier series of the functions $\varphi(x)$ and $\psi(x)$, respectively, i. e.

$$\varphi_{k1} = \int_{-1}^{1}\varphi(x)\cos(k-1/2)\pi x\,dx, \quad \varphi_{k2} = \int_{-1}^{1}\varphi(x)\sin\pi kx\,dx, \tag{2.5}$$
$$\psi_{k1} = \int_{-1}^{1}\psi(x)\cos(k-1/2)\pi x\,dx, \quad \psi_{k2} = \int_{-1}^{1}\psi(x)\sin\pi kx\,dx. \tag{2.6}$$

Now conditions (0.2) imply the equalities

$$\begin{cases} C_{k1} + \dfrac{f_{k1}}{(1+\alpha)(\pi k-\pi/2)^4} = \varphi_{k1}, \\ C_{k1}\cos\sqrt{1+\alpha}(\pi k-\pi/2)^2 T + \dfrac{f_{k1}}{(1+\alpha)(\pi k-\pi/2)^4} = \psi_{k1}, \\ C_{k2} = 0, \end{cases}$$
$$\begin{cases} D_{k1} + \dfrac{f_{k2}}{(1-\alpha)(\pi k)^4} = \varphi_{k2}, \\ D_{k1}\cos\sqrt{1-\alpha}(\pi k)^2 T + \dfrac{f_{k2}}{(1-\alpha)(\pi k)^4} = \psi_{k2}, \\ D_{k2} = 0. \end{cases}$$

Solving these systems of algebraic equations, we obtain

$$C_{k1} = \frac{\varphi_{k1}-\psi_{k1}}{1-\cos\sqrt{1+\alpha}(\pi k-\pi/2)^2 T}, \quad f_{k1} = (\varphi_{k1}-C_{k1})(1+\alpha)(\pi k-\pi/2)^4,$$
$$D_{k1} = \frac{\varphi_{k2}-\psi_{k2}}{1-\cos\sqrt{1-\alpha}(\pi k)^2 T}, \quad f_{k2} = (\varphi_{k2}-D_{k1})(1-\alpha)(\pi k)^4.$$

Substituting the found values of $u_{k1}(t)$, $u_{k2}(t)$, f_{k1}, f_{k2} into (2.1) and (2.2), we get

$$\begin{aligned} u(x,t) = \varphi(x) &+ \sum_{k=1}^{\infty}\frac{1-\cos\sqrt{1+\alpha}(\pi k-\pi/2)^2 t}{1-\cos\sqrt{1+\alpha}(\pi k-\pi/2)^2 T}(\psi_{k1}-\varphi_{k1})\cos(k-1/2)\pi x + \\ &+ \sum_{k=1}^{\infty}\frac{1-\cos\sqrt{1-\alpha}(\pi k)^2 t}{1-\cos\sqrt{1-\alpha}(\pi k)^2 T}(\psi_{k2}-\varphi_{k2})\sin\pi kx \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} f(x) = \varphi^{IV}(x) &+ \alpha\cdot\varphi^{IV}(-x) + \\ &+ \sum_{k=1}^{\infty}\frac{\psi_{k1}-\varphi_{k1}}{1-\cos\sqrt{1+\alpha}(\pi k-\pi/2)^2 T}(1+\alpha)(\pi k-\pi/2)^4\cos(k-1/2)\pi x + \\ &+ \sum_{k=1}^{\infty}\frac{\psi_{k2}-\varphi_{k2}}{1-\cos\sqrt{1-\alpha}(\pi k)^2 T}(1-\alpha)(\pi k)^4\sin\pi kx. \end{aligned} \tag{2.8}$$

Thus, we have finally obtained a formal solution of problem (0.1)–(0.3) in the form of series (2.7), (2.8).

§ 3. Main results

Before formulating the main results, we note that the solvability conditions for problems (0.1)–(0.4) depend on the values of the parameter in the equations. Therefore, we specify three cases: $\alpha < -1$, $-1 < \alpha < 1$, $\alpha > 1$. Let us formulate the main results of this work. The solvability of inverse problems (0.1)–(0.3) can be presented in the form of the following theorems.

Theorem 3.1. *Let* $|\alpha| < 1$ *and the following conditions be satisfied:*

(1) $\varphi(x),\ \psi(x) \in C^5[-1,1]$;

(2) *there exist positive numbers* δ_0, δ_1 *such that* $\cos\sqrt{1+\alpha}(\pi k - \pi/2)^2 T \le \delta_0 < 1$, $\cos\sqrt{1-\alpha}(\pi k)^2 T \le \delta_1 < 1$;

(3) *the derivatives of the functions* $\varphi(x)$ *and* $\psi(x)$ *have the properties* $\frac{d^j\varphi(\mp 1)}{dx^j} = 0$, $\frac{d^j\psi(\mp 1)}{dx^j} = 0$, $j = 0, 2, 4$.

Then there is a unique solution to the inverse problem (0.1)–(0.3), *which can be represented as a Fourier series* (2.7), (2.8).

Theorem 3.2. *Let* $\alpha > 1$ *and the following conditions be satisfied:*

(1) $\varphi(x),\ \psi(x) \in C^5[-1,1]$;

(2) *there exists a positive number* δ_0 *such that* $\cos\sqrt{1+\alpha}(\pi k - \pi/2)^2 T \le \delta_0 < 1$;

(3) *the derivatives of the functions* $\varphi(x)$ *and* $\psi(x)$ *have the properties* $\frac{d^j\varphi(\mp 1)}{dx^j} = 0$, $\frac{d^j\psi(\mp 1)}{dx^j} = 0$, $j = 0, 2, 4$.

Then there is a unique solution to the inverse problem (0.1)–(0.3), *which can be represented as a Fourier series* (2.7), (2.8).

Theorem 3.3. *Let* $\alpha < -1$ *and the following conditions be satisfied:*

(1) $\varphi(x),\ \psi(x) \in C^5[-1,1]$;

(2) *there exists a positive number* δ_1 *such that* $\cos\sqrt{1-\alpha}(\pi k)^2 T \le \delta_1 < 1$;

(3) *the derivatives of the functions* $\varphi(x)$ *and* $\psi(x)$ *have the properties* $\frac{d^j\varphi(\mp 1)}{dx^j} = 0$, $\frac{d^j\psi(\mp 1)}{dx^j} = 0$, $j = 0, 2, 4$.

Then there is a unique solution to the inverse problem (0.1)–(0.3), *which can be represented as a Fourier series* (2.7), (2.8).

Let us consider formulation of the theorems on the solvability of inverse problems (0.1), (0.2), (0.4). Proceeding as above, we first write out the formal solution of our inverse problem:

$$
\begin{aligned}
u(x,t) = \varphi(x) + \frac{t^2}{T^2}(\psi_0 - \varphi_0) + \sum_{k=1}^{\infty} \frac{1-\cos\sqrt{1+\alpha}(\pi k)^2 t}{1-\cos\sqrt{1+\alpha}(\pi k)^2 T}(\psi_{k1} - \varphi_{k1})\cos \pi k x + \\
+ \sum_{k=1}^{\infty} \frac{1-\cos\sqrt{1-\alpha}(\pi k - \pi/2)^2 t}{1-\cos\sqrt{1-\alpha}(\pi k - \pi/2)^2 T}(\psi_{k2} - \varphi_{k2})\sin(k - 1/2)\pi x
\end{aligned} \tag{3.1}
$$

and

$$
\begin{aligned}
f(x) = \varphi^{IV}(x) + \alpha\cdot\varphi^{IV}(-x) + \frac{1}{T}(\psi_0 - \varphi_0) + \\
+ \sum_{k=1}^{\infty} \frac{\psi_{k1} - \varphi_{k1}}{1-\cos\sqrt{1+\alpha}(\pi k)^2 T}(1+\alpha)(\pi k)^4 \cos \pi k x + \\
+ \sum_{k=1}^{\infty} \frac{\psi_{k2} - \varphi_{k2}}{1-\cos\sqrt{1-\alpha}(\pi k - \pi/2)^2 T}(1-\alpha)(\pi k - \pi/2)^4 \sin(k - 1/2)\pi x,
\end{aligned} \tag{3.2}
$$

where

$$\varphi_0 = \frac{1}{2}\int_{-1}^{1} \varphi(x)\,dx, \quad \psi_0 = \frac{1}{2}\int_{-1}^{1} \psi(x)\,dx,$$
$$\varphi_{k1} = \int_{-1}^{1} \varphi(x)\cos\pi kx\,dx, \quad \varphi_{k2} = \int_{-1}^{1} \varphi(x)\sin(k-1/2)\pi x\,dx,$$
$$\psi_{k1} = \int_{-1}^{1} \psi(x)\cos\pi kx\,dx, \quad \psi_{k2} = \int_{-1}^{1} \psi(x)\sin(k-1/2)\pi x\,dx.$$

The following theorems give conditions for the solvability of inverse problems (0.1), (0.2), (0.4).

Theorem 3.4. *Let* $|\alpha| < 1$ *and the following conditions be satisfied:*
(1) $\varphi(x)$, $\psi(x) \in C^5[-1,1]$;
(2) *there exist positive numbers* δ_0, δ_1 *such that*

$$\cos\sqrt{1+\alpha}(\pi k)^2 T \le \delta_0 < 1, \qquad \cos\sqrt{1-\alpha}(\pi k - \pi/2)^2 T \le \delta_1 < 1;$$

(3) *the derivatives of the functions* $\varphi(x)$ *and* $\psi(x)$ *have the properties* $\frac{d^j\varphi(\mp 1)}{dx^j} = 0$, $\frac{d^j\psi(\mp 1)}{dx^j} = 0$, $j = 1, 3$.

Then there is a unique solution to the inverse problem (0.1)–(0.2) *and* (0.4)*, which can be represented as a Fourier series* (3.1)*,* (3.2)*.*

Theorem 3.5. *Let* $\alpha > 1$ *and the following conditions be satisfied:*
(1) $\varphi(x)$, $\psi(x) \in C^5[-1,1]$;
(2) *there exists a positive number* δ_0 *such that* $\cos\sqrt{1+\alpha}(\pi k)^2 T \le \delta_0 < 1$;
(3) *the derivatives of the functions* $\varphi(x)$ *and* $\psi(x)$ *have the properties* $\frac{d^j\varphi(\mp 1)}{dx^j} = 0$, $\frac{d^j\psi(\mp 1)}{dx^j} = 0$, $j = 1, 3$.

Then there is a unique solution to the inverse problem (0.1)–(0.2) *and* (0.4)*, which can be represented as a Fourier series* (3.1)*,* (3.2)*.*

Theorem 3.6. *Let* $\alpha < -1$ *and the following conditions be satisfied:*
(1) $\varphi(x)$, $\psi(x) \in C^5[-1,1]$;
(2) *there exists a positive number* δ_1 *such that* $\cos\sqrt{1-\alpha}(\pi k - \pi/2)^2 T \le \delta_1 < 1$;
(3) *the derivatives of the functions* $\varphi(x)$ *and* $\psi(x)$ *have the properties* $\frac{d^j\varphi(\mp 1)}{dx^j} = 0$, $\frac{d^j\psi(\mp 1)}{dx^j} = 0$, $j = 1, 3$.

Then there is a unique solution to the inverse problem (0.1)–(0.2) *and* (0.4)*, which can be represented as a Fourier series* (3.1)*,* (3.2)*.*

§ 4. Proof of the main results

Let us prove Theorem 3.1. We must prove that the resulting formal solution in the form of series (2.7), (2.8) satisfies equation (0.1) and conditions (0.2), (0.3). Let us first show that the series (2.7), (2.8), as well as the formal derivatives with respect to the variables and of the series (2.7) up to the fourth and second orders, respectively, converge uniformly in E. Let $-1 < \alpha < 1$. Then from (2.7) it follows the estimate

$$|u(x,t)| \le |\varphi(x)| + M_1 \sum_{k=1}^{\infty} \big(|\psi_{k1} - \varphi_{k1}| + |\psi_{k2} - \varphi_{k2}|\big), \tag{4.1}$$

where M_1 is a positive number.

The smoothness of the functions $\varphi(x)$, $\psi(x)$ makes it possible to carry out double integration by parts in expressions (2.5) and (2.6). In this case, by virtue of condition (0.3) of Theorem 3.1, the integrated terms vanish and we obtain the equalities

$$\varphi_{k1} = -\frac{\varphi_{k1}^{(2)}}{(k-1/2)^2\pi^2}, \quad \varphi_{k2} = -\frac{\varphi_{k2}^{(2)}}{(\pi k)^2}, \quad \psi_{k1} = -\frac{\psi_{k1}^{(2)}}{(k-1/2)^2\pi^2}, \quad \psi_{k2} = -\frac{\psi_{k2}^{(2)}}{(\pi k)^2}, \quad k \in N,$$

where $\varphi_{kj}^{(2)}$, $\psi_{kj}^{(2)}$, $j = 1, 2$, are the Fourier coefficients of bounded functions $\varphi''(x)$, $\psi''(x)$, respectively. Using the obtained equalities in (4.1), we obtain the estimate

$$|u(x,t)| \leq |\varphi(x)| + M_2 \sum_{k=1}^{\infty} \left(\frac{2\left|\psi_{k1}^{(2)} - \varphi_{k1}^{(2)}\right|}{(2k-1)^2\pi^2} + \frac{\left|\psi_{k2}^{(2)} - \varphi_{k2}^{(2)}\right|}{(\pi k)^2} \right),$$

from which the uniform convergence of the series (2.5) in E follows. Similarly, the uniform convergence of the series $u_t(x,t)$, $u_x(x,t)$ is proved by triple integration by parts in expressions (2.5) and (2.6). The uniform convergences of the series $u_{tt}(x,t)$, $u_{xx}(x,t)$, $u_{xxx}(x,t)$ are proved in the same way. Let us prove the uniform convergence of the series $u_{xxxx}(x,t)$. The function $u_{xxxx}(x,t)$ has the estimate

$$|u_{xxxx}(x,t)| \leq |\varphi^{IV}(x)| + M_1 \sum_{k=1}^{\infty} \big(|\psi_{k1} - \varphi_{k1}| + |\psi_{k2} - \varphi_{k2}| \big) (k-1/2)^4 \pi^4. \tag{4.2}$$

After fivefold integration by parts in expressions (2.5) and (2.6), we obtain the equalities

$$\varphi_{k1} = -\frac{\varphi_{k1}^{(5)}}{(k-1/2)^5\pi^5}, \quad \varphi_{k2} = \frac{\varphi_{k2}^{(5)}}{(\pi k)^5}, \quad \psi_{k1} = -\frac{\psi_{k1}^{(5)}}{(k-1/2)^5\pi^5}, \quad \psi_{k2} = \frac{\psi_{k2}^{(5)}}{(\pi k)^5}, \quad k \in N,$$

where $\varphi_{kj}^{(5)}$, $\psi_{kj}^{(5)}$, $j = 1, 2$, are the Fourier coefficients of bounded functions $\varphi^V(x)$, $\psi^V(x)$, respectively. Applying these equalities to (4.2), we obtain

$$|u_{xxxx}(x,t)| \leq |\varphi^{IV}(x)| + \frac{M_2}{\pi} \sum_{k=1}^{\infty} \left(\frac{2\varphi_{k1}^{(5)}}{2k-1} + \frac{2\psi_{k1}^{(5)}}{2k-1} + \frac{\varphi_{k2}^{(5)}}{k} + \frac{\psi_{k2}^{(5)}}{k} \right).$$

We apply the inequality $|ab| \leq \frac{a^2}{2} + \frac{b^2}{2}$ to each term of the general term of the resulting series. Then

$$|u_{xxxx}(x,t)| \leq |\varphi^{IV}(x)| + \frac{M_2}{\pi} \sum_{k=1}^{\infty} \left(\frac{\left|\varphi_{k1}^{(5)}\right|^2}{2} + \frac{\left|\psi_{k1}^{(5)}\right|^2}{2} + \frac{\left|\varphi_{k2}^{(5)}\right|^2}{2} + \frac{\left|\psi_{k2}^{(5)}\right|^2}{2} + \frac{8}{(2k-1)^2} + \frac{1}{k^2} \right).$$

Due to the Bessel inequality for the Fourier coefficients, this implies the uniform convergence of the series $u_{xxxx}(x,t)$ in the domain E. The uniform convergence of series (2.8) is proved in a similar way. Boundary conditions (0.3) for series (2.7) are satisfied, since each term of the series satisfies this condition. The fulfillment of conditions (0.2) is also obvious.

§ 5. Let us prove the uniqueness of the solution

Suppose that there are two sets of solutions $\{u_1(x,t),\ f_1(x)\}$ and $\{u_2(x,t),\ f_2(x)\}$ of the inverse problem (0.1)–(0.3). Denote

$$u(x,t) = u_1(x,t) - u_2(x,t)$$

and

$$f(x) = f_1(x) - f_2(x).$$

Then the functions $u(x,t)$ and $f(x)$ satisfy Equation (0.1), boundary conditions (0.2), and homogeneous conditions

$$u(x,0) = 0, \quad u(x,T) = 0, \quad u_t(x,0) = 0, \quad x \in [-1,1]. \tag{5.1}$$

Consider the following expressions:

$$u_{k1}(t) = \int_{-1}^{1} u(x,t)\cos(k-1/2)\pi x\,dx, \quad k \in N, \tag{5.2}$$

$$u_{k2}(t) = \int_{-1}^{1} u(x,t)\sin \pi kx\,dx, \quad k \in N, \tag{5.3}$$

$$f_{k1} = \int_{-1}^{1} f(x)\cos(k-1/2)\pi x\,dx, \quad k \in N, \tag{5.4}$$

$$f_{k2} = \int_{-1}^{1} f(x)\sin \pi kx\,dx, \quad k \in N. \tag{5.5}$$

The homogeneous conditions (5.1) and expressions (5.2), (5.3) imply the equalities

$$u_{ki}(x,0) = u_{ki}(x,T) = 0, \quad \left.\frac{\partial u_{ki}(x,t)}{\partial t}\right|_{t=0} = 0, \quad i = 1,2.$$

Let us differentiate expression (5.2) twice with respect to the variable and use Equation (0.1) on the right side. Then, we get the equality

$$u''_{k1}(t) = \int_{-1}^{1} \big(-u_{xxxx}(x,t) - \alpha u_{xxxx}(-x,t)\big)\cos(k-1/2)\pi x\,dx + f_{k1},$$

which we integrate by parts four times. As a result, we get

$$u''_{k1}(t) + (k-1/2)^4\pi^4(1+\alpha)u_{k1}(t) = f_{k1}.$$

Now it is easy to see that, from this equation and conditions $u_{k1}(0) = u_{k1}(T) = 0$, $u_t(0) = 0$, we get the relations

$$f_{k1} = 0, \quad u_{k1}(t) \equiv 0.$$

Similarly, we obtain the equalities $f_{k2} = 0$, $u_{k2}(t) \equiv 0$. From here, due to the completeness of the system of eigenfunctions (1.7) and relations (5.4), (5.5), we obtain

$$f(x) \equiv 0, \quad u(x,t) \equiv 0, \quad (x,t) \in E.$$

Theorem 3.1 is proved.

§ 6. Proof of Theorem 3.2

If $\alpha > 1$, then series (2.7) and (2.8) can be written as follows:

$$\begin{aligned} u(x,t) = \varphi(x) + \sum_{k=1}^{\infty} \frac{1-\cos\sqrt{1+\alpha}(\pi k - \pi/2)^2 t}{1-\cos\sqrt{1+\alpha}(\pi k-\pi/2)^2 T}(\psi_{k1}-\varphi_{k1})\cos(k-1/2)\pi x + \\ + \sum_{k=1}^{\infty} \frac{1-\cos i\sqrt{\alpha-1}(\pi k)^2 t}{1-\cos i\sqrt{\alpha-1}(\pi k)^2 T}(\psi_{k2}-\varphi_{k2})\sin \pi kx \end{aligned} \tag{6.1}$$

and

$$f(x) = \varphi^{IV}(x) + \alpha \cdot \varphi^{IV}(-x) + \sum_{k=1}^{\infty} \frac{\psi_{k1} - \varphi_{k1}}{1 - \cos\sqrt{1+\alpha}(\pi k - \pi/2)^2 T} \times$$
$$\times (1+\alpha)(\pi k - \pi/2)^4 \cos(k - 1/2)\pi x + \sum_{k=1}^{\infty} \frac{\psi_{k2} - \varphi_{k2}}{1 - \cos i\sqrt{\alpha - 1}(\pi k)^2 T}(1-\alpha)(\pi k)^4 \sin \pi k x. \tag{6.2}$$

As the function

$$\cos i\sqrt{\alpha - 1}(\pi k)^2 T = \frac{\exp(\sqrt{\alpha - 1}(\pi k)^2 T) + \exp(-\sqrt{\alpha - 1}(\pi k)^2 T)}{2}$$

has an exponential growth, it is easy to see that the second series in (6.2) will converge uniformly. The uniform convergence of the first series in (6.1) and (6.2) is proved by the above method. The uniform convergence of the second series in (6.1) is ensured by the presence of the term in the series containing the expression $\exp(\sqrt{\alpha - 1}(\pi k)^2(t - T))$. Theorem 3.2 is proved.

The proof of Theorem 3.3 is similar to the proof of Theorem 3.2.

We will not consider the proof of Theorems 3.4–3.6, since the main ideas of the proof of Theorems 3.1–3.3 can be easily transferred to the case of problem (0.1), (0.2), (0.4).

§ 7. Example

For inverse Dirichlet problems, we present a simple example of solution. Consider the following choice of conditions (0.2):

$$u(x,0) = \cos(\pi/2x), \quad u(x,T) = 0, \quad u_t(x,0 = 0, \quad x \in [-1,1],$$

i. e., here $\varphi(x) = \cos(\pi/2x)$ and $\psi(x) = 0$. Let us calculate the coefficients of the series using the property of Theorem 3.1, then

$$u(x,t) = \cos(\pi/2x) - \frac{1 - \cos\sqrt{1+\alpha}(\pi^2/4)t}{1 - \cos\sqrt{1+\alpha}(\pi^2/4)T}\cos(\pi/2x),$$
$$f(x) = (\pi/2)^4\cos(\pi/2x) + \alpha(\pi/2)^4\cos(\pi/2x) - \frac{\pi^4}{16} \cdot \frac{1+\alpha}{1 - \cos\sqrt{1+\alpha}(\pi^2/4)T}\cos(\pi/2x).$$

§ 8. Conclusion

The inverse problem of determining the right side of the perturbed beam vibration equation with involution is solved. The dependence of the solvability conditions of the problem on the values of the parameter contained in the studied equation is established. Dirichlet and Neumann boundary conditions are considered. The basic properties of eigenfunctions of spectral problems for the fourth-order differential equation with involution with the specified boundary conditions are investigated. The main ideas of the paper can be extended to the cases of periodic and antiperiodic boundary conditions.

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А. Б. Иманбетова, А. А. Сарсенби, Б. Н. Сейлбеков

Обратные задачи для уравнения колебания балки с инволюцией

Ключевые слова: дифференциальные уравнения с инволюцией, обратная задача, собственное значение, собственная функция, метод Фурье.

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В этой статье рассматриваются обратные задачи для уравнения гиперболического вида четвертого порядка с инволюцией. Существование и единственность решения изучаемых обратных задач устанавливается методом разделения переменных. Для применения метода разделения переменных доказываем базисность Рисса собственных функций дифференциального оператора четвертого порядка с инволюцией в пространстве $L_2(-1,1)$. При доказательстве теорем о существовании и единственности решения широко используем неравенство Бесселя для коэффициентов разложений в ряд Фурье в пространстве $L_2(-1,1)$. Показана существенная зависимость существования решения от коэффициента уравнения α. В каждом из случаев $\alpha < -1$, $\alpha > 1$, $-1 < \alpha < 1$ выписаны представления решений в виде рядов Фурье по собственным функциям краевых задач для уравнения четвертого порядка с инволюцией.

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