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INVERSE IMAGE OF PRECOMPACT SETS AND REGULAR SOLUTIONS TO THE NAVIER–STOKES EQUATIONS

We consider the initial value problem for the Navier–Stokes equations over $\mathbb{R}^3 \times [0,T]$ with time $T>0$ in the spatially periodic setting. We prove that it induces open injective mappings $\mathcal{A}_s \colon B^s_1 \to B^{s-1}_2$ where B^s_1, B^{s-1}_2 are elements from scales of specially constructed function spaces of Bochner–Sobolev type parametrized with the smoothness index $s \in \mathbb{N}$. Finally, we prove that a map $\mathcal{A}_s$ is surjective if and only if the inverse image $\mathcal{A}_s^{-1}(K)$ of any precompact set K from the range of the map $\mathcal{A}_s$ is bounded in the Bochner space $L^{\mathfrak{s}}([0,T], L^{\mathfrak{r}}(\mathbb{T}^3))$ with the Ladyzhenskaya–Prodi–Serrin numbers $\mathfrak{s}$, $\mathfrak{r}$.

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Introduction

The problem of describing the dynamics of incompressible viscous fluid is of great importance in applications. The dynamics is described by the Navier–Stokes equations and the problem consists in finding a sufficiently regular solution to the equations for which a uniqueness theorem is available. Essential contributions have been published in the research articles [11, 17], as well as surveys and books [15, 18, 19, 29], etc. The topic is actively discussed in various aspects: regularity of solutions, possible blow-up conditions, solvability criterion, etc., in various function spaces and situations, including bounded domains, the half space $\mathbb{R}^n \times \mathbb{R}_+$ and the periodic setting, see, for instance, [3–7, 9, 12, 13, 24, 31].

More precisely, let $\Delta = \partial^2_{x_1} + \partial^2_{x_2} + \partial^2_{x_3}$ be the Laplace operator, ∇ and div be the gradient operator and the divergence operator, respectively, in the Euclidean space $\mathbb{R}^3$. In the sequel we consider the following initial problem. Given any sufficiently regular vector-valued functions $f = (f^1, f^2, f^3)$ and $u_0 = (u_0^1, u_0^2, u_0^3)$ on $\mathbb{R}^3 \times [0,T]$ and $\mathbb{R}^3$, respectively, find a pair (u,p) of sufficiently regular functions $u = (u^1, u^2, u^3)$ and p on $\mathbb{R}^3 \times [0,T]$ satisfying

$$\begin{cases} \partial_t u - \mu \Delta u + (u \cdot \nabla) u + \nabla p = f, & (x,t) \in \mathbb{R}^3 \times (0,T), \\ \operatorname{div} u = 0, & (x,t) \in \mathbb{R}^3 \times (0,T), \\ u = u_0, & (x,t) \in \mathbb{R}^3 \times \{0\}, \end{cases} \tag{0.1}$$

with positive fixed numbers T and μ. We additionally assume that the data f and u_0 are spatially periodic with a period $\ell > 0$, i. e., for any $1 \le j \le 3$ we have

$$f(x + \ell e_j, t) = f(x,t), \qquad u_0(x + \ell e_j) = u_0(x)$$

whenever $x \in \mathbb{R}^3$ and $t \in [0,T]$, where e_j is as usual the j-th unit basis vector in $\mathbb{R}^3$. Then, the solution (u,p) is also looked for in the space of spatially periodic functions with period ℓ on $\mathbb{R}^3 \times [0,T]$. Relations (0.1) are usually referred to as the Navier–Stokes equations for an incompressible fluid with given dynamical viscosity μ of the fluid under consideration, density vector of outer forces f, initial velocity u_0 and the search-for velocity vector field u and pressure p of the flow, see for instance [29] for the classical setting or [25, 30] for the periodic setting. In the present paper we use the method of energy type estimates to obtain an open mapping theorem and a criterion of the surjectivity for the mapping induced by (0.1) over scales of specially constructed function spaces of Bochner–Sobolev type parametrized with the smoothness index $s \in \mathbb{N}$.

First, we note that the integrability is a kind of regularity, too. Due to [15,23,26] and [18,19], it is known that the uniqueness theorem and improvement of regularity actually follow from the existence of a weak solution in the Bochner class $L^{\mathfrak{s}}([0,T], L^{\mathfrak{r}}(\mathbb{R}^3))$ with the Ladyzhenskaya–Prodi–Serrin numbers $\mathfrak{r}$, $\mathfrak{s}$ satisfying $2/\mathfrak{s} + 3/\mathfrak{r} = 1$ and $\mathfrak{r} > 3$ (the limit case $\mathfrak{r} = 3$ was added to the list in [8]). Thus, avoiding weak solutions to the Navier–Stokes equations, we define two scales $\{B_1^s\}_{s\in\mathbb{Z}_+}$, $\{B_2^s\}_{s\in\mathbb{Z}_+}$ of separable Banach spaces such that: (a) each space of the scale $\{B_1^s\}_{s\in\mathbb{Z}_+}$ is continuously embedded into the spaces $L^{\mathfrak{s}}([0,T], L^{\mathfrak{r}}(\mathbb{R}^3))$ with $2/\mathfrak{s} + 3/\mathfrak{r} = 1$; (b) the Navier–Stokes equations induce non-linear continuous mappings $\mathcal{A}_s \colon B_1^s \to B_2^{s-1}$ for all $s \in \mathbb{N}$; (c) the components of vector fields belonging to the intersections $\cap_{s=1}^{\infty} B_1^s$, $\cap_{s=0}^{\infty} B_2^s$ are infinitely differentiable functions on the torus $\mathbb{T}^3$.

Second, we note that the existence of regular solutions to the Navier–Stokes equations for sufficiently small data in different spaces is known since J. Leray. In addition to these results, O. A. Ladyzhenskaya discovered the so-called stability property for the Navier–Stokes equations in some Bochner type spaces (see [15, Ch. 4, § 4, Theorems 10 and 11]). Hence, using in full this property, we extend it to the mappings $\mathcal{A}_s \colon B_1^s \to B_2^{s-1}$ with arbitrary $s \in \mathbb{N}$, expressing it as open mapping theorem for (0.1), cf. [27] for the Navier–Stokes type equations in $\mathbb{R}^3$ or [22] in a more general context of Elliptic Complexes on closed manifolds.

Finally, we prove that a map $\mathcal{A}_s$ is surjective if and only if the inverse image $\mathcal{A}_s^{-1}(K)$ of any precompact set K from the range of the map $\mathcal{A}_s$ is bounded in the Bochner space $L^{\mathfrak{s}}([0,T], L^{\mathfrak{r}}(\mathbb{T}^3))$ with the Ladyzhenskaya–Prodi–Serrin numbers $\mathfrak{s}$, $\mathfrak{r}$. This echoes the idea of using the properness property to study nonlinear operator equations, see for instance [28].

§ 1. Preliminaries

As usual, we denote by $\mathbb{Z}_+$ the set of all nonnegative integers including zero, and by $\mathbb{R}^n$ the Euclidean space of dimension $n \geq 2$ with coordinates $x = (x^1, \dots, x^n)$.

For a domain $\mathcal{X}$ in $\mathbb{R}^n$, and $s = 1, 2, \dots$, we write $H^s(\mathcal{X})$ for the Sobolev space of all functions $u \in L^2(\mathcal{X})$ whose generalised partial derivatives up to order s belong to $L^2(\mathcal{X})$. This is a Hilbert space with the standard inner product $(\cdot,\cdot)_{H^s(\mathcal{X})}$. The space $H^s_{\mathrm{loc}}(\mathcal{X})$ consists of functions belonging to $H^s(U)$ for each relatively compact domain $U \subset \mathcal{X}$.

Next, for $s = 0, 1, \dots$, and $0 \leq \lambda < 1$, we denote by $C^{s,\lambda}(\overline{\mathcal{X}})$ the so-called Hölder spaces, see for instance [16, Ch. 1, § 1], [15, Ch. 1, § 1]. The normed spaces $C^{s,\lambda}(\overline{\mathcal{X}})$ with $s \in \mathbb{Z}_+$ and $\lambda \in [0, 1)$ are known to be Banach spaces which admit the standard embedding theorems.

We are now ready to define proper spaces of periodic functions on $\mathbb{R}^n$. For this purpose, fix any $\ell > 0$ and denote by $\mathcal{Q}$ the cube $(0, \ell)^n$ of side length ℓ. Suppose $s \in \mathbb{Z}_+$. We denote by H^s the space of all functions $u \in H^s_{\mathrm{loc}}(\mathbb{R}^n)$ which satisfy the periodicity condition

$$u(x + \ell e_j) = u(x) \tag{1.1}$$

for all $x \in \mathbb{R}^n$ and $1 \leq j \leq n$, where e_j is the j-th unit basis vector in $\mathbb{R}^n$. The space H^s is a Hilbert space endowed with the inner product $(u, v)_{H^s} = (u, v)_{H^s(\mathcal{Q})}$. The functions from H^s can be characterised by their Fourier series expansions with respect to the orthogonal system $\{e^{\iota (k,z)(2\pi/\ell)}\}_{k\in\mathbb{Z}^n}$ in $L^2(\mathcal{Q})$ where ι is the imaginary unit. As the system consists of eigenfunctions of the Laplace operator Δ corresponding to eigenvalues $\{\lambda_k = -(k, k)(2\pi/\ell)^2\}_{k\in\mathbb{Z}^n}$, we see that the above scale of Sobolev spaces may be defined for all $s \in \mathbb{R}$ by

$$H^s = \{u = \sum_{k\in\mathbb{Z}^n} c_k(u) e^{\iota(k,z)(2\pi/\ell)} \colon |c_0(u)|^2 + \sum_{k\neq 0} (k, k)^s |c_k(u)|^2 < \infty\} \tag{1.2}$$

where $c_k(u)$ are the Fourier coefficients of u with respect to an orthonormal system of eigenfunctions of the Laplace operator in the space $\mathbf{L}^2$ corresponding to the eigenvalues λ_k. Traditionally, $\dot{H}^s$ stands for the subspace of H^s consisting of the elements u with $c_0(u) = 0$ in (1.2). Actually, this discussion leads us to the identification of the space H^s with Sobolev functions on the torus $\mathbb{T}^n$, to wit, $H^s \cong H^s(\mathbb{T}^n)$, see [2, § 2.4] and elsewhere.

We also need efficient tools for obtaining a priori estimates. Namely, it is the Gagliardo–Nirenberg inequality, see [21] for functions on $\mathbb{R}^n$. Its analogue for the torus reads for periodic functions as follows (see for instance [30, § 2.3]). For $1 \le p \le \infty$, set $\|\nabla^j u\|_{L^p(\mathcal{Q})} := := \max_{|\alpha|=j} \|\partial^\alpha u\|_{L^p(\mathcal{Q})}$. Then for any function $u \in L^{q_0} \cap L^{s_0}$ satisfying $\nabla^{j_0} u \in L^{p_0}$ and $\nabla^{k_0} u \in L^{r_0}$ it follows that

$$\|\nabla^{j_0} u\|_{L^{p_0}(\mathcal{Q})} \le c_1 \|\nabla^{k_0} u\|^{a_0}_{L^{r_0}(\mathcal{Q})} \|u\|^{1-a_0}_{L^{q_0}(\mathcal{Q})} + c_2 \|u\|_{L^{s_0}(\mathcal{Q})} \tag{1.3}$$

whenever $s_0 \ge 1$ and $0 \le a_0 \le 1$, where $\frac{1}{p_0} = \frac{j_0}{n} + a_0 \left(\frac{1}{r_0} - \frac{k_0}{n}\right) + (1 - a_0) \frac{1}{q_0}$, $\frac{j_0}{k_0} \le a_0$, the constants c_1 and c_2 depend on j_0, k_0, s_0, p_0, q_0 and r_0 but not on u ($a_0 < 1$ if $k_0 - j_0 - \frac{n}{r_0} \in \mathbb{Z}_+$).

Next, for $s \in \mathbb{Z}_+$ and $\lambda \in [0, 1)$, denote by $C^{s,\lambda}$ the space of all functions on $\mathbb{R}^n$ which belong to $C^{s,\lambda}(\overline{\mathcal{X}})$ for any bounded domain $\mathcal{X} \subset \mathbb{R}^n$ and satisfy (1.1). The space C^∞ of spatially periodic C^∞-functions reduces to the intersection of the spaces $C^{s,0}$ over $s \in \mathbb{Z}_+$. Let D' stand for the space of distributions on $\mathbb{T}^n$.

We will use the symbol $\mathbf{L}^p$ for the space of periodic vector fields $u = (u^1, u^2, u^3)$ on $\mathbb{R}^3$ with components u_i in L^p. The space is endowed with the standard norm. In a similar way we designate the spaces of periodic vector fields on $\mathbb{R}^n$ whose components are of Sobolev or Hölder class. We thus get $\mathbf{H}^s$ and $\mathbf{C}^{s,\lambda}$, respectively. By $\mathbf{C}^\infty$ and $\mathbf{D}'$ are meant the spaces of infinitely smooth periodic vector fields or distribution vector fields on $\mathbb{T}^n$.

To continue, we recall basic formulas of vector analysis saying that

$$\operatorname{rot} \nabla = 0, \quad \operatorname{div} \nabla = \Delta, \quad \operatorname{div} \operatorname{rot} = 0, \quad -\operatorname{rot} \operatorname{rot} + \nabla \operatorname{div} = E_3 \Delta \tag{1.4}$$

where E_3 is the unit (3×3)-matrix and rot is the usual rotation operator.

Given any differential operator A with C^∞ coefficients on the space of vector fields, we denote by $\ker(A)$ the subspace of $\mathbf{D}'$ consisting of all vector fields satisfying $Au = 0$ in the sense of distributions in $\mathbb{R}^n$. Furthermore, for an integer s, we write V_s for the space $\mathbf{H}^s \cap \ker(\operatorname{div})$ and V_s' for the dual spaces. The designations H and V are usually used for V_0 and V_1, respectively, see [30, § 2.1].

In order to characterize the space V_s, we denote by $\mathbb{N}_{2,3}$ the set of all natural numbers that can be represented as $(k, k) = k_1^2 + k_2^2 + k_3^2$, where $k = (k_1, k_2, k_3)$ is a triple of natural numbers. For $m \in \mathbb{N}_{2,3}$, let S_m be the finite-dimensional linear span of the system $\{e^{\iota(k,z)(2\pi/\ell)}\}_{(k,k)=m}$ of eigenfunctions of the Laplace operator, and let $\mathbf{S}_m$ be the corresponding space of vector fields on $\mathbb{R}^n$. Thus, we arrive at the following useful well-known statements.

Lemma 1. *Let $m \in \mathbb{N}_{2,n}$. There are $\mathbf{L}^2$-orthonormal bases $\{v_{m,j}\}_{j=1}^{J_m}$ and $\{w_{m,k}\}_{k=1}^{K_m}$ in the spaces $\mathbf{S}_m \cap \ker(\operatorname{div})$ and $\mathbf{S}_m \cap \ker(\operatorname{rot})$ such that*

$$\begin{cases} \operatorname{rot} \circ \operatorname{rot} v_{m,j} = -\Delta v_{m,j} = m(2\pi/\ell)^2 v_{m,j}, \\ \nabla \circ \operatorname{div} w_{m,k} = \Delta w_{m,k} = -m(2\pi/\ell)^2 w_{m,k} \end{cases}$$

for all $j = 1, \dots, J_m$ and $k = 1, \dots, K_m$. In particular, the system

$$\{e_1, e_2, \dots, e_n, v_{m,j}\}_{m \in \mathbb{N}, j=1,\dots,J_m}$$

is an orthonormal basis in V_s.

Denote by $\mathbf{P}$ the orthogonal projection of $\mathbf{L}^2$ onto V_0 which is usually referred to as the Helmholtz projection. By Lemma 1, we get

$$\mathbf{P} = \Pi + \mathrm{rot}^* \,\mathrm{rot}\, \varphi$$

where for $u = \sum_{k \neq 0} c_k(u) e^{\iota(k,z)(2\pi/\ell)}$ the operators Π and φ are given by

$$\Pi u = c_0(u), \quad \varphi u = - \sum_{k \neq 0} \frac{c_k(u)}{(k,k)(2\pi/\ell)^2} e^{\iota(k,z)(2\pi/\ell)}.$$

In particular, $\mathbf{P}$ is actually the $\mathbf{H}^s$-orthogonal projection onto V_s whenever $s \in \mathbb{Z}_+$.

We use also the Bochner spaces of functions of (x,t) in the strip $\mathbb{R}^n \times I$, where $I = [0,T]$. Namely, if $\mathcal{B}$ is a Banach space (possibly, a space of functions on $\mathbb{R}^n$) and $p \geq 1$, we denote by $L^p(I,\mathcal{B})$ the Banach space of all measurable mappings $u \colon I \to \mathcal{B}$ with finite norm

$$\|u\|_{L^p(I,\mathcal{B})} := \| \|u(\cdot,t)\|_{\mathcal{B}} \|_{L^p(I)},$$

see for instance [29, Ch. III, § 1]. In the same line stays the space $C(I,\mathcal{B})$, i. e., it is the Banach space of all continuous mappings $u \colon I \to \mathcal{B}$ with finite norm

$$\|u\|_{C(I,\mathcal{B})} := \sup_{t \in I} \|u(\cdot,t)\|_{\mathcal{B}}.$$

We are now in a position to introduce appropriate function spaces for solutions and for the data in order to obtain existence theorems for regular solutions to the Navier–Stokes type equations (0.1), cf. similar spaces for non-periodic vector-fields over $\mathbb{R}^n$ in [27]. Namely, for $s,k \in \mathbb{Z}_+$ we denote by $B^{k,2s,s}_{\mathrm{vel}}(I)$ the set of all vector fields u in the space $C(I,V_{k+2s}) \cap L^2(I,V_{k+1+2s})$ such that

$$\partial_x^\alpha \partial_t^j u \in C(I,V_{k+2s-|\alpha|-2j}) \cap L^2(I,V_{k+1+2s-|\alpha|-2j})$$

provided $|\alpha| + 2j \leq 2s$. We endow each space $B^{k,2s,s}_{\mathrm{vel,a}}(I)$ with the norm

$$\|u\|_{B^{k,2s,s}_{\mathrm{vel}}(I)} := \Big(\sum_{i=0}^{k} \sum_{|\alpha|+2j \leq 2s} \|\partial_x^\alpha \partial_t^j u\|^2_{i,\mu,T} \Big)^{1/2}$$

where $\|u\|_{i,\mu,T} = \Big(\|\nabla^i u\|^2_{C(I,\mathbf{L}^2)} + \mu \|\nabla^{i+1} u\|^2_{L^2(I,\mathbf{L}^2)} \Big)^{1/2}$ are seminorms on the space $B^{k,2s,s}_{\mathrm{vel}}(I)$, too. Similarly, for $s,k \in \mathbb{Z}_+$, we define the space $B^{k,2s,s}_{\mathrm{for}}(I)$ to consist of all fields f in $C(I,\mathbf{H}^{2s+k}) \cap L^2(I,\mathbf{H}^{2s+k+1})$ with the property that $\partial_x^\alpha \partial_t^j f \in C(I,\mathbf{H}^k) \cap L^2(I,\mathbf{H}^{k+1})$ provided $|\alpha| + 2j \leq 2s$. If $f \in B^{k,2s,s}_{\mathrm{for}}(I)$, then actually

$$\partial_x^\alpha \partial_t^j f \in C(I,\mathbf{H}^{k+2(s-j)-|\alpha|}) \cap L^2(I,\mathbf{H}^{k+1+2(s-j)-|\alpha|})$$

for all α and j satisfying $|\alpha| + 2j \leq 2s$. We endow each space $B^{k,2s,s}_{\mathrm{for}}(I)$ with the norm

$$\|f\|_{B^{k,2s,s}_{\mathrm{for}}(I)} := \Big(\sum_{i=0}^{k} \sum_{|\alpha|+2j \leq 2s} \|\nabla^i \partial_x^\alpha \partial_t^j f\|^2_{C(I,\mathbf{L}^2(\mathcal{Q}))} + \|\nabla^{i+1} \partial_x^\alpha \partial_t^j f\|^2_{L^2(I,\mathbf{L}^2(\mathcal{Q}))} \Big)^{1/2}.$$

Obviously, the pressure p is defined by (0.1) up to a summand independent of x. However, for the methods we use, the uniqueness of the map, defined by (0.1), is important. For this mathematical reason, let the space $B^{k+1,2s,s}_{\mathrm{pre}}(I)$ for the pressure p consist of all functions

$p \in C(I, \dot{H}^{2s+k+1}) \cap L^2(I, \dot{H}^{2s+k+2})$ such that $\nabla p \in B^{k,2s,s}_{\rm for}(I)$. The space does not contain functions depending on t only, and this allows us to equip it with the norm

$$\|p\|_{B^{k+1,2s,s}_{\rm pre}(I)} = \|\nabla p\|_{B^{k,2s,s}_{\rm for}(I)}.$$

Physically, this means that we allow for the pressure to be negative in order to achieve the uniqueness. Of course, if $2s + k \geq 1$ then $p \in B^{k+1,2s,s}_{\rm pre}(I)$ is bounded because of the Sobolev embedding theorem and we may grant the positivity of a pressure adding a large constant to p. If $0 \leq 2s + k < 1$ then we may follow [15, Ch 2, § 5] to replace the positivity of the pressure by other reasonable interpretations.

Clearly, $B^{k,2s,s}_{\rm vel}(I)$, $B^{k,2s,s}_{\rm for}(I)$, $B^{k+1,2s,s}_{\rm pre}(I)$ are Banach spaces with the following properties.

Lemma 2. *Suppose that $s \in \mathbb{N}$, $k \in \mathbb{Z}_+$. The following mappings are continuous:*

$$\begin{aligned}
\nabla\colon B^{k+1,2(s-1),s-1}_{\rm pre}(I) &\to B^{k,2(s-1),s-1}_{\rm for}(I),\\
\partial_j\partial_i\colon B^{k,2s,s}_{\rm vel}(I) &\to B^{k,2(s-1),s-1}_{\rm for}(I),\\
\partial_j\partial_i\colon B^{k+2,2(s-1),s-1}_{\rm vel}(I) &\to B^{k,2(s-1),s-1}_{\rm for}(I),\\
\partial_t\colon B^{k,2s,s}_{\rm vel}(I) &\to B^{k,2(s-1),s-1}_{\rm for}(I),\\
\delta_t\colon B^{k,2s,s}_{\rm vel}(I) &\to V_{k+2s},
\end{aligned}$$

where $\delta_t(u(\cdot,t)) = u(\cdot,0)$ is the initial value functional (or the delta-function in t).

P r o o f. Follows immediately from the definition of the spaces. □

Lemma 3. *If $s \in \mathbb{N}$ then the following embeddings are continuous:*

$$\begin{gathered}
B^{k,2s,s}_{\rm vel}(I) \hookrightarrow B^{k+2,2(s-1),s-1}_{\rm vel}(I), \qquad B^{k,2s,s}_{\rm for}(I) \hookrightarrow B^{k+2,2(s-1),s-1}_{\rm for}(I),\\
B^{k+1,2s,s}_{\rm pre}(I) \hookrightarrow B^{k+3,2(s-1),s-1}_{\rm pre}(I), \qquad B^{k,2s,s}_{\rm vel} \hookrightarrow L^\infty(I, \mathbf{L}^3), \qquad B^{k,2s,s}_{\rm vel}(I) \hookrightarrow L^{\mathfrak{s}}(I, \mathbf{L}^{\mathfrak{r}})
\end{gathered}$$

for all $\mathfrak{s}$, $\mathfrak{r}$ satisfying $2/\mathfrak{s} + 3/\mathfrak{r} = 1$.

P r o o f. The continuity of the first three embeddings follows immediately from the definition of the spaces. The other embeddings follow from (1.3) and the Sobolev embedding theorem (see, for instance, [1, Ch. 4, Theorem 4.12]). For the fifth embedding with $\mathfrak{s} = 2$, $\mathfrak{r} = +\infty$, we use the following: if $k, s \in \mathbb{Z}_+$ and $\lambda \in (0, 1)$ satisfying $k - s - \lambda > 3/2$, then there exists a constant $c(k, s, \lambda)$ depending on the parameters, such that

$$\|u\|_{C^{s,\lambda}} \leq c(k, s, \lambda)\, \|u\|_{H^k} \ \text{ for all } \ u \in H^k, \tag{1.5}$$

cf. [27, Lemma 3.4] for vector-fields over $\mathbb{R}^n$ without periodicity assumptions. □

§ 2. An open mapping theorem

This section is devoted to the so-called stability property for solutions to the Navier–Stokes type equations (0.1). One of the first statements of this kind was obtained by O. A. Ladyzhenskaya [15, Ch. 4, § 4, Theorem 11] for flows in bounded domains in $\mathbb{R}^3$ with C^2 smooth boundaries. In order to extend the property to the spaces of high smoothness, we consider the standard linearisation of problem (0.1) at the zero solution $(0, 0)$. Namely, let

$$\mathbf{D}(u) = u \cdot \nabla u, \qquad \mathbf{B}(w, u) = u \cdot \nabla w + w \cdot \nabla u,$$

for vector fields $u = (u^1, u^2, u^3)$ and $w = (w^1, w^2, w^3)$.

Lemma 4. *Suppose that* $s \in \mathbb{N}$, $k \in \mathbb{Z}_+$, *and* $w \in B^{k,2s,s}_{\mathrm{vel}}(I)$. *The following mappings are continuous:*

$$\begin{aligned} \mathbf{B}(w,\cdot)\colon B^{k+2,2(s-1),s-1}_{\mathrm{vel}}(I) &\to B^{k,2(s-1),s-1}_{\mathrm{for}}(I),\\ \mathbf{D}\colon B^{k+2,2(s-1),s-1}_{\mathrm{vel}}(I) &\to B^{k,2(s-1),s-1}_{\mathrm{for}}(I),\\ \mathbf{B}(w,\cdot)\colon B^{k,2s,s}_{\mathrm{vel}}(I) &\to B^{k,2(s-1),s-1}_{\mathrm{for}}(I),\\ \mathbf{D}\colon B^{k,2s,s}_{\mathrm{vel}}(I) &\to B^{k,2(s-1),s-1}_{\mathrm{for}}(I). \end{aligned}$$

Moreover, with positive constants $c_{s,k}$ *independent of* u, w,

$$\|\mathbf{B}(w,u)\|_{B^{k,2(s-1),s-1}_{\mathrm{for}}(I)} \le c_{s,k}\|w\|_{B^{k+2,2(s-1),s-1}_{\mathrm{vel,a}}(I)}\|u\|_{B^{k+2,2(s-1),s-1}_{\mathrm{vel,a}}(I)}. \tag{2.1}$$

P r o o f. Follows easily from the definition of the spaces and inequality (1.3), cf. [27, Lemma 3.5] for vector-fields over $\mathbb{R}^n$ without periodicity assumptions. □

Now, let us consider a linearisation of problem (0.1): given spatially periodic functions $f = (f^1, f^2, f^3) \in B^{k,2(s-1),s-1}_{\mathrm{for}}(I)$, $w = (w^1, w^2, w^3) \in B^{k,2s,s}_{\mathrm{vel}}(I)$ on $\mathbb{R}^3 \times [0, T]$ and $u_0 = (u^1_0, u^2_0, u^3_0) \in V_{2s+k}$ on $\mathbb{R}^3$ with values in $\mathbb{R}^3$, find spatially periodic functions $u = (u^1, u^2, u^3) \in B^{k,2s,s}_{\mathrm{vel}}(I)$ and $p \in B^{k+1,2s,s}_{\mathrm{pre}}(I)$ in the strip $\mathbb{R}^3 \times [0, T]$ which satisfy

$$\begin{cases} \partial_t u - \mu\Delta u + \mathbf{B}(w,u) + \nabla p = f, & (x,t) \in \mathbb{R}^3 \times (0,T),\\ \operatorname{div} u = 0, & (x,t) \in \mathbb{R}^3 \times (0,T),\\ u = u_0, & (x,t) \in \mathbb{R}^3 \times \{0\}. \end{cases} \tag{2.2}$$

Let us obtain an expected existence and uniqueness theorem.

Theorem 1. *Let* $s \in \mathbb{N}$, $k \in \mathbb{Z}_+$, *and* $w \in B^{k,2s,s}_{\mathrm{vel}}(I)$. *Then* (2.2) *induces a bijective continuous linear mapping*

$$\mathcal{A}_w\colon B^{k,2s,s}_{\mathrm{vel}}(I) \times B^{k+1,2(s-1),s-1}_{\mathrm{pre}}(I) \to B^{k,2(s-1),s-1}_{\mathrm{for}}(I) \times V_{2s+k}, \tag{2.3}$$

which admits a continuous inverse $\mathcal{A}_w^{-1}$.

P r o o f. The continuity of $\mathcal{A}_w$ follows from Lemmata 2, 4. Next, one usually follows a rather standard scheme beginning with the notion of a weak solution, see, for instance [15, Ch. VI, § 5], [29, Ch. 3, § 1] or [27, Theorems 3.1 and 3.2] for this particular type of spaces in the case of vector fields over $\mathbb{R}^n \times (0, T)$ without periodicity assumptions.

Namely, one usually begins with the following statement.

Proposition 1. *Suppose* $w \in C(I, V_0) \cap L^2(I, V_1) \cap L^2(I, \mathbf{L}^\infty)$. *Given any pair* $(f, u_0) \in \in L^2(I, V_1') \times V_0$, *there is a unique vector field* $u \in C(I, V_0) \cap L^2(I, V_1)$ *with* $\partial_t u \in L^2(I, V_1')$, *satisfying for all* $v \in V_1$

$$\begin{cases} \dfrac{d}{dt}(u,v)_{\mathbf{L}^2} + \mu(\nabla u, \nabla v)_{\mathbf{L}^2} = \langle f - \mathbf{B}(w,u), v\rangle,\\ u(\cdot,0) = u_0. \end{cases} \tag{2.4}$$

P r o o f. It is similar to the proof of the uniqueness and existence theorem for the Stokes problem and the Navier–Stokes problem, see [30, § 2.3, § 2.4] (or [19, Ch. II, Theorem 6.1 and Theorem 6.9] or [29, Ch. III, Theorem 1.1, Theorem 3.1 and Theorem 3.4] for domains in $\mathbb{R}^3$). It is based on Grönwall type Lemma, see [20, Ch. XII, p. 360, formulas (9.1′) and (9.1″)], inequality (1.3) and the following useful lemmata and formulas.

Lemma 5. *Let V, H and V' be Hilbert spaces such that V' is the dual to V and the embeddings $V \subset H \subset V'$ are continuous and everywhere dense. If $u \in L^2(I,V)$ and $\partial_t u \in L^2(I,V')$ then $\frac{d}{dt}\|u(\cdot,t)\|_H^2 = 2\langle \partial_t u, u\rangle$ and u is equal almost everywhere to a continuous mapping from $[0,T]$ to H.*

P r o o f. See [29, Ch. III, § 1, Lemma 1.2]. □

The following standard statement, where

$$\|u\|_{k,\mu,T} = \Big(\|\nabla^k u\|^2_{C(I,\mathbf{L}^2)} + \mu\,\|\nabla^{k+1}u\|^2_{L^2(I,\mathbf{L}^2)}\Big)^{1/2},$$
$$\|(f,u_0)\|_{0,\mu,T} = \Big(\|u_0\|^2_{\mathbf{L}^2} + \frac{2}{\mu}\,\|f\|^2_{L^2(I,V_1')} + \|f\|^2_{L^1(I,V_1')}\Big)^{1/2},$$

gives a basic a priori estimate for regular solutions to (2.4).

Lemma 6. *Let $w \in L^2(I,V_1)\cap C(I,V_0)\cap L^2(I,\mathbf{L}^\infty)$. If $u \in C(I,V_0)\cap L^2(I,V_1)$ and $(f,u_0) \in L^2(I,V_1')\times V_0$ satisfy*

$$\begin{cases} \frac{1}{2}\frac{d}{d\tau}\,\|u(\cdot,\tau)\|^2_{\mathbf{L}^2} + \mu\,\|\nabla u\|^2_{\mathbf{L}^2} \le \langle f - \mathbf{B}(w,u), u\rangle \\ u(\cdot,0) = u_0 \end{cases} \tag{2.5}$$

for all $t \in [0,T]$, then, with positive constant c_j independent of w and u,

$$\begin{aligned} \|u\|^2_{0,\mu,T} \le \|(f,u_0)\|^2_{0,\mu,T}\Big(1 + c_1 \exp\Big(\frac{c_2}{\mu}\int_0^T \|w(\cdot,t)\|^2_{\mathbf{L}^\infty}dt\Big) + \\ + \frac{c_3}{\mu}\Big(\int_0^T \|w(\cdot,t)\|^2_{\mathbf{L}^\infty}dt\Big)\exp\Big(\frac{c_4}{\mu}\int_0^T \|w(\cdot,t)\|^2_{\mathbf{L}^\infty}dt\Big)\Big). \end{aligned} \tag{2.6}$$

The rest of the proof runs the standard scheme with the use of Faedo–Galerkin approximations, see *ibid*. □

Of course, we also have to recover the pressure p via known velocity u.

Proposition 2. *Let $n \ge 3$, $s \in \mathbb{N}$, $k \in \mathbb{Z}_+$, and $F \in B^{k,2(s-1),s-1}_{\text{for}}$ satisfy $\mathbf{P}F = 0$. Then there is a unique function $p \in B^{k+1,2(s-1),s-1}_{\text{pre}}$, satisfying $\nabla p = F$ in $\mathbb{T}^n \times [0,T]$.*

P r o o f. The arguments are straightforward. □

This finishes our sketch of the proof of Theorem 1. □

Since problem (2.2) is a linearisation of the Navier–Stokes type equations at an arbitrary vector field w, it follows from Theorem 1 that the nonlinear mapping given by (0.1) is locally invertible. The implicit function theorem for Banach spaces even implies that the local inverse mappings can be obtained from the contraction principle of Banach. In this way we obtain what we shall call the open mapping theorem for problem (0.1).

Theorem 2. *Let $s \in \mathbb{N}$ and $k \in \mathbb{Z}_+$. Then (0.1) induces an injective continuous nonlinear open mapping*

$$\mathcal{A}\colon B^{k,2s,s}_{\text{vel}}(I)\times B^{k+1,2(s-1),s-1}_{\text{pre}}(I) \to B^{k,2(s-1),s-1}_{\text{for}}(I)\times V_{2s+k}. \tag{2.7}$$

The significance of the theorem is in the assertion that for each point $(u_0,p_0) \in B^{k,2s,s}_{\text{vel}}(I)\times B^{k+1,2(s-1),s-1}_{\text{pre}}(I)$ there is a neighbourhood $\mathcal{V}$ of the image $\mathcal{A}(u_0,p_0)$ in $B^{k,2(s-1),s-1}_{\text{for}}(I)\times V_{2s+k}$, such that $\mathcal{A}$ is a homeomorphism of the open set $\mathcal{U} := \mathcal{A}^{-1}(\mathcal{V})$ onto $\mathcal{V}$.

P r o o f. Since the bilinear form $\mathbf{B}$ is symmetric and $\mathbf{B}(u,u) = 2\mathbf{D}(u)$, we obtain

$$\mathbf{D}(u') - \mathbf{D}(u'') = \mathbf{B}(u', u'-u'') + (1/2)\,\mathbf{B}(u'-u'', u'-u''). \tag{2.8}$$

Then the continuity of the mapping $\mathcal{A}$ is clear from (2.1) and (2.8).

Suppose that $(u,p) \in B^{k,2s,s}_{\mathrm{vel}}(I) \times B^{k+1,2(s-1),s-1}_{\mathrm{pre}}(I)$, $\mathcal{A}(u,p) = (f, u_0) \in B^{k,2(s-1),s-1}_{\mathrm{for}}(I) \times \times V_{k+2s}$. Obviously, an integration by parts implies that u is a weak solution to equations (0.1), see, for instance, [30, (2.41)]. By the Sobolev embedding theorem, see (1.5), the space $B^{k,2s,s}_{\mathrm{vel}}(I)$ is continuously embedded into $L^2(I, \mathbf{L}^\infty(\mathbb{R}^3))$. Thus, u is a strong solution that is unique, see, for instance, [30, Remark 3.1] or [29, Theorem III.3.9]. Thus, if (u',p') and (u'',p'') belong to $B^{k,2s,s}_{\mathrm{vel}}(I) \times B^{k+1,2(s-1),s-1}_{\mathrm{pre}}(I)$ and $\mathcal{A}(u',p') = \mathcal{A}(u'',p'')$ then $u' = u''$ and $\nabla(p'-p'')(\cdot,t) = 0$ for all $t \in [0,T]$. It follows that the difference $p'-p''$ is identically equal to a function $c(t)$ on the segment $[0,T]$. Since $p'-p'' \in C(I, \dot{H}^k)$, we conclude by Proposition 2 that $p'-p'' \equiv 0$. So, the operator $\mathcal{A}$ of (2.7) is injective.

Finally, (2.8) makes it evident that the Fréchet derivative $\mathcal{A}'_{(w,p_0)}$ of the nonlinear mapping $\mathcal{A}$ at an arbitrary point $(w,p_0) \in B^{k,2s,s}_{\mathrm{vel}}(I) \times B^{k+1,2(s-1),s-1}_{\mathrm{pre}}(I)$ coincides with the continuous linear mapping $\mathcal{A}_w$ of (2.3). By Theorem 1, $\mathcal{A}_w$ is an invertible continuous linear mapping from $B^{k,2s,s}_{\mathrm{vel}}(I) \times B^{k+1,2(s-1),s-1}_{\mathrm{pre}}(I)$ to $B^{k,2(s-1),s-1}_{\mathrm{for}}(I) \times V_{k+2s}$. Both the openness of the mapping $\mathcal{A}$ and the continuity of its local inverse mapping now follow from the implicit function theorem for Banach spaces, see for instance [10, Theorem 5.2.3, p. 101]. □

Corollary 1. *The range of the mapping* (2.7) *is closed if and only if it coincides with the whole destination space.*

P r o o f. Since the destination space is convex, it is connected. The only closed and open sets in a connected topological vector space are the empty set and the space itself. Hence, the range of $\mathcal{A}$ is closed if and only if it coincides with the whole destination space. □

§ 3. A surjectivity criterion

Inspired by [15, 23, 26] and [18, 19], let us obtain a surjectivity criterion for mapping (2.7) in terms of $L^{\mathfrak{s}}(I, \mathbf{L}^{\mathfrak{r}})$-estimates for solutions to (0.1) via the data.

Theorem 3. *Let $s \in \mathbb{N}$, $k \in \mathbb{Z}_+$ and the numbers $\mathfrak{r}$, $\mathfrak{s}$ satisfy $2/\mathfrak{s} + 3/\mathfrak{r} = 1$. Then mapping* (2.7) *is surjective if and only if, given subset $S = S_{\mathrm{vel}} \times S_{\mathrm{pre}}$ of the product $B^{k,2s,s}_{\mathrm{vel}}(I) \times B^{k+1,2(s-1),s-1}_{\mathrm{pre}}(I)$ such that the image $\mathcal{A}(S)$ is precompact in the space $B^{k,2(s-1),s-1}_{\mathrm{for}}(I) \times V_{2s+k}$, the set S_{vel} is bounded in the space $L^{\mathfrak{s}}(I, \mathbf{L}^{\mathfrak{r}})$.*

P r o o f. Let mapping (2.7) be surjective. Then the range of this mapping is closed according to Theorem 2. Fix a subset $S = S_{\mathrm{vel}} \times S_{\mathrm{pre}}$ of the product $B^{k,2s,s}_{\mathrm{vel}}(I) \times B^{k+1,2(s-1),s-1}_{\mathrm{pre}}(I)$ such that the image $\mathcal{A}(S)$ is precompact in the space $B^{k,2(s-1),s-1}_{\mathrm{for}}(I) \times V_{2s+k}$. If the set S_{vel} is unbounded in the space $L^{\mathfrak{s}}(I, \mathbf{L}^{\mathfrak{r}})$ then there is a sequence $\{(u_k, p_k)\} \subset S$ such that

$$\lim_{k\to\infty} \|u_k\|_{L^{\mathfrak{s}}(I,\mathbf{L}^{\mathfrak{r}})} = \infty. \tag{3.1}$$

As the set $\mathcal{A}(S)$ is precompact in $B^{k,2(s-1),s-1}_{\mathrm{for}}(I) \times V_{2s+k}$, we conclude that the corresponding sequence of data $\{\mathcal{A}(u_k,p_k) = (f_k, u_{k,0})\}$ contains a subsequence $\{(f_{k_m}, u_{k_m,0})\}$ which converges to an element (f, u_0) in this space. But the range of the map is closed and hence for the data (f, u_0) there is a unique solution (u,p) to (0.1) in the space $B^{k,2s,s}_{\mathrm{vel}}(I) \times B^{k+1,2(s-1),s-1}_{\mathrm{pre}}(I)$ and the sequence $\{(u_{k_m}, p_{k_m})\}$ converges to (u,p) in this space. Therefore, $\{(u_{k_m}, p_{k_m})\}$ is bounded in

$B^{k,2s,s}_{\rm vel}(I) \times B^{k+1,2(s-1),s-1}_{\rm pre}(I)$ and this contradicts (3.1) because the space $B^{k,2s,s}_{\rm vel}(I)$ is embedded continuously into the space $L^{\mathfrak{s}}(I,\mathbf{L}^{\mathfrak{r}})$ for any pair $\mathfrak{r}$, $\mathfrak{s}$ satisfying $2/\mathfrak{s}+3/\mathfrak{r}=1$.

We continue with typical estimates for solutions to the Navier–Stokes equations (0.1). We emphasize again that the elements of the spaces under consideration are already sufficiently regular. So, we need the estimates for proving the surjectivety of mapping (2.7) but not for improving regularity of weak (Leray–Hopf) solutions.

Lemma 7. *If $(u,p) \in B^{0,2,1}_{\rm vel}(I) \times B^{1,0,0}_{\rm pre}(I)$ is a solution to the Navier–Stokes equations (0.1) with data $(f,u_0) \in B^{0,0,0}_{\rm for}(I) \times V_2$, then $\|u\|_{0,\mu,T} \le \|(f,u_0)\|_{0,\mu,T}$.*

P r o o f. For a solution (u,p) to (0.1) related to data (f,u_0) within the declared function classes, the component u belongs to $C(I,H^2)\cap L^2(I,H^3)$, and both $\partial_t u$ and f belong to $C(I,L^2)\cap L^2(I,H^1)$. Then we may calculate the inner product $(\mathcal{A}u,u)_{\mathbf{L}^2}$ with the use of integration by parts and Lemma 5, obtaining

$$\frac{d}{dt}\|u\|^2_{\mathbf{L}^2} + \mu\|\nabla u\|^2_{\mathbf{L}^2} = \langle f,u\rangle.$$

Finally, applying Lemma 6 with $w=0$, we conclude the estimate follow. □

Let's obtain estimates for the derivatives of vector fields with respect to space variables.

Lemma 8. *Let $k\in\mathbb{Z}_+$ and $\mathfrak{s}$, $\mathfrak{r}$ satisfy $2/\mathfrak{s}+3/\mathfrak{r}=1$. Then for any $\varepsilon>0$ and for all $u\in\mathbf{H}^{2+k}$ it follows that*

$$\begin{aligned} \|(-\Delta)^{\frac{k}{2}}\mathbf{D}u\|^2_{\mathbf{L}^2} \le\ & \varepsilon\,\|\nabla^{k+2}u\|^2_{\mathbf{L}^2} + \\ & + c(k,\mathfrak{s},\mathfrak{r},\varepsilon)\,\|u\|^{\mathfrak{s}}_{\mathbf{L}^{\mathfrak{r}}}\|\nabla^{k+1}u\|^2_{\mathbf{L}^2} + c(k,\mathfrak{s},\mathfrak{r})\,\|u\|^2_{\mathbf{L}^2}\|u\|^2_{\mathbf{L}^{\mathfrak{r}}} + c(k,\mathfrak{s},\mathfrak{r})\,\|u\|^2_{\mathbf{L}^2} \end{aligned} \tag{3.2}$$

with positive constants depending on the parameters in parentheses and not necessarily the same in diverse applications, the constants being independent of u.

P r o o f. On using the Leibniz rule, the Hölder inequality we deduce that

$$\|(-\Delta)^{\frac{k}{2}}\mathbf{D}u\|^2_{\mathbf{L}^{\mathfrak{r}}} \le \sum_{j=0}^{k} C^k_j\,\|\nabla^{k+1-j}u\|^2_{\mathbf{L}^{\frac{2q}{q-1}}}\|\nabla^j u\|^2_{\mathbf{L}^{2q}} \tag{3.3}$$

with binomial type coefficients C^j_k and any $q\in(1,\infty)$.

For $k=0$ there are no other summands than that with $j=0$. But for $k\ge 1$ we have to consider the items corresponding to $1\le j\le k$, too. The standard interpolation inequalities on compact manifolds (see, for instance, [10, Theorem 2.2.1]) hint us that those summands which correspond to $1\le j\le k$ could actually be estimated by the item with $j=0$. We realize this as follows: For any j satisfying $1\le j\le k$ there are numbers $q>1$ and $c>0$ depending on k and j but not on u, such that

$$\|\nabla^{k+1-j}u\|_{\mathbf{L}^{\frac{2q}{q-1}}}\|\nabla^j u\|_{\mathbf{L}^{2q}} \le c\Big(\|\nabla^{k+1}u\|_{\mathbf{L}^{\frac{2\mathfrak{r}}{\mathfrak{r}-2}}}\|u\|_{\mathbf{L}^{\mathfrak{r}}} + \|u\|_{\mathbf{L}^2}\Big). \tag{3.4}$$

Indeed, we may apply Gagliardo–Nirenberg inequality (1.3) if we prove that for each $1\le j\le k$ there is a $q>1$ depending on k and j, such that the system of algebraic equations

$$\begin{cases} \dfrac{1}{2q} = \dfrac{j}{3} + \Big(\dfrac{\mathfrak{r}-2}{2\mathfrak{r}} - \dfrac{k+1}{3}\Big)\vartheta_1 + \dfrac{1-\vartheta_1}{\mathfrak{r}}, \\ \dfrac{q-1}{2q} = \dfrac{k+1-j}{3} + \Big(\dfrac{\mathfrak{r}-2}{2\mathfrak{r}} - \dfrac{k+1}{3}\Big)\vartheta_2 + \dfrac{1-\vartheta_2}{\mathfrak{r}} \end{cases}$$

admits solutions $\vartheta_1 \in [\frac{j}{k+1}, 1)$, $\vartheta_2 \in [\frac{k+1-j}{k+1}, 1)$. On adding these equations we see that

$$\frac{1}{2} - \frac{k+1}{3} - \frac{2}{\mathfrak{r}} = \Big(\frac{1}{2} - \frac{k+1}{3} - \frac{2}{\mathfrak{r}}\Big)(\vartheta_1 + \vartheta_2),$$

i. e., the system is reduced to

$$\begin{cases} \vartheta_1 q(2(k+1)\mathfrak{r} + 12 - 3\mathfrak{r}) = 2j\,\mathfrak{r}q + 6q - 3\mathfrak{r}, \\ \vartheta_1 + \vartheta_2 = 1. \end{cases}$$

Choose $\vartheta_1 = \dfrac{j}{k+1}$ and $\vartheta_2 = \dfrac{k+1-j}{k+1}$ to obtain $q = q(k,j) = \frac{(k+1)\mathfrak{r}}{2(k+1)+j(\mathfrak{r}-4)}$. Since $\mathfrak{r} > n \geq 2$ and $1 \leq j \leq k$, an easy calculation shows that

$$2(k+1) + j(\mathfrak{r}-4) > 2(k+1) - 2j \geq 2 > 0,$$
$$(k+1)\mathfrak{r} - (2(k+1) + j(\mathfrak{r}-4)) = (k+1)(\mathfrak{r}-2) - j(\mathfrak{r}-4) > 0,$$

i. e., $q(k,j) > 1$ in this case, and so (3.4) holds true.

Therefore, if we choose $q(k,0) = \mathfrak{r}/2 > 1$, the estimates of (3.3) and (3.4) readily yield

$$\|(-\Delta)^{\frac{k}{2}}\mathbf{D}u\|^2_{\mathbf{L}^2} \leq c(k,\mathfrak{r})\Big(\|\nabla^{k+1}u\|^2_{\mathbf{L}^{\frac{2\mathfrak{r}}{\mathfrak{r}-2}}}\|u\|^2_{\mathbf{L}^{\mathfrak{r}}} + \|u\|^2_{\mathbf{L}^2}\Big) \tag{3.5}$$

with a constant $c(k,\mathfrak{r})$ independent of u.

Now, if $\mathfrak{s} = 2$ and $\mathfrak{r} = +\infty$, then, obviously, we get

$$c(k,\mathfrak{r})\,\|\nabla^{k+1}u\|^2_{\mathbf{L}^{\frac{2\mathfrak{r}}{\mathfrak{r}-2}}}\|u\|^2_{\mathbf{L}^{\mathfrak{r}}} = c(k,\mathfrak{r})\,\|\nabla^{k+1}u\|^2_{\mathbf{L}^2}\|u\|^2_{\mathbf{L}^{\infty}}. \tag{3.6}$$

If $\mathfrak{s} > 2$ and $3 < \mathfrak{r} < \infty$, then we may again apply Gagliardo–Nirenberg inequality (1.3) to achieve

$$\|\nabla^{k+1}u\|_{\mathbf{L}^{\frac{2\mathfrak{r}}{\mathfrak{r}-2}}}\|u\|_{\mathbf{L}^{\mathfrak{r}}} \leq c(\mathfrak{r})\Big(\|\nabla^{k+2}u\|^{\frac{3}{\mathfrak{r}}}_{\mathbf{L}^2}\|\nabla^{k+1}u\|^{\frac{\mathfrak{r}-3}{\mathfrak{r}}}_{\mathbf{L}^2} + \|u\|_{\mathbf{L}^2}\Big)\|u\|_{\mathbf{L}^{\mathfrak{r}}} \tag{3.7}$$

with an appropriate Gagliardo-Nirenberg constant $c(\mathfrak{r})$ independent of u.

Since $\mathfrak{s} = \frac{2\mathfrak{r}}{\mathfrak{r}-3}$, it follows from (3.7) that

$$\begin{aligned} \|\nabla^{k+1}u\|^2_{\mathbf{L}^{\frac{2\mathfrak{r}}{\mathfrak{r}-2}}}\|u\|^2_{\mathbf{L}^{\mathfrak{r}}} &\leq 2\Big(\|\nabla^{k+2}u\|^{\frac{6}{\mathfrak{r}}}_{\mathbf{L}^2}\|\nabla^{k+1}u\|^{\frac{2(\mathfrak{r}-3)}{\mathfrak{r}}}_{\mathbf{L}^2}\|u\|^2_{\mathbf{L}^{\mathfrak{r}}} + \|u\|^2_{\mathbf{L}^2}\|u\|^2_{\mathbf{L}^{\mathfrak{r}}}\Big) \leq \\ &\leq \frac{1}{c(k,\mathfrak{r})}\Big(\varepsilon\,\|\nabla^{k+2}u\|^2_{\mathbf{L}^2} + \frac{c(k,\mathfrak{r})}{\varepsilon}\|\nabla^{k+1}u\|^2_{\mathbf{L}^2}\|u\|^{\mathfrak{s}}_{\mathbf{L}^{\mathfrak{r}}} + 2c(k,\mathfrak{r})\,\|u\|^2_{\mathbf{L}^2}\|u\|^2_{\mathbf{L}^{\mathfrak{r}}}\Big) \end{aligned} \tag{3.8}$$

with some positive constants independent of u because of Young's inequality applied with $p_1 = \mathfrak{r}/3$ and $p_2 = \mathfrak{r}/(\mathfrak{r}-3)$.

Now, inequalities (3.5), (3.6) and (3.8) imply (3.2) for all $3 < \mathfrak{r} \leq \infty$ and $2 \leq \mathfrak{s} = 2\mathfrak{r}/(\mathfrak{r}-3) < < \infty$, as desired. □

For $k \geq 1$, we now introduce

$$\|(f,u_0)\|_{k,\mu,T} = \Big(\|\nabla^k u_0\|^2_{\mathbf{L}^2} + 4\mu^{-1}\|\nabla^{k-1}f\|^2_{L^2(I,\mathbf{L}^2)}\Big)^{1/2}.$$

Lemma 9. *Let $k \in \mathbb{Z}_+$ and the pair $\mathfrak{s}$, $\mathfrak{r}$ satisfy $\frac{2}{\mathfrak{s}} + \frac{3}{\mathfrak{r}} = 1$. If $(u,p) \in B^{k,2,1}_{\mathrm{vel}}(I) \times B^{k+1,0,0}_{\mathrm{pre}}(I)$ is a solution to (0.1) corresponding to data (f,u_0) in $B^{k,0,0}_{\mathrm{for}}(I) \times V_{k+2}$ then*

$$\begin{gathered} \|u\|_{j+1,\mu,T} \leq c_j((f,u_0),u),\ \|\nabla^j\mathbf{D}u\|_{L^2(I,\mathbf{L}^2)} \leq c_j((f,u_0),u), \\ \|\nabla^j\partial_t u\|^2_{L^2(I,\mathbf{L}^2)} + \|\nabla^{j+1}p\|^2_{L^2(I,\mathbf{L}^2)} \leq c_j((f,u_0),u), \end{gathered} \tag{3.9}$$

for all $0 \leq j \leq k+1$, where the constants on the right-hand side depend on the norms $\|(f,u_0)\|_{0,\mu,T}$, $\|(f,u_0)\|_{j+1,\mu,T}$ and $\|u\|_{L^{\mathfrak{s}}(I,\mathbf{L}^{\mathfrak{r}})}$ and need not be the same in diverse applications.

P r o o f. We first recall that $u \in C(I, \mathbf{H}^{k+2}) \cap L^2(I, \mathbf{H}^{k+3})$, $u_0 \in \mathbf{H}^{k+2}$ and $\nabla p, f \in$ $\in C(I, \mathbf{H}^k) \cap L^2(I, \mathbf{H}^{k+1})$ under the hypotheses of the lemma. Next, we see that in the sense of distributions we have

$$\begin{cases} (-\Delta)^{\frac{j}{2}}(\partial_t u - \mu \Delta u + \mathbf{D}u + \nabla p) = (-\Delta)^{\frac{j}{2}} f \text{ in } \mathbb{R}^3 \times (0,T), \\ (-\Delta)^{\frac{j}{2}} u(x,0) = (-\Delta)^{\frac{j}{2}} u_0(x) \text{ for } x \in \mathbb{R}^3 \end{cases} \tag{3.10}$$

for all $0 \le j \le k+1$, if (u,p) is a solution to (0.1). Integration by parts yields

$$((-\Delta)^{\frac{j}{2}} u, (-\Delta)^{\frac{j+2}{2}} u)_{\mathbf{L}^2} = \|(-\Delta)^{\frac{j+1}{2}} u\|^2_{\mathbf{L}^2} = \|\nabla^{j+1} u\|^2_{\mathbf{L}^2} \tag{3.11}$$

and similarly

$$2\,(\partial_t (-\Delta)^{\frac{j}{2}} u, (-\Delta)^{\frac{j+2}{2}} u)_{\mathbf{L}^2} = \frac{d}{dt}\|\nabla^{j+1} u\|^2_{\mathbf{L}^2}, \tag{3.12}$$

cf. Lemma 5. Furthermore, using (1.4) we conclude that, for all $t \in [0,T]$,

$$\begin{aligned} ((-\Delta)^{\frac{j}{2}} \nabla p(\cdot,t), (-\Delta)^{\frac{j+2}{2}} u(\cdot,t))_{\mathbf{L}^2} &= \lim_{i\to\infty} ((-\Delta)^{\frac{j}{2}} \nabla p_i(\cdot,t), (\mathrm{rot})^* \mathrm{rot}\,(-\Delta)^{\frac{j}{2}} u(\cdot,t))_{\mathbf{L}^2} = \\ &= \lim_{i\to\infty} ((-\Delta)^{\frac{j}{2}} \mathrm{rot}\,\nabla p_i(\cdot,t), \mathrm{rot}\,(-\Delta)^{\frac{j}{2}} u(\cdot,t))_{\mathbf{L}^2} = 0, \end{aligned} \tag{3.13}$$

where $p_i(\cdot,t) \in H^{j+2}$ is any sequence approximating $p(\cdot,t)$ in H^{j+1}.

On combining (3.10), (3.11), (3.12) and (3.13) we get

$$\begin{aligned} &2\,((-\Delta)^{\frac{j}{2}}(\partial_t u - \mu\Delta u + \mathbf{D}u + \nabla p)(\cdot,t), (-\Delta)^{\frac{j+2}{2}} u(\cdot,t))_{\mathbf{L}^2} = \\ &= \frac{d}{dt}\|\nabla^{j+1} u(\cdot,t)\|^2_{\mathbf{L}^2} + 2\mu\|\nabla^{j+2} u(\cdot,t)\|^2_{\mathbf{L}^2} + 2((-\Delta)^{\frac{j}{2}} \mathbf{D}u(\cdot,t), (-\Delta)^{\frac{j+2}{2}} u(\cdot,t))_{\mathbf{L}^2} \end{aligned} \tag{3.14}$$

for all $0 \le j \le k+1$. Next, according to the Hölder inequality, we get

$$2|((-\Delta)^{\frac{j}{2}} \mathbf{D}u, (-\Delta)^{\frac{j+2}{2}} u)_{\mathbf{L}^2}| \le \frac{2}{\mu}\|(-\Delta)^{\frac{j}{2}} \mathbf{D}u\|^2_{\mathbf{L}^2} + \frac{\mu}{2}\|(-\Delta)^{\frac{j+2}{2}} u(\cdot,t)\|_{\mathbf{L}^2}, \tag{3.15}$$

$$\begin{aligned} 2\,((-\Delta)^{\frac{j}{2}} f(\cdot,t), (-\Delta)^{\frac{j+2}{2}} u(\cdot,t))_{\mathbf{L}^2} &\le 2\,\|(-\Delta)^{\frac{j}{2}} f(\cdot,t)\|_{\mathbf{L}^2}\|(-\Delta)^{\frac{j+2}{2}} u(\cdot,t)\|_{\mathbf{L}^2} \le \\ &\le \frac{4}{\mu}\|(-\Delta)^{\frac{j}{2}} f(\cdot,t)\|^2_{\mathbf{L}^2} + \frac{\mu}{4}\|(-\Delta)^{\frac{j+2}{2}} u(\cdot,t)\|^2_{\mathbf{L}^2} \end{aligned} \tag{3.16}$$

for all $t \in [0,T]$. By the Hölder inequality with $q_1 = \frac{\mathfrak{r}}{3}$ and $q_2 = \frac{\mathfrak{r}}{\mathfrak{r}-3}$,

$$\int_0^t \|u(\cdot,s)\|^2_{\mathbf{L}^2}\|u(\cdot,s)\|^2_{\mathbf{L}^{\mathfrak{r}}}\,ds \le \|u\|^2_{L^{\frac{2}{3}\mathfrak{r}}([0,t],\mathbf{L}^2)}\|u\|^2_{L^{\mathfrak{s}}([0,t],\mathbf{L}^{\mathfrak{r}})}. \tag{3.17}$$

On summarising inequalities (3.10), (3.14), (3.15), (3.2), (3.17) and (3.16) we obtain

$$\begin{aligned} &\|\nabla^{j+1} u(\cdot,t)\|^2_{\mathbf{L}^2} + \mu\int_0^t \|\nabla^{j+2} u(\cdot,s)\|^2_{\mathbf{L}^2}\,ds \le \\ &\le \|\nabla^{j+1} u_0\|^2_{\mathbf{L}^2} + \frac{4}{\mu}\|\nabla^j f\|^2_{L^2(I,\mathbf{L}^2)} + c(j,\mathfrak{s},\mathfrak{r})\|u\|^2_{L^{\frac{2\mathfrak{r}}{n}}([0,t],\mathbf{L}^2)}\|u\|^2_{L^{\mathfrak{s}}([0,t],\mathbf{L}^{\mathfrak{r}})} + \\ &+ c(j,\mathfrak{s},\mathfrak{r})\frac{1}{\mu}\int_0^t \|u(\cdot,s)\|^{\mathfrak{s}}_{\mathbf{L}^{\mathfrak{r}}}\|\nabla^{j+1} u(\cdot,s)\|^2_{\mathbf{L}^2} ds + c(j,\mathfrak{s},\mathfrak{r})\|u\|^2_{\mathbf{L}^2} \end{aligned} \tag{3.18}$$

for all $t \in [0,T]$. By (2.6), and (3.18), given any $0 \leq j \leq k+1$, we get for all $t \in I$:

$$\begin{gathered}\|\nabla^{j+1}u(\cdot,t)\|^2_{\mathbf{L}^2} + \mu \int_0^t \|\nabla^{j+2}u(\cdot,s)\|^2_{\mathbf{L}^2}\,ds \leq \\ \leq \|(f,u_0)\|^2_{j+1,\mu,T} + c(j,\mathfrak{s},\mathfrak{r})T^{\frac{3}{\mathfrak{r}}}\|(f,u_0)\|^2_{0,\mu,T}\|u\|^2_{L^{\mathfrak{s}}([0,t],\mathbf{L}^{\mathfrak{r}})} + \\ + c(j,\mathfrak{s},\mathfrak{r})\frac{1}{\mu}\int_0^t \|u(\cdot,s)\|^{\mathfrak{s}}_{\mathbf{L}^{\mathfrak{r}}}\|\nabla^{j+1}u(\cdot,s)\|^2_{\mathbf{L}^2}ds + c(j,\mathfrak{s},\mathfrak{r})T\,\|(f,u_0)\|^2_{0,\mu,T}.\end{gathered} \tag{3.19}$$

On applying Grönwall type Lemma to (3.19) we conclude that, for all $t \in [0,T]$ and $0 \leq j \leq k+1$,

$$\|\nabla^{j+1}u(\cdot,t)\|^2_{\mathbf{L}^2} \leq c(j,\mathfrak{s},\mathfrak{r},T,\mu,(f,u_0))\exp\left(c(j,\mathfrak{s},\mathfrak{r})\frac{1}{\mu}\int_0^t \|u(\cdot,s)\|^{\mathfrak{s}}_{\mathbf{L}^{\mathfrak{r}}}ds\right) \tag{3.20}$$

with a positive constant $c(j,\mathfrak{s},\mathfrak{r},T,\mu,(f,u_0))$ independent of u. Obviously, (3.19) and (3.20) imply the first estimate of (3.9). Next, applying (3.2) and (3.17) we see that

$$\begin{gathered}\|(-\Delta)^{\frac{j}{2}}\mathbf{D}u\|^2_{L^2([0,t],\mathbf{L}^2)} \leq \|\nabla^{j+2}u\|^2_{L^2([0,t],\mathbf{L}^2)} + 2c(j,\mathfrak{r})\,\|u\|^2_{L^2([0,t],\mathbf{L}^2)} + \\ + c(j,\mathfrak{s},\mathfrak{r},\varepsilon\!=\!1)\,\|u\|^{\mathfrak{s}}_{L^{\mathfrak{s}}([0,t],\mathbf{L}^{\mathfrak{r}})}\|\nabla^{j+1}u\|^2_{C([0,t],\mathbf{L}^2)} + 2c(j,\mathfrak{r})\,\|u\|^2_{L^{\frac{2\mathfrak{r}}{3}}([0,t],\mathbf{L}^2)}\|u\|^2_{L^{\mathfrak{s}}([0,t],\mathbf{L}^{\mathfrak{r}})},\end{gathered}$$

the constants being independent of u. The second bound of (3.9) follows from (2.6) and (3.9).

We are now ready to establish the estimates on $\partial_t u$ and p. Indeed, since $\operatorname{div} u = 0$, we get

$$\|(-\Delta)^{\frac{j}{2}}(\partial_t u + \nabla p)\|^2_{\mathbf{L}^2} = \|\nabla^j\partial_t u\|^2_{\mathbf{L}^2} + \|\nabla^{j+1}p\|^2_{\mathbf{L}^2} \tag{3.21}$$

for all j satisfying $0 \leq j \leq k+1$. From (3.10) it follows that

$$\begin{gathered}\frac{1}{2}\|(-\Delta)^{\frac{j}{2}}(\partial_t u + \nabla p)\|^2_{L^2(I,\mathbf{L}^2)} \leq \\ \leq \|\nabla^j f\|^2_{L^2(I,\mathbf{L}^2)} + \mu\,\|\nabla^{j+2}u\|^2_{L^2(I,\mathbf{L}^2)} + \|(-\Delta)^{\frac{j}{2}}\mathbf{D}u\|^2_{L^2(I,\mathbf{L}^2)}\end{gathered} \tag{3.22}$$

for all $0 \leq j \leq k+1$. Therefore, the third estimate of (3.9) follows from the first and second estimates of (3.9), (3.21) and (3.22), showing the lemma. □

Clearly, we may obtain additional information on $\partial_t u$ and p.

Lemma 10. *Under the hypotheses of Lemma 9, for all* $0 \leq j \leq k$

$$\begin{gathered}\|\nabla^j\mathbf{D}u\|_{C(I,\mathbf{L}^2)} \leq c_j((f,u_0),u), \\ \|\nabla^j\partial_t u\|^2_{C(I,\mathbf{L}^2)} + \|\nabla^{j+1}p\|^2_{C(I,\mathbf{L}^2)} \leq c_j((f,u_0),u)\end{gathered} \tag{3.23}$$

with a positive constant $c_j((f,u_0),u)$ *depending on the norms* $\|(f,u_0)\|_{0,\mu,T},\dots,\|(f,u_0)\|_{k+2,\mu,T}$, $\|\nabla^j f\|_{C(I,\mathbf{L}^2)}$ *and* $\|u\|_{L^{\mathfrak{s}}(I,\mathbf{L}^{\mathfrak{r}})}$.

P r o o f. Using (3.10), we get

$$\begin{gathered}\sup_{t\in[0,T]}\|(-\Delta)^{\frac{j}{2}}(\partial_t u + \nabla p)(\cdot,t)\|^2_{\mathbf{L}^2} \leq \sup_{t\in[0,T]}\|(-\Delta)^{\frac{j}{2}}(f + \mu\Delta u + \mathbf{D}u)(\cdot,t)\|^2_{\mathbf{L}^2} \leq \\ \leq 2\sup_{t\in[0,T]}\left(\|\nabla^j f(\cdot,t)\|^2_{\mathbf{L}^2} + \|\nabla^{j+2}u(\cdot,t)\|^2_{\mathbf{L}^2} + \|\nabla^j\mathbf{D}u(\cdot,t)\|^2_{\mathbf{L}^2}\right)\end{gathered} \tag{3.24}$$

for all $0 \leq j \leq k$. The first two summands in the last line of (3.24) can be estimated via the data (f,u_0) and $\|u\|_{L^{\mathfrak{s}}(I,\mathbf{L}^{\mathfrak{r}})}$ using Lemma 9.

On applying Lemma 8 to the third summand in (3.24) we see that

$$\begin{aligned} \|\nabla^j \mathbf{D}u\|^2_{C(I,\mathbf{L}^2)} \le \|\nabla^{j+2}u\|^2_{C(I,\mathbf{L}^2)} + c(j,\mathfrak{s},\mathfrak{r},\varepsilon{=}1)\, \|u\|^{\mathfrak{s}}_{C(I,\mathbf{L}^{\mathfrak{r}})} \|\nabla^{j+1}u\|^2_{C(I,\mathbf{L}^2)} + \\ + c(j,\mathfrak{s},\mathfrak{r})\, \|u\|^2_{C(I,\mathbf{L}^2)} \|u\|^2_{C(I,\mathbf{L}^{\mathfrak{r}})} + c(j,\mathfrak{s},\mathfrak{r})\, \|u\|^2_{C(I,\mathbf{L}^2)} \end{aligned} \tag{3.25}$$

for all $0 \le j \le k$, the constants being independent of u. On the other hand, we may use the Sobolev embedding theorem (see, for instance, [1, Ch. 4, Theorem 4.12] or (1.5)) to conclude that for any $\lambda \in [0, 1/2)$ there exists a constant $c(\lambda)$ independent of u and t, such that

$$\|u(\cdot,t)\|_{\mathbf{C}^{0,\lambda}} \le c(\lambda)\, \|u(\cdot,t)\|_{\mathbf{H}^2}$$

for all $t \in [0,T]$. Then energy estimate (2.6) and Lemma 9 imply immediately

$$\sup_{t\in[0,T]} \|u(\cdot,t)\|_{\mathbf{C}^{0,\lambda}} \le c((f,u_0),u), \tag{3.26}$$

where the constant $c((f,u_0),u)$ depends on $\|(f,u_0)\|_{j',\mu,T}$ with $j' = 0,1,2$ and $\|u\|_{L^{\mathfrak{s}}(I,\mathbf{L}^{\mathfrak{r}})}$, if inequality (1.5) is fulfilled. In particular,

$$\|u\|^{\mathfrak{s}}_{C(I,\mathbf{L}^{\mathfrak{r}})} \le T \ell^{\frac{3\mathfrak{s}}{\mathfrak{r}}} \sup_{t\in[0,T]} \|u(\cdot,t)\|^{\mathfrak{s}}_{\mathbf{C}} \le T \ell^{\frac{3\mathfrak{s}}{\mathfrak{r}}} c((f,u_0),u) \tag{3.27}$$

with constant $c((f,u_0),u)$ from (3.26). Hence, the first estimate of (3.23) is fulfilled.

At this point Lemma 9 and (3.24), (3.25) and (3.27) allow us to conclude that

$$\sup_{t\in[0,T]} \|(-\Delta)^{\frac{j}{2}}(\partial_t u + \nabla p)(\cdot,t)\|^2_{\mathbf{L}^2} \le c(j,(f,u_0),u) \tag{3.28}$$

for all $j = 0,1,\dots,k$, where $c(j,(f,u_0),u)$ is a positive constant depending on $\|(f,u_0)\|_{j',\mu,T}$ with $0 \le j' \le k+2$, $\|u\|_{L^{\mathfrak{s}}(I,\mathbf{L}^{\mathfrak{r}})}$ and T. Hence, the second estimate of (3.23) follows from (3.21) and (3.28). □

Our next objective is to evaluate the derivatives of u and p with respect to x and t.

Lemma 11. *Suppose that $s \in \mathbb{N}$, $k \in \mathbb{Z}_+$ and $\mathfrak{s}$, $\mathfrak{r}$ satisfy $2/\mathfrak{s} + 3/\mathfrak{r} = 1$. If $(u,p) \in B^{k,2s,s}_{\mathrm{vel}}(I) \times B^{k+1,2(s-1),s-1}_{\mathrm{pre}}(I)$ is a solution to the Navier-Stokes equations of (0.1) with data $(f,u_0) \in B^{k,2(s-1),s-1}_{\mathrm{for}}(I) \times V_{k+2s}$ then it is subjected to an estimate of the form*

$$\|(u,p)\|_{B^{k,2s,s}_{\mathrm{vel}}(I)\times B^{k+1,2(s-1),s-1}_{\mathrm{pre}}(I)} \le c(k,s,(f,u_0),u), \tag{3.29}$$

the constant on the right-hand side depending on $\|f\|_{B^{k,2(s-1),s-1}_{\mathrm{for}}(I)}$, $\|u_0\|_{V_{2s+k}}$ and $\|u\|_{L^{\mathfrak{s}}(I,\mathbf{L}^{\mathfrak{r}})}$ as well as on $\mathfrak{r}$, T, μ, etc.

P r o o f. For $s = 1$ and any $k \in \mathbb{Z}_+$, the statement of the lemma was proved in Lemmata 9 and 10. Then the statement follows by induction with respect to s from the recurrent formulas

$$\begin{aligned} \partial^\alpha \partial_t^j (\partial_t u + \nabla p) &= \partial^\alpha \partial_t^j (f + \mu \Delta u - \mathbf{D}u), \\ \|\partial^\alpha \partial_t^j (\partial_t u + \nabla p)\|^2_{\mathbf{L}^2} &= \|\partial^\alpha \partial_t^{j+1} u\|^2_{\mathbf{L}^2} + \|\partial^\alpha \partial_t^j \nabla p\|^2_{\mathbf{L}^2} \end{aligned} \tag{3.30}$$

provided that $\operatorname{div} u = 0$ and $j \in \mathbb{Z}_+$, $\alpha \in \mathbb{Z}^n_+$ are fit for the assumptions.

Indeed, suppose the assertion of the lemma is valid for $s = s_0$ and any $k \in \mathbb{Z}_+$. We then prove that it is fulfilled for $s = s_0 + 1$ and any $k \in \mathbb{Z}_+$. As $(u,p) \in B^{k,2(s_0+1),s_0+1}_{\mathrm{vel}}(I) \times B^{k+1,2s_0,s_0}_{\mathrm{pre}}(I)$, $(f,u_0) \in B^{k,2s_0,s_0}_{\mathrm{for}}(I) \times V_{2(s_0+1)+k}$, then, by the definition of the spaces, $(u,p) \in B^{k+2,2s_0,s_0}_{\mathrm{vel}}(I) \times B^{k+3,2(s_0-1),s_0-1}_{\mathrm{pre}}(I)$, $(f,u_0) \in B^{k+2,2(s_0-1),s_0-1}_{\mathrm{for}}(I) \times V_{2s_0+(k+2)}$. Thus, by the induction assumption,

$$\|(u,p)\|_{B^{k+2,2s_0,s_0}_{\mathrm{vel}}(I)\times B^{k+1,2(s_0-1),s_0-1}_{\mathrm{pre}}(I)} \le c(k,s_0,(f,u_0),u), \tag{3.31}$$

where the properties of the constant $c(k, s_0, (f, u_0), u)$ are similar to those described in the statement of the lemma.

It follows from the first equality of (3.30) that for all suitable j we get

$$\begin{aligned} \|\nabla^j \partial_t^{s_0} (\partial_t u + \nabla p)\|^2_{\mathbf{L}^2} &= \|\nabla^j \partial_t^{s_0} (f + \mu \Delta u - \mathbf{D} u)\|^2_{\mathbf{L}^2} \leq \\ &\leq 2 \left(\|\nabla^j \partial_t^{s_0} f\|^2_{\mathbf{L}^2} + \mu \,\|\nabla^{j+2} \partial_t^{s_0} u\|^2_{\mathbf{L}^2} + \|\nabla^j \partial_t^{s_0} \mathbf{D} u\|^2_{\mathbf{L}^2}\right). \end{aligned} \tag{3.32}$$

By the induction assumption, if $0 \leq j \leq k+1$ and $0 \leq i \leq k$, then the norms $\|\nabla^j \partial_t^{s_0} f\|^2_{L^2(I,\mathbf{L}^2)}$ and $\|\nabla^i \partial_t^{s_0} f\|^2_{C(I,\mathbf{L}^2)}$ are finite and

$$\|\nabla^{j+2} \partial_t^{s_0} u\|^2_{L^2(I,\mathbf{L}^2)} \leq c\, \|u\|^2_{B^{k+2,2s_0,s_0}_{\mathrm{vel}}(I)}, \qquad \|\nabla^{i+2} \partial_t^{s_0} u\|^2_{C(I,\mathbf{L}^2)} \leq c\, \|u\|^2_{B^{k+2,2s_0,s_0}_{\mathrm{vel}}(I)} \tag{3.33}$$

with constants c independent of u and not necessarily the same in diverse applications. Besides, (2.1) with $w = u$ yields

$$\|\nabla^j \partial_t^{s_0} \mathbf{D} u\|^2_{L^2(I,\mathbf{L}^2)} \leq c\, \|u\|^4_{B^{k+2,2s_0,s_0}_{\mathrm{vel}}(I)}, \qquad \|\nabla^i \partial_t^{s_0} \mathbf{D} u\|^2_{C(I,\mathbf{L}^2)} \leq c\, \|u\|^4_{B^{k+2,2s_0,s_0}_{\mathrm{vel}}(I)} \tag{3.34}$$

provided $0 \leq j \leq k+1$ and $0 \leq i \leq k$, the constants being independent of u.

Finally, combining (3.31), (3.32), (3.33), (3.34) with the second equality of (3.30), we conclude that

$$\|(u,p)\|_{B^{k,2(s_0+1),s_0+1}_{\mathrm{vel}}(I) \times B^{k+1,2s_0,s_0}_{\mathrm{pre}}(I)} \leq c(k, s_0+1, (f, u_0), u),$$

where the constant on the right-hand side depends on $\|f\|_{B^{k,2s_0,s_0}_{\mathrm{for}}(I)}$, $\|u_0\|_{V_{2(s_0+1)+k}}$ and $\|u\|_{L^{\mathfrak{s}}(I,\mathbf{L}^{\mathfrak{r}})}$ as well as on $\mathfrak{r}$, T, μ, etc. This proves the lemma. □

Keeping in mind Corollary 1, we are now in a position to show that the range of mapping (2.7) is closed if given subset $S = S_{\mathrm{vel}} \times S_{\mathrm{pre}}$ of the product $B^{k,2s,s}_{\mathrm{vel}}(I) \times B^{k+1,2(s-1),s-1}_{\mathrm{pre}}(I)$ such that the image $\mathcal{A}(S)$ is precompact in the space $B^{k,2(s-1),s-1}_{\mathrm{for}}(I) \times V_{2s+k}$, the set S_{vel} is bounded in the space $L^{\mathfrak{s}}(I, \mathbf{L}^{\mathfrak{r}})$ with a pair $\mathfrak{s}$, $\mathfrak{r}$ satisfying $2/\mathfrak{s} + 3/\mathfrak{r} = 1$.

Let a pair $(f, u_0) \in B^{k,2(s-1),s-1}_{\mathrm{for}}(I) \times V_{2s+k}$ belongs to the closure of the range of values of the mapping $\mathcal{A}$. Then there is a sequence $\{(u_i, p_i)\}$ in $B^{k,2s,s}_{\mathrm{vel}}(I) \times B^{k+1,2(s-1),s-1}_{\mathrm{pre}}(I)$ such that the sequence $\{(f_i, u_{i,0}) = \mathcal{A}(u_i, p_i)\}$ converges to (f, u_0) in the space $B^{k,2(s-1),s-1}_{\mathrm{for}}(I) \times V_{2s+k}$.

Consider the set $S = \{(u_i, p_i)\}$. As the image $\mathcal{A}(S) = \{(f_i, u_{i,0})\}$ is precompact in $B^{k,2(s-1),s-1}_{\mathrm{for}}(I) \times V_{2s+k}$, it follows from our assumption that the subset $S_{\mathrm{vel}} = \{u_i\}$ of $B^{k,2s,s}_{\mathrm{vel}}(I)$ is bounded in the space $L^{\mathfrak{s}}(I, \mathbf{L}^{\mathfrak{r}})$.

Applying Lemmata 7 and 11 we conclude that the sequence $\{(u_i, p_i)\}$ is bounded in the space $B^{k,2s,s}_{\mathrm{vel}}(I) \times B^{k+1,2(s-1),s-1}_{\mathrm{pre}}(I)$. By the definition of $B^{k,2s,s}_{\mathrm{vel}}(I)$, the sequence $\{u_i\}$ is bounded in $C(I, \mathbf{H}^{k+2s})$ and $L^2(I, \mathbf{H}^{k+2s+1})$, and the partial derivatives $\{\partial_t^j u_i\}$ in time with $1 \leq j \leq s$ are bounded in $C(I, \mathbf{H}^{k+2(s-j)})$ and $L^2(I, \mathbf{H}^{k+2(s-j+1)})$. Therefore, there is a subsequence $\{u_{i_k}\}$ such that:

(1) the sequence $\{\partial_x^{\alpha+\beta} \partial_t^j u_{i_k}\}$ converges weakly in $L^2(I, \mathbf{L}^2)$ provided that $|\alpha| + 2j \leq 2s$ and $|\beta| \leq k+1$;

(2) the sequence $\{\partial_x^{\alpha+\beta} \partial_t^j u_{i_k}\}$ converges weakly-* in $L^\infty(I, \mathbf{L}^2)$ provided that $|\alpha| + 2j \leq 2s$ and $|\beta| \leq k$.

It is clear that the limit u of $\{u_{i_k}\}$ is a solution to the Navier–Stokes equations (0.1) such that:

(1) each derivative $\partial_x^{\alpha+\beta} \partial_t^j u$ belongs to $L^2(I, V_0)$ provided that $|\alpha| + 2j \leq 2s$ and $|\beta| \leq k+1$;

(2) each derivative $\partial_x^{\alpha+\beta} \partial_t^j u$ belongs to $L^\infty(I, V_0)$ provided that $|\alpha| + 2j \leq 2s$ and $|\beta| \leq k$.

As we already mentioned, such a strong solution is unique, see, for instance, [30, Remark 3.1] or [29, Theorem III.3.9]. In addition, if

$$0 \le j \le s-1, \quad |\alpha| + 2j \le 2s, \quad |\beta| \le k, \tag{3.35}$$

then $\partial_x^{\alpha+\beta}\partial_t^j u \in L^2(I, V_1)$ and $\partial_x^{\alpha+\beta}\partial_t^{j+1} u \in L^2(I, V_1')$. Applying Lemma 5 we readily conclude that $\partial_x^{\alpha+\beta}\partial_t^j u \in C(I, V_0)$ for all j and α, β satisfying (3.35). Hence it follows that u belongs to the space $B^{k+2,2(s-1),s-1}_{\mathrm{vel}}(I)$. Moreover, using formula (2.1) with $w = u$ implies that the derivatives $\partial_x^{\alpha+\beta}\partial_t^j \mathbf{D}u$ belong to $C(I, \mathbf{L}^2)$ for all j and α, β which satisfy inequalities (3.35).

Besides, the operator $\mathbf{P}$ maps $C(I, \mathbf{L}^2)$ continuously into $C(I, \mathbf{L}^2)$. Therefore, since u is a solution to (0.1) we deduce that

$$\partial_x^\beta \partial_t^s u = \partial_x^\beta \partial_t^{s-1} \mu \Delta u - \partial_x^\beta \partial_t^{s-1} \mathbf{P}\mathbf{D}u + \partial_x^\beta \partial_t^{s-1} \mathbf{P} f$$

belongs to $C(I, V_0)$ for all multi-indices β such that $|\beta| \le k$. In other words, $u \in B^{k,2s,s}_{\mathrm{vel}}(I)$. Finally, applying Proposition 2, we conclude that there is $p \in B^{k+1,2(s-1),s-1}_{\mathrm{pre}}(I)$ such that

$$\nabla p = (I - \mathbf{P})(f - \mathbf{D}u),$$

i. e., the pair $(u, p) \in B^{k,2s,s}_{\mathrm{vel}}(I) \times B^{k+1,2(s-1),s-1}_{\mathrm{pre}}(I)$ and it is a solution to (0.1).

Thus, we have proved that the image of the mapping in (2.7) is closed. Then the statement of the theorem related to the surjectivity of the mapping follows from Corollary 1. □

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Прообраз предкомпактных множеств и регулярные решения уравнений Навье–Стокса

Ключевые слова: уравнения Навье–Стокса, регулярные решения.

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Рассматривается задача Коши для уравнений Навье–Стокса над полосой $\mathbb{R}^3 \times [0, T]$ с временем $T > 0$ в пространственно-периодической постановке. Доказывается, что задача индуцирует открытые инъективные отображения $\mathcal{A}_s \colon B_1^s \to B_2^{s-1}$, где B_1^s, B_2^{s-1} суть элементы шкал специально построенных функциональных пространств Бохнера–Соболева, параметризованных индексом гладкости $s \in \mathbb{N}$. Наконец, мы доказываем, что отображение $\mathcal{A}_s$ сюръективно тогда и только тогда, когда прообраз $\mathcal{A}_s^{-1}(K)$ любого предкомпактного множества K из образа отображения $\mathcal{A}_s$ ограничен в пространстве Бохнера $L^{\mathfrak{s}}([0,T], L^{\mathfrak{r}}(\mathbb{T}^3))$ с показателями Ладыженской–Проди–Серрина $\mathfrak{s}$, $\mathfrak{r}$.

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