VESTNIK UDMURTSKOGO UNIVERSITETA. MATEMATIKA. MEKHANIKA. KOMP'UTERNYE NAUKI
MATHEMATICS 2022. Vol. 32. Issue 2. Pp. 153–170.

MSC2020: 34L25, 35Q41, 37K10, 35R30, 34M46

© ***B. A. Babajanov, A. Sh. Azamatov***

INTEGRATION OF THE KAUP–BOUSSINESQ SYSTEM WITH A SELF-CONSISTENT SOURCE VIA INVERSE SCATTERING METHOD

In this study we consider the Kaup–Boussinesq system with a self-consistent source. We show that the Kaup–Boussinesq system with a self-consistent source can be integrated by the method of inverse scattering theory. For a solving the problem under consideration, we use the direct and inverse scattering problem of the Sturm–Liouville equation with an energy-dependent potential. The time evolution of the scattering data for the Sturm–Liouville equation with an energy-dependent potentials associated with the solution of the Kaup–Boussinesq system with a self-consistent source is determined. The obtained equalities completely determine the scattering data for any t, which makes it possible to apply the method of the inverse scattering problem to solve the Cauchy problem for the Kaup–Boussinesq system with a self-consistent source.

Keywords: nonlinear soliton equation, Kaup–Boussinesq system, self-consistent source, inverse scattering method, quadratic pencil of Sturm–Liouville equations.

DOI: 10.35634/vm220201

Introduction

The Kaup–Boussinesq system describing wave propagation in shallow water was first derived by Boussinesq [1]. In [2], D. J. Kaup proved that this system is completely integrable. The next important step was taken in [3], in which complex finite-gap multiphase solutions of this system were obtained, expressed in terms of the Riemann theta functions, multisoliton solutions were found, and the asymptotic behavior of these solutions was investigated. In papers [4, 5], real finite-zone regular solutions of the Kaup–Boussinesq system are studied. In general, it is difficult to study the exact dynamical behavior for nonlinear evolution problems. Therefore, different techniques have been used to construct its solutions, such as inverse scattering method [6–9], Hirota bilinear method [10], Lie group analysis method [11], trial equation method [12], first integral method [13], bifurcation method [14], etc.

Soliton equations with self-consistent sources were proposed by V. K. Melnikov [15]. In such models, the sources may change some properties of the physical system, for example, the velocities of the solitons. These equations have essential applications in plasma physics, hydrodynamics, solid state physics, etc. For instance, the Korteweg–de Vries equation, which has an integral type self-consistent source, was studied in [16]. Via this kind of equation, the interaction of long and short capillary-gravity waves can be defined [17]. Another major soliton equations with self-consistent source are the nonlinear Schrödinger equations with self-consistent sources which identify the nonlinear relation of an ion acoustic wave in the two-component homogeneous plasma with the electrostatic high frequency wave [18]. Various methods have been applied to formulate their solutions, such as inverse scattering [19–29], Darboux transformation [30–34] or Hirota bilinear methods [35–39]. In papers [40, 41], the Kaup–Boussinesq system and its hierarchy with a self-consistent source were studied in the class of periodic functions and real periodic infinite-zone solutions of the these system are obtained expressed in the form of uniformly converging functional series.

In the present paper, we study the integration of the Kaup–Boussinesq system with a self-consistent source via the inverse scattering method in the class of rapidly decreasing functions.

The time evolution of the scattering data for the Sturm–Liouville equation with an energy-dependent potential associated with the solution of the Kaup–Boussinesq system with a self-consistent source is determined. Using the solution of the inverse scattering problem with respect to the time-dependent scattering data we construct the solution of the Kaup–Boussinesq system with a self-consistent source. We show that the Kaup–Boussinesq system with a self-consistent source is also an important theoretical model as it is a completely integrable system. The inverse scattering problem for the Sturm–Liouville equation with an energy-dependent potential in the class of "rapidly decreasing" functions was solved in the works [42–49].

§ 1. Formulation of the problem

We consider the Kaup–Boussinesq system with a self-consistent source

$$\begin{cases} v_t = u_{xxx} - 4vu_x - 2uv_x + 2\sum\limits_{m=1}^{N}\left[-u_x\phi_m^2 + (k_m - 2u)\dfrac{\partial}{\partial x}\phi_m^2\right], \\ u_t = -6uu_x - v_x + \sum\limits_{m=1}^{N}\dfrac{\partial}{\partial x}\phi_m^2, \qquad x \in \mathbb{R}, \quad t > 0, \\ (\phi_m)_{xx} + [k_m^2 - v - 2k_m u]\,\phi_m = 0, \qquad m = 1, 2, \ldots, N, \end{cases} \tag{1.1}$$

under the initial condition

$$v(x,t)|_{t=0} = v_0(x), \qquad u(x,t)|_{t=0} = u_0(x), \quad x \in \mathbb{R}, \tag{1.2}$$

and the normalizing conditions

$$\int_{-\infty}^{\infty}(2k_m - 2u)\phi_m^2\,dx = A_m(t), \qquad m = 1, 2, \ldots, N, \tag{1.3}$$

where $\phi_m = \phi_m(x,t)$ is an eigenfunction corresponding to the eigenvalue $k_m = k_m(t)$, $\operatorname{Im} k_m < 0$ of the Sturm–Liouville equation with an energy-dependent potential

$$T(t,k)y \equiv -y_{xx} + v(x,t)y + 2ku(x,t)y = k^2 y, \qquad x \in \mathbb{R}. \tag{1.4}$$

Moreover, $A_m(t)$ are given arbitrary continuous functions for all $m \in \{1, 2, \ldots, N\}$ and the functions $v_0(x)$, $u_0(x)$ satisfy the following conditions:

(i) where $v_0(x)$, $u_0'(x)$ are continuously differentiable complex valued functions on $\mathbb{R}$ and $u_0(x) \to 0$ when $|x| \to \infty$, moreover, the inequalities hold

$$\int_{-\infty}^{\infty} x^2\,[|v_0(x)| + |u_0'(x)|]\,dx < \infty, \qquad \int_{-\infty}^{\infty} |x|\,[|v_0'(x)| + |u_0''(x)|]\,dx < \infty, \tag{1.5}$$

(ii) the operator generated by the differential expression

$$T(0,k) := -\frac{d^2}{dx^2} + v_0(x) + 2ku_0(x)$$

has only simple eigenvalues.

The main aim of this work is to derive representations for the solutions

$$\{v(x,t),\ u(x,t),\ \phi_1(x,t),\ \phi_2(x,t), \ldots, \phi_N(x,t)\}$$

of the problem (1.1)–(1.5) with the use of the inverse scattering method for the operator $T(t,k)$.

§ 2. The basic facts from scattering problem

In this section we give basic information about the scattering theory for the Sturm–Liouville equation with an energy-dependent potential [6, 42–44]. For convenience, we momentarily omit the dependence of the functions $v(x,t)$ and $u(x,t)$ on variable t.

Consider the differential equation with an energy-dependent potential $V(x,k) = v(x) + 2ku(x)$:

$$y'' + (k^2 - V)y = 0, \quad x \in \mathbb{R}, \tag{2.1}$$

where $v(x)$, $u'(x)$ are continuously differentiable complex valued functions on $\mathbb{R}$ and $u(x) \to 0$ as $|x| \to \infty$, moreover, the inequalities hold:

$$\int_{-\infty}^{\infty} x^2 \left[|v(x)| + |u'(x)|\right] dx < \infty, \qquad \int_{-\infty}^{\infty} |x| \left[|v'(x)| + |u''(x)|\right] dx < \infty. \tag{2.2}$$

Under condition (2.2), equation (2.1) for all $k \in \mathbb{C}$ has Jost solutions $\{f_1(x,k), g_1(x,k)\}$ and $\{f_2(x,k), g_2(x,k)\}$ which satisfy the conditions

$$[f_1(x,k), g_1(x,k)] \sim \left[e^{-ikx}, e^{ikx}\right], \quad x \to \infty, \tag{2.3}$$

$$[f_2(x,k), g_2(x,k)] \sim \left[e^{ikx}, e^{-ikx}\right], \quad x \to -\infty. \tag{2.4}$$

Obviously, for each $x \in (-\infty, \infty)$ the functions $f_1(x,k)$, $f_2(x,k)$ $(g_1(x,k), g_2(x,k))$ are regular in the half-plane $\operatorname{Im} k < 0$ $(\operatorname{Im} k > 0)$. For real $k \neq 0$, the pairs $\{f_1(x,k), g_1(x,k)\}$ and $\{f_2(x,k), g_2(x,k)\}$ form two fundamental systems of solutions to equation (2.1). The following relations hold:

$$f_2 = c_{11} f_1 + c_{12} g_1, \qquad g_2 = d_{12} f_1 + d_{11} g_1, \tag{2.5}$$

$$f_1 = c_{22} f_2 + c_{21} g_2, \qquad g_1 = d_{21} f_2 + d_{22} g_2, \tag{2.6}$$

where the functions c_{ij} and d_{ij} are independent of x and the following equality is fulfilled

$$c_{12} = c_{21} = (2ik)^{-1} W[f_1, f_2], \qquad c_{11} = -d_{22} = (2ik)^{-1} W[f_2, g_1], \tag{2.7}$$

$$d_{12} = d_{21} = (2ik)^{-1} W[g_2, g_1], \qquad d_{11} = -c_{22} = (2ik)^{-1} W[f_1, g_2]. \tag{2.8}$$

Moreover, $c_{11}(k)$ $(k \in \mathbb{R}^*)$ and $c_{21}(k)$ $(\operatorname{Im} k \leq 0,\ k \neq 0)$ are continuous while $c_{21}(k)$ $(\operatorname{Im} k < 0)$ is analytic. In (2.5), $W[f,g]$ is a Wronskian of functions f and g.

To keep all the interesting analytic properties in $\operatorname{Im} k < 0$ we can take the point of view that (1.1) is a pair of equations $(1.1)^{\pm}$ having potentials $V^{\pm}(x,k) = V(x, \pm k)$. This corresponds to [34]. Now all the above equations can be understood to have superscripts "±" and the Jost solutions are related by $g_j^{\pm}(x,k) = f_j^{\mp}(x,-k)$, $j = 1, 2$.

We now make a further somewhat technical "bound state" assumption: $c_{21}^{\pm}(k) \neq 0$ $(k \in \mathbb{R}^*)$; $[c_{21}^{\pm}(k)]^{-1} = O(1)$, $|k| \to 0$, $k \in \mathbb{R}$, and the zeros of $c_{21}^{\pm}(k)$ $(\operatorname{Im} k < 0)$ are simple.

One can now prove that $c_{21}^{\pm}(k)$ $(\operatorname{Im} k < 0)$ each have a finite number of zeros $N^{\pm}$, located at the points $k = k_n^{\pm}$, $n = 1, 2, \ldots, N^{\pm}$, and the following equality holds:

$$f_1^{\pm}(x, k_n^{\pm}) = B_n^{\pm} f_2^{\pm}(x, k_n^{\pm}), \tag{2.9}$$

here the quantities $B_n^{\pm}$ are independent of x. The corresponding functions $f_1^{\pm}(x, k_n^{\pm})$ are the only $L^2(\mathbb{R})$ solutions of $(1.1)^{\pm}$ for $\operatorname{Im} k < 0$ and are the "bound states". It is clear that

$$\phi_n(x) = D_n^{+} f_2^{+}(x, k_n^{+}), \qquad n = 1, 2, \ldots, N^{+}, \tag{2.10}$$

where the quantities D_n^+ are independent of x. It follows from (2.10) that $k_n = k_n^+$, $N^+ = N$.

The set of the quantities

$$\left\{R^\pm(k) = \frac{c_{11}^\pm(-k)}{c_{21}^\mp(-k)},\ k \in \mathbb{R}^*,\ k_n^\pm,\ C_n^\pm,\ n = 1, 2, \ldots, N^\pm\right\}$$

is called the left and right scattering data of equation (2.1), where

$$C_n^\pm = \left[c_{11}^\pm\left(k_n^\pm\right)\right]^{-1}\left[i\frac{d}{dk}c_{21}^\pm(k)\right]_{k=k_n^\pm}.$$

Now we show how to construct v and u from scattering data of equation (2.1).

Using the methods of [42] we find $F^\pm(x) = \exp\left(\mp\frac{i}{2}\int_x^\infty u(y)\,dy\right)$, i. e.,

$$u(x) = \mp 2i(\ln F^\pm(x))_x \tag{2.11}$$

and

$$v(x) = h^\pm(x)(F^\pm(x))^{-1}, \tag{2.12}$$

where

$$h^\pm(x) = \left(F^\pm(x)\right)_{xx} - 2\left(A^\pm(x,x)\right)_x + 2A^\pm(x,x)\left(\ln F^\pm(x)\right)_x. \tag{2.13}$$

The function $(F^+(x), F^-(x), A^+(x,y), A^-(x,y))$ ($x \in \mathbb{R}$, $y \geq x$) is a solution of the following system:

$$h^+(x)F^+(x) = h^-(x)F^+(x), \qquad F^+(x)F^-(x) = 1, \tag{2.14}$$

$$A^\pm(x,y) = F^\mp(x)r^\pm(x+y) + \int_x^\infty r^\pm(y+s)\,A^\mp(x,s)\,ds, \tag{2.15}$$

$$r^\pm(x) = \sum_n \left(C_n^\mp\right)^{-1} e^{-ik_n^\mp x} - \frac{1}{2\pi}\int_{-\infty}^\infty e^{ikx}R^\pm(k)\,dk. \tag{2.16}$$

In order to have a unique solution for the system (2.14)–(2.16), we add the following condition. For any $x \in \mathbb{R}$,

$$a^\pm(y) = \int_x^\infty r^\pm(y+s)a^\mp(s)\,ds \Longrightarrow \left(a^+(y), a^-(y)\right) = (0,0) \qquad (y \geq x). \tag{2.17}$$

To solve the system (2.14)–(2.16) we then proceed as follows. For fixed $F^+(x)$ and $F^-(x)$ equations (2.15) have a unique solution because of (2.17) and it can be expressed as

$$A^\pm(x,y) = F^\mp(x)\alpha^\pm(x,y) + F^\pm(x)\beta^\mp(x,y), \quad y \geq x, \quad x \in \mathbb{R}, \tag{2.18}$$

with $\alpha^\pm$ and $\beta^\mp$ well determined. Inserting this representation into (2.15) we obtain the following equation for $z(x)$ where $F^\pm(x) = \exp(\mp iz(x))$:

$$z_x = 2i\alpha^+(x,x)e^{iz} - 2i\alpha^-(x,x)e^{-iz} - 2i\beta^+(x,x) + 2i\beta^-(x,x), \quad z(\infty) = 0. \tag{2.19}$$

Equation (2.19) has a unique bounded solution which can be obtained by a method of successive approximations. From $z(x)$ we obtain $F^\pm(x)$ and then $A^\pm(x,y)$ from (2.18).

It is easy to see that the following statement is true.

Lemma 1. *If $y(x,\lambda)$ and $z(x,\mu)$ are solutions of equations $T(\lambda)y = \lambda^2 y$ and $T(\mu)z = \mu^2 z$. Then the identity*

$$(\lambda + \mu - 2u)yz = \frac{(yz' - y'z)'}{\lambda - \mu} \tag{2.20}$$

holds.

The Lemma 1 is proved by direct verification.

§ 3. Evolution of the scattering data

In this section we derive time evolution of the scattering data which allows us to provide the algorithm for solution of the problem (1.1)–(1.3).

We set

$$U=\begin{pmatrix} v\\ u\end{pmatrix},\ G=\begin{pmatrix} G_1\\ G_2\end{pmatrix},\ G_1=2\sum_{m=1}^{N}\left[-u_x\phi_m^2+(k_m-2u)\frac{\partial}{\partial x}\phi_m^2\right],\ G_2=\sum_{m=1}^{N}\frac{\partial}{\partial x}\phi_m^2, \tag{3.1}$$

$$L^*=\begin{pmatrix} 0 & -\frac{\partial^2}{\partial x^2}+4v-2v_x\int_x^\infty d\tau\\ 1 & 4u-2u_x\int_x^\infty d\tau\end{pmatrix}. \tag{3.2}$$

Then the first two equations in (1.1) can be rewritten as follows

$$U_t+L^*U_x=G. \tag{3.3}$$

Now we introduce the "scalar product" notation

$$\langle V(x),W(x)\rangle=\int_{-\infty}^{\infty}[V_1(x)W_1(x)+V_2(x)W_2(x)]\,dx \tag{3.4}$$

for $V(x)=(V_1(x),V_2(x))^T$, and the vector functions

$$\Phi_1^\pm(x,k)=\left(f_1^\pm(x,k)f_2^\pm(x,k),\ \pm 2kf_1^\pm(x,k)f_2^\pm(x,k)\right)^T, \tag{3.5}$$

$$\Phi_2^\pm(x,k)=\left(g_1^\pm(x,k)f_2^\pm(x,k),\ \pm 2kg_1^\pm(x,k)f_2^\pm(x,k)\right)^T. \tag{3.6}$$

Notice that the components are products of Jost solutions.

Lemma 2. *The following equalities hold*

$$-2ik\frac{d}{dt}c_{21}^\pm(t,k)=\left\langle U_t(x,t),\Phi_1^\pm(x,t,k)\right\rangle,\quad \operatorname{Im}k\le 0,\quad k\ne 0, \tag{3.7}$$

$$-2ik\frac{d}{dt}c_{11}^\pm(t,k)=\left\langle U_t(x,t),\Phi_2^\pm(x,t,k)\right\rangle,\quad k\in\mathbb{R}^*. \tag{3.8}$$

P r o o f. Let $y^\pm=y^\pm(x,t,k)$ and $z^\pm=z^\pm(x,t,k)$ are two elements of the set

$$\left\{f_1^\pm(x,t,k),f_2^\pm(x,t,k),g_1^\pm(x,t,k),g_2^\pm(x,t,k)\right\}.$$

For $x\in\mathbb{R}$, we have following equations with $V^\pm=v(x,t)\pm 2ku(x,t)$:

$$y_{xx}^\pm+(k^2-V^\pm)y^\pm=0, \tag{3.9}$$

$$z_{xx}^\pm+(k^2-V^\pm)z^\pm=0. \tag{3.10}$$

Multiplying (3.9) by $z_t^\pm$ and (3.10) by $y_t^\pm$ and then adding yields an equation

$$y_{xx}^\pm z_t^\pm+z_{xx}^\pm y_t^\pm+(k^2-V^\pm)(y^\pm z^\pm)_t=0. \tag{3.11}$$

Differentiate (3.9) (resp. ((3.10)) with respect to t and multiply the result by $z^\pm$ (resp. $y^\pm$), then adding the two results yields

$$y_{xxt}^\pm z^\pm+z_{xxt}^\pm y^\pm+(k^2-V^\pm)(y^\pm z^\pm)_t-2V_t^\pm y^\pm z^\pm=0. \tag{3.12}$$

Subtracting (3.12) from (3.11) gives

$$\frac{\partial}{\partial x}\left[W(z_t^\pm, y^\pm) + W(y_t^\pm, z^\pm)\right] = -2V_t^\pm y^\pm z^\pm. \tag{3.13}$$

Integrating (3.13) in x from $-\infty$ to ∞ we obtain

$$W(z_t^\pm, y^\pm)\big|_{-\infty}^{\infty} + W(y_t^\pm, z^\pm)\big|_{-\infty}^{\infty} = -2\int_{-\infty}^{\infty} V_t^\pm y^\pm z^\pm\, dx. \tag{3.14}$$

Now to obtain (3.7) and (3.8) we use the asymptotics (2.3), (2.4) and their time derivatives to calculate the L. H. S. of (3.14) (with $y^\pm = f_1^\pm(x,t,k)$, $z^\pm = f_2^\pm(x,t,k)$, ($\operatorname{Im} k \le 0$, $k \neq 0$), or $y^\pm = g_1^\pm(x,t,k)$, $z^\pm = f_2^\pm(x,t,k)$, ($k \in \mathbb{R}^*$)). Then using notations (3.5) and (3.6) we get (3.7) and (3.8). □

Lemma 3. *For all t the following equalities hold*

$$0 = \left\langle U_x(x,t), \Phi_1^\pm(x,t,k)\right\rangle, \qquad \operatorname{Im} k \le 0, \quad k \neq 0, \tag{3.15}$$

$$-4k^2 c_{11}^\pm(t,k) = \left\langle U_x(x,t), \Phi_2^\pm(x,t,k)\right\rangle, \qquad \text{for } k \in \mathbb{R}^* \text{ and for } k = k_n^\pm. \tag{3.16}$$

P r o o f. Add equations (3.9) and (3.10), after having multiply them by $z_x^\pm$ and $y_x^\pm$ respectively, to successively obtain

$$(y_x^\pm z_x^\pm + k^2 y^\pm z^\pm)_x = (V^\pm y^\pm z^\pm)_x - V_x^\pm y^\pm z^\pm, \tag{3.17}$$

$$(y_x^\pm z_x^\pm + k^2 y^\pm z^\pm)\big|_{-\infty}^{\infty} = -\int_{-\infty}^{\infty} V_x^\pm y^\pm z^\pm\, dx. \tag{3.18}$$

In (3.18), if we take $y^\pm = f_1^\pm(x,t,k)$, $z^\pm = f_2^\pm(x,t,k)$ ($\operatorname{Im} k \le 0$, $k \neq 0$), or $y^\pm = g_1^\pm(x,t,k)$, $z^\pm = f_2^\pm(x,t,k)$ ($k \in \mathbb{R}^*$, $k = k_n^\pm$) then using (2.3), (2.4), (2.5), (2.6) and notations (3.5), (3.6) we get (3.15) and (3.16). □

Lemma 4. *L, the "adjoint" of L^* defined in (3.2), for fixed t, has the properties*

$$\left\langle U_x(x,t), L\Phi_1^\pm(x,t,k)\right\rangle = \pm 2k\left\langle U_x(x,t), \Phi_1^\pm(x,t,k)\right\rangle, \qquad \operatorname{Im} k \le 0, \quad k \neq 0, \tag{3.19}$$

$$\left\langle U_x(x,t), L\Phi_2^\pm(x,t,k)\right\rangle = \pm 2k\left\langle U_x(x,t), \Phi_2^\pm(x,t,k)\right\rangle, \qquad \operatorname{Im} k \le 0, \quad k \neq 0. \tag{3.20}$$

P r o o f. Add equations (3.9) and (3.10), after multiplying them by $z^\pm$ and $y^\pm$ respectively, to get

$$\left(y^\pm z^\pm\right)_{xx} - 2y_x^\pm z_x^\pm + 2\left[k^2 - V^\pm\right] y^\pm z^\pm = 0. \tag{3.21}$$

Now integrate both (3.17) and (3.21) from $-\infty$ to x and then adding yields the equation

$$\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2} + V^\pm - \frac{1}{2}\int_{-\infty}^{x} V_x^\pm\, dx\right) y^\pm z^\pm = k^2 y^\pm z^\pm + n(k),$$

where $n(k) = -\frac{1}{2}\left(y_x^\pm z_x^\pm + k^2 y^\pm z^\pm\right)\big|_{x=-\infty}$. After introducing these results into the scalar product and taking $y^\pm = f_1^\pm(x,t,k)$, $z^\pm = f_2^\pm(x,t,k)$ or $y^\pm = g_1^\pm(x,t,k)$, $z^\pm = f_2^\pm(x,t,k)$ we obtain (3.19) and (3.20). □

The main result of the paper is included in the theorem below.

Theorem 1. *If the functions $v = v(x,t)$, $u = u(x,t)$ and $\phi_m = \phi_m(x,t)$ are solutions of the problem (1.1)–(1.4), then the scattering data of the operator*

$$T(t, \pm k)y \equiv -y_{xx} + v(x,t)y \pm 2ku(x,t)y = k^2 y, \quad x \in \mathbb{R},$$

depend on t as follows

$$\frac{dR^{\pm}(t,k)}{dt} = \mp 4ik^2 R^{\pm}(t,k), \quad k \in \mathbb{R}, \tag{3.22}$$

$$\frac{dk_n^{\pm}(t)}{dt} = 0, \quad n = 1, 2, \dots, N^{\pm}, \tag{3.23}$$

$$\frac{dC_n^{+}(t)}{dt} = -\left[4i(k_n^{+})^2 + 2ik_n^{+}A_n(t)\right] C_n^{+}(t), \quad n = 1, 2, \dots, N^{+}, \tag{3.24}$$

$$\frac{dC_n^{-}(t)}{dt} = 4i(k_n^{-})^2 C_n^{-}(t), \quad n = 1, 2, \dots, N^{-}. \tag{3.25}$$

P r o o f. From Lemmas 1, 2, 3 and notations (3.1), it follows easily that

$$\frac{d}{dt} c_{21}^{\pm}(t,k) = -(2ik)^{-1} \left\langle U_t + L^* U_x, \Phi_1^{\pm} \right\rangle, \tag{3.26}$$

$$\frac{d}{dt} c_{11}^{\pm}(t,k) \mp 4ik^2 c_{11}^{\pm}(t,k) = -(2ik)^{-1} \left\langle U_t + L^* U_x, \Phi_2^{\pm} \right\rangle. \tag{3.27}$$

If $U(x,t)$ satisfies (3.3), then (3.26) and (3.27) will take the following form

$$\frac{d}{dt} c_{21}^{\pm}(t,k) = -(2ik)^{-1} \left\langle G, \Phi_1^{\pm} \right\rangle, \tag{3.28}$$

$$\frac{d}{dt} c_{11}^{\pm}(t,k) \mp 4ik^2 c_{11}^{\pm}(t,k) = -(2ik)^{-1} \left\langle G, \Phi_2^{\pm} \right\rangle. \tag{3.29}$$

Now, we calculate R. H. S. of (3.28). According to (3.1), R.H.S. of (3.28) can be rewritten as follows

$$\begin{gathered} \left\langle G, \Phi_1^{\pm} \right\rangle = \int_{-\infty}^{\infty} (G_1 f_1^{\pm} f_2^{\pm} \pm 2kG_2 f_1^{\pm} f_2^{\pm})\, dx = \\ = \sum_{m=1}^{N} \left(-2 \int_{-\infty}^{\infty} u_x \phi_m^2 f_1^{\pm} f_2^{\pm}\, dx + 2 \int_{-\infty}^{\infty} (\pm k + k_m - 2u) f_1^{\pm} f_2^{\pm} \frac{\partial}{\partial x} \phi_m^2\, dx \right) = \sum_{m=1}^{N} I_m^{\pm}, \end{gathered} \tag{3.30}$$

where

$$I_m^{\pm} = -2 \int_{-\infty}^{\infty} u_x \phi_m^2 f_1^{\pm} f_2^{\pm}\, dx + 2 \int_{-\infty}^{\infty} (\pm k + k_m - 2u) f_1^{\pm} f_2^{\pm} \frac{\partial}{\partial x} \phi_m^2\, dx.$$

We will calculate I_m^{+}.

$$\begin{gathered} I_m^{+} = -2 \int_{-\infty}^{\infty} u_x \phi_m^2 f_1^{+} f_2^{+}\, dx + \int_{-\infty}^{\infty} (k + k_m - 2u) f_1^{+} f_2^{+} \frac{\partial}{\partial x} \phi_m^2\, dx + \\ + \int_{-\infty}^{\infty} (k + k_m - 2u) f_1^{+} f_2^{+} \frac{\partial}{\partial x} \phi_m^2\, dx = \\ = -\int_{-\infty}^{\infty} (k + k_m - 2u) \phi_m^2 (f_1^{+} f_2^{+})'\, dx + \int_{-\infty}^{\infty} (k + k_m - 2u) f_1^{+} f_2^{+} \frac{\partial}{\partial x} \phi_m^2\, dx = \\ = \int_{-\infty}^{\infty} (k + k_m - 2u) f_1^{+} \phi_m \cdot (f_2^{+} \phi_m' - (f_2^{+})' \phi_m)\, dx + \\ + \int_{-\infty}^{\infty} (k + k_m - 2u) f_2^{+} \phi_m (f_1^{+} \phi_m' - (f_1^{+})' \phi_m)\, dx. \end{gathered} \tag{3.31}$$

In (3.31), using equality (2.20) we get

$$I_m^+ = \frac{1}{k-k_m}\int_{-\infty}^{\infty}\left[\frac{dW\{f_1^+,\ \phi_m\}}{dx}W\{f_2^+,\ \phi_m\} + \frac{dW\{f_2^+,\ \phi_m\}}{dx}W\{f_1^+,\ \phi_m\}\right]dx = \\ = \frac{1}{k-k_m}\left[W\{f_1^+,\phi_m\}W\{f_2^+,\phi_m\}\right]\Big|_{-\infty}^{\infty} = 0. \tag{3.32}$$

Similarly, we obtain

$$I_m^- = \frac{1}{k_m+k}\int_{-\infty}^{\infty}\left[\frac{dW\{f_1^-,\phi_m\}}{dx}W\{f_2^-,\phi_m\} + \frac{dW\{f_2^-,\phi_m\}}{dx}W\{f_1^-,\phi_m\}\right]dx = \\ = \frac{1}{k_m+k}\left[W\{f_1^-,\phi_m\}W\{f_2^-,\phi_m\}\right]\Big|_{-\infty}^{\infty} = 0. \tag{3.33}$$

Putting (3.32) and (3.33) into (3.30), and taking into account (3.28) we obtain

$$\frac{d}{dt}c_{21}^{\pm}(t,k) = 0, \qquad \operatorname{Im} k \le 0, \quad k \ne 0. \tag{3.34}$$

It follows that the zeros $k_n^{\pm} = k_n^{\pm}(t)$ of $c_{21}^{\pm}(t,k)$ are also independent of time, which implies (3.23).

Same as the above calculations, it is easy to see that

$$\frac{d}{dt}c_{11}^{\pm}(t,k) = \pm 4ik^2 c_{11}^{\pm}(t,k). \tag{3.35}$$

From (3.34), (3.35) and view of $R^{\pm}(t,k)$, we obtain (3.22).

Next, we deduce time dependents of $C_n^{\pm}(t)$. For this purpose, we calculate R. H. S. of (3.29) for $k = k_n^{\pm}$:

$$-(2ik_n^{\pm})^{-1}\left\langle G, \Phi_2^{\pm}(x,t,k_n^{\pm})\right\rangle = \\ = -(2ik_n^{\pm})^{-1}\sum_{m=1}^{N}\left(-2\int_{-\infty}^{\infty}u_x\phi_m^2 g_1^{\pm} f_2^{\pm}\,dx + 2\int_{-\infty}^{\infty}(\pm k_n^{\pm} + k_m - 2u)g_1^{\pm} f_2^{\pm}\frac{\partial}{\partial x}\phi_m^2\,dx\right) = \\ = -(2ik_n^{\pm})^{-1}\sum_{m=1}^{N} J_m^{\pm}, \tag{3.36}$$

where

$$J_m^{\pm} = -2\int_{-\infty}^{\infty}u_x\phi_m^2 g_1^{\pm} f_2^{\pm}\,dx + 2\int_{-\infty}^{\infty}(\pm k_n^{\pm} + k_m - 2u)g_1^{\pm} f_2^{\pm}\frac{\partial}{\partial x}\phi_m^2\,dx.$$

We will calculate J_m^{+}. Let $m \ne n$.

$$J_m^{+} = -2\int_{-\infty}^{\infty}u_x\phi_m^2 g_1^{+} f_2^{+}\,dx + 2\int_{-\infty}^{\infty}(k_n^{+} + k_m - 2u)g_1^{+} f_2^{+}\frac{\partial}{\partial x}\phi_m^2\,dx = \\ = \frac{1}{k_n^{+}-k_m}\int_{-\infty}^{\infty}\left[\frac{dW\{g_1^{+},\phi_m\}}{dx}W\{f_2^{+},\phi_m\} + \frac{dW\{f_2^{+},\phi_m\}}{dx}W\{g_1^{+},\phi_m\}\right]dx = \\ = \frac{1}{k_n^{+}-k_m}\left[W\{g_1^{+},\phi_m\}W\{f_2^{+},\phi_m\}\right]\Big|_{-\infty}^{\infty} = 0. \tag{3.37}$$

The same way, we obtain

$$J_m^- = \frac{1}{k_n^- + k_m} \int_{-\infty}^{\infty} \left[\frac{dW\{g_1^-, \phi_m\}}{dx} W\{f_2^-, \phi_m\} + \frac{dW\{f_2^-, \phi_m\}}{dx} W\{g_1^-, \phi_m\} \right] dx = \\ = \frac{1}{k_n^- + k_m} \left[W\{g_1^-, \phi_m\} W\{f_2^-, \phi_m\} \right] \Big|_{-\infty}^{\infty} = 0. \quad (3.38)$$

Let now $m = n$.

$$J_n^+ = -2 \int_{-\infty}^{\infty} u_x \phi_n^2 g_1^+ f_2^+ \, dx + 2 \int_{-\infty}^{\infty} (2k_n - 2u) g_1^+ f_2^+ \frac{\partial}{\partial x} \phi_n^2 \, dx = \\ = -2 \int_{-\infty}^{\infty} u_x \phi_n^2 g_1^+ f_2^+ \, dx + \int_{-\infty}^{\infty} (2k_n - 2u) g_1^+ f_2^+ \frac{\partial}{\partial x} \phi_n^2 \, dx - \\ - \int_{-\infty}^{\infty} (-2u_x) g_1^+ f_2^+ \phi_n^2 \, dx - \int_{-\infty}^{\infty} (2k_n - 2u)(g_1^+ f_2^+)' \phi_n^2 \, dx = \\ = \int_{-\infty}^{\infty} (2k_n - 2u) g_1^+ f_2^+ \frac{\partial}{\partial x} \phi_n^2 \, dx - \int_{-\infty}^{\infty} (2k_n - 2u)(g_1^+ f_2^+)' \phi_n^2 \, dx = \\ = \int_{-\infty}^{\infty} (2k_n - 2u) \phi_n f_2^+ W\{g_1^+, \phi_n\} \, dx + \int_{-\infty}^{\infty} (2k_n - 2u) \phi_n g_1^+ W\{f_2^+, \phi_n\} \, dx.$$

Using (2.9) and (2.10) we deduce

$$J_n^+ = \int_{-\infty}^{\infty} (2k_n - 2u) \phi_n^2 W\{g_1^+, f_2^+\} \, dx = W\{g_1^+, f_2^+\} \int_{-\infty}^{\infty} (2k_n - 2u) \phi_n^2 \, dx. \quad (3.39)$$

Substituting (1.3) and (2.7) into (3.39) we obtain

$$J_n^+ = 2ik_n c_{11}^+(t, k_n) A_n(t). \quad (3.40)$$

Now, we will calculate J_n^-.

$$J_n^- = -2 \int_{-\infty}^{\infty} u_x \phi_n^2 g_1^- f_2^- \, dx + 2 \int_{-\infty}^{\infty} (-k_n^- + k_n - 2u) g_1^- f_2^- \frac{\partial}{\partial x} \phi_n^2 \, dx = \\ = -2 \int_{-\infty}^{\infty} u_x \phi_n^2 g_1^- f_2^- \, dx + \int_{-\infty}^{\infty} (-k_n^- + k_n - 2u) g_1^- f_2^- \frac{\partial}{\partial x} \phi_n^2 \, dx + \\ + \int_{-\infty}^{\infty} (-k_n^- + k_n - 2u) g_1^- f_2^- \frac{\partial}{\partial x} \phi_n^2 \, dx = \\ = \int_{-\infty}^{\infty} (k_n^- + k_n - 2u) g_1^- f_2^- \frac{\partial}{\partial x} \phi_n^2 \, dx - \int_{-\infty}^{\infty} (-k_n^- + k_n - 2u) \phi_n^2 (g_1^- f_2^-)' \, dx = \\ = \int_{-\infty}^{\infty} (-k_n^- + k_n - 2u) g_1^- \phi_n \cdot (f_2^- \phi_n' - (f_2^-)' \phi_n) \, dx + \\ + \int_{-\infty}^{\infty} (-k_n^- + k_n - 2u) f_2^- \phi_n (g_1^- \phi_n' - (g_1^-)' \phi_n) \, dx = \\ = \frac{1}{k_n + k_n^-} \int_{-\infty}^{\infty} \left[\frac{dW\{g_1^-, \phi_n\}}{dx} W\{f_2^-, \phi_n\} + \frac{dW\{f_2^-, \phi_n\}}{dx} W\{g_1^-, \phi_n\} \right] dx = \\ = \frac{1}{k_n + k_n^-} \left[W\{g_1^-, \phi_n\} W\{f_2^-, \phi_n\} \right] \Big|_{-\infty}^{\infty} = 0. \quad (3.41)$$

Substituting (3.37), (3.38), (3.40) and (3.41) into (3.36) we get

$$-(2ik_n)^{-1}\left\langle G, \Phi_2^+(x,t,k_n)\right\rangle = 2ik_n c_{11}^+(t,k_n)A_n(t), \tag{3.42}$$

$$(2ik_n^-)^{-1}\left\langle G, \Phi_2^-(x,t,k_n^-)\right\rangle = 0. \tag{3.43}$$

Putting (3.42) and (3.43) into (3.29) we obtain

$$\frac{d}{dt}c_{11}^+(t,k_n) - 4i(k_n)^2 c_{11}^+(t,k_n) = 2ik_n c_{11}^+(t,k_n)A_n(t), \qquad n = 1,2,\ldots,N, \tag{3.44}$$

$$\frac{d}{dt}c_{11}^-(t,k_n^-) + 4i(k_n^-)^2 c_{11}^-(t,k_n^-) = 0, \qquad n = 1,2,\ldots,N^- \tag{3.45}$$

From (3.44), (3.45) and view of $C_n^{\pm}(t)$ we conclude (3.24) and (3.25). The theorem is proved. □

§ 4. Conclusion

The obtained results completely define the time evolution of the spectral data, which allows us to solve the problem (1.1)–(1.4) by using the following algorithm. Let us given $v_0(x)$ and $u_0(x)$.

1. With the given $v_0(x)$ and $u_0(x)$, we find scattering data

$$\left\{R^{\pm}(k),\ k \in \mathbb{R}^*,\ k_n^{\pm},\ C_n^{\pm},\ n = 1,2,\ldots,N^{\pm}\right\}$$

   for $T(0,\pm k)$.

2. According to the results of theorem 1, we obtain the time evolution of the scattering data

$$\left\{R^{\pm}(t,k),\ k \in \mathbb{R}^*,\ k_n^{\pm}(t),\ C_n^{\pm}(t),\ n = 1,2,\ldots,N^{\pm}\right\}$$

   for $T(t,\pm k)$.

3. With the obtained scattering data, we uniquely define the function $r^{\pm}(x,t)$ from the equality (2.16).

4. Substituting $r^{\pm}(x,t)$ into the Gelfand–Levitan and Marchenko integral equation (2.15) and solving the system (2.14), (2.15) taking into account (2.18), (2.19) we define $F^+(x,t)$, $F^-(x,t)$, $A^+(x,y,t)$, $A^-(x,y,t)$ then the potentials $v(x,t)$ and $u(x,t)$ can be obtain via the formulas (2.11) and (2.12).

5. Solving the equation (1.4), we will construct the eigenfunctions

$$\phi_m = \phi_m(x,t), \quad m = 1,2,\ldots,N.$$

REFERENCES

1. Boussinesq J. Théorie de l'intumescence liquide appelée onde solitaire ou de translation, se propageant dans un canal rectangulaire, *Comptes rendus hebdomadaires des séances de l'Académie des Sciences*, 1871, vol. 72, pp. 755–759.
2. Kaup D. J. A higher-order water-wave equation and the method for solving it, *Progress of Theoretical Physics*, 1975, vol. 54, issue 2, pp. 396–408. https://doi.org/10.1143/PTP.54.396

3. Matveev V. B., Yavor M. I. Solutions presque périodiques et a N-solitons de l'equation hydrodynamique non linéaire de Kaup, *Annales de l'I. H. P. Physique théorique*, 1979, vol. 31, no. 1, pp. 25–41. http://eudml.org/doc/76040
4. Smirnov A. O. Real finite-gap regular solutions of the Kaup–Boussinesq equation, *Theoretical and Mathematical Physics*, 1986, vol. 66, no. 1, pp. 19–31. https://doi.org/10.1007/BF01028935
5. Mitropol'skii Yu. A., Bogolyubov N. N. (Jr.), Prikarpatskii A. K., Samoilenko V. G. *Integriruemaya dinamicheskaya sistema: spektral'nye i differentsial'no-geometricheskie aspecty* (Integrable dynamical system: spectral and differential-geometric aspects), Kiev: Naukova dumka, 1987.
6. Jaulent M., Miodek I. Nonlinear evolution equation associated with 'energy-dependent Schrödinger potentials', *Letters in Mathematical Physics*, 1976, vol. 1, no. 3, pp. 243–250. https://doi.org/10.1007/BF00417611
7. Sattinger D. H., Szmigielski J. Energy dependent scattering theory, *Differential and Integral Equations*, 1995, vol. 8, no. 5, pp. 945–959.
8. Ivanov R., Lyons T. Integrable models for shallow water with energy dependent spectral problems, *Journal of Nonlinear Mathematical Physics*, 2012, vol. 19, suppl. 1, pp. 72–88. https://doi.org/10.1142/S1402925112400086
9. Haberlin J., Lyons T. Solitons of shallow-water models from energy-dependent spectral problems, *The European Physical Journal Plus*, 2018, vol. 133, article number: 16. https://doi.org/10.1140/epjp/i2018-11848-8
10. Wazwaz A.-M. The generalized Kaup–Boussinesq equation: multiple soliton solutions, *Waves in Random and Complex Media*, 2015, vol. 25, issue 4, pp. 473–481. https://doi.org/10.1080/17455030.2015.1016474
11. Chen Ch., Jiang Y.-L. Invariant solutions and conservation laws of the generalized Kaup–Boussinesq equation, *Waves in Random and Complex Media*, 2019, vol. 29, issue 1, pp. 138–152. https://doi.org/10.1080/17455030.2017.1418098
12. Li W.-H., Wang Y. Exact dynamical behavior for a dual Kaup–Boussinesq system by symmetry reduction and coupled trial equations method, *Advances in Difference Equations*, 2019, vol. 2019, article number: 451. https://doi.org/10.1186/s13662-019-2357-0
13. Hosseini K., Ansari R., Gholamin P. Exact solutions of some nonlinear systems of partial differential equations by using the first integral method, *Journal of Mathematical Analysis and Applications*, 2012, vol. 387, issue 2, pp. 807–814. https://doi.org/10.1016/j.jmaa.2011.09.044
14. Zhou J., Tian L., Fan X. Solitary-wave solutions to a dual equation of the Kaup–Boussinesq system, *Nonlinear Analysis: Real World Applications*, 2010, vol. 11, issue 4, pp. 3229–3235. https://doi.org/10.1016/j.nonrwa.2009.11.017
15. Mel'nikov V. K. Integration method of the Korteweg–de Vries equation with a self-consistent source, *Physics Letters A*, 1988, vol. 133, issue 9, pp. 493–496. https://doi.org/10.1016/0375-9601(88)90522-1
16. Mel'nikov V. K. Integration of the Korteweg–de Vries equation with a source, *Inverse Problems*, 1990, vol. 6, no. 2, pp. 233–246. https://doi.org/10.1088/0266-5611/6/2/007
17. Leon J., Latifi A. Solution of an initial-boundary value problem for coupled nonlinear waves, *Journal of Physics A: Mathematical and General*, 1990, vol. 23, issue 8, pp. 1385–1403. https://doi.org/10.1088/0305-4470/23/8/013
18. Claude C., Latifi A., Leon J. Nonlinear resonant scattering and plasma instability: an integrable model, *Journal of Mathematical Physics*, 1991, vol. 32, issue 12, pp. 3321–3330. https://doi.org/10.1063/1.529443
19. Mel'nikov V. K. Integration of the nonlinear Schrödinger equation with a self-consistent source, *Communications in Mathematical Physics*, 1991, vol. 137, issue 2, pp. 359–381. https://doi.org/10.1007/BF02431884
20. Gerdjikov V. S., Grahovski G., Ivanov R. On the integrability of KdV hierarchy with self-consistent sources, *Communications on Pure and Applied Analysis*, 2012, vol. 11, issue 4, pp. 1439–1452. https://doi.org/10.3934/cpaa.2012.11.1439

21. Li Q., Zhang J. B., Chen D. Y. The eigenfunctions and exact solutions of discrete mKdV hierarchy with self-consistent sources via the inverse scattering transform, *Advances in Applied Mathematics and Mechanics*, 2015, vol. 7, issue 5, pp. 663–674. https://doi.org/10.4208/aamm.2013.m450
22. Bondarenko N., Freiling G., Urazboev G. Integration of the matrix KdV equation with self-consistent sources, *Chaos, Solitons and Fractals*, 2013, vol. 49, pp. 21–27. https://doi.org/10.1016/j.chaos.2013.02.010
23. Babadjanova A., Kriecherbauer T., Urazboev G. The periodic solutions of the discrete modified KdV equation with a self-consistent source, *Applied Mathematics and Computation*, 2020, vol. 376, 125136. https://doi.org/10.1016/j.amc.2020.125136
24. Urazboev G. U., Babadjanova A. K., Saparbaeva D. R. Integration of the Harry Dym equation with an integral type source, *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki*, 2021, vol. 31, issue 2, pp. 285–295. https://doi.org/10.35634/vm210209
25. Grinevich P. G., Taimanov I. A. Spectral conservation laws for periodic nonlinear equations of the Melnikov type, *Geometry, Topology, and Mathematical Physics: S. P. Novikov's Seminar: 2006–2007*, AMS, 2008, pp. 125–138. https://doi.org/10.48550/arXiv.0801.4143
26. Urazboev G., Babadjanova A. On the integration of the matrix modified Korteweg–de Vries equation with self-consistent source, *Tamkang Journal of Mathematics*, 2019, vol. 50, no. 3, pp. 281–291. https://doi.org/10.5556/j.tkjm.50.2019.3355
27. Yakhshimuratov A.. The nonlinear Schrödinger equation with a self-consistent source in the class of periodic functions, *Mathematical Physics, Analysis and Geometry*, 2011, vol. 14, issue 2, pp. 153–169. https://doi.org/10.1007/s11040-011-9091-5
28. Babajanov B., Fečkan M., Urazbaev G. On the periodic Toda lattice hierarchy with an integral source, *Communications in Nonlinear Science and Numerical Simulation*, 2017, vol. 52, pp. 110–123. https://doi.org/10.1016/j.cnsns.2017.04.023
29. Babajanov B. A., Khasanov A. B. Periodic Toda chain with an integral source, *Theoretical and Mathematical Physics*, 2015, vol. 184, issue 2, pp. 1114–1128. https://doi.org/10.1007/s11232-015-0321-z
30. Huang Y., Zeng Y., Ragnisco O. The Degasperis–Procesi equation with self-consistent sources, *Journal of Physics A: Mathematical and Theoretical*, 2008, vol. 41, no. 35, 355203. https://doi.org/10.1088/1751-8113/41/35/355203
31. Zeng Y.-B., Ma W.-X., Shao Y.-J. Two binary Darboux transformations for the KdV hierarchy with self-consistent sources, *Journal of Mathematical Physics*, 2001, vol. 42, issue 5, pp. 2113–2128. https://doi.org/10.1063/1.1357826
32. Zeng Y.-B., Shao Y.-J., Ma W.-X. Integral-type Darboux transformations for the mKdV hierarchy with self-consistent sources, *Communications in Theoretical Physics*, 2002, vol. 38, no. 6, pp. 641–648. https://doi.org/10.1088/0253-6102/38/6/641
33. Zeng Y.-B., Shao Y.-J., Xue W.-M. Positon solutions of the KdV equation with self-consistent sources, *Theoretical and Mathematical Physics*, 2003, vol. 137, issue 2, pp. 1622–1631. https://doi.org/10.1023/A:1027378306253
34. Xiao T., Zeng Y.-B. Generalized Darboux transformations for the KP equation with self-consistent sources, *Journal of Physics A: Mathematical and General*, 2004, vol. 37, no. 28, pp. 7143–7162. https://doi.org/10.1088/0305-4470/37/28/006
35. Gegenhasi, Xiaorong Bai. On the modified discrete KP equation with self-consistent sources, *Journal of Nonlinear Mathematical Physics*, 2017, vol. 24, issue 2, pp. 224–238. https://doi.org/10.1080/14029251.2017.1313476
36. Zhang D. The N-soliton solutions of some soliton equations with self-consistent sources, *Chaos, Solitons and Fractals*, 2003, vol. 18, issue 1, pp. 31–43. https://doi.org/10.1016/S0960-0779(02)00636-7
37. Matsuno Y. Bilinear Backlund transformation for the KdV equation with a source, *Journal of Physics A: Mathematical and General*, 1991, vol. 24, no. 6, pp. 273–277. https://doi.org/10.1088/0305-4470/24/6/005

38. Deng S.-F., Chen D.-Y., Zhang D.-J. The multisoliton solutions of the KP equation with self-consistent sources, *Journal of the Physical Society of Japan*, 2003, vol. 72, issue 9, pp. 2184–2192. https://doi.org/10.1143/JPSJ.72.2184
39. Zhang D.-J., Chen D.-Y. The N-soliton solutions of the sine-Gordon equation with self-consistent sources, *Physica A: Statistical Mechanics and its Applications*, 2003, vol. 321, issues 3–4, pp. 467–481. https://doi.org/10.1016/S0378-4371(02)01742-9
40. Cabada A., Yakhshimuratov A. The system of Kaup equations with a self-consistent source in the class of periodic functions, *Journal of Mathematical Physics, Analysis, Geometry*, 2013, vol. 9, no. 3, pp. 287–303.
41. Yakhshimuratov A., Kriecherbauer T., Babajanov B. On the construction and integration of a hierarchy for the Kaup system with a self-consistent source in the class of periodic functions, *Journal of Mathematical Physics, Analysis, Geometry*, 2021, vol. 17, no. 2, pp. 233–257. https://doi.org/10.15407/mag17.02.233
42. Jaulent M. On an inverse scattering problem with an energy-dependent potential, *Annales de l'I. H. P. Physique théorique*, 1972, vol. 17, no. 4, pp. 363–378. http://eudml.org/doc/75763
43. Jaulent M., Jean C. The inverse problem for the one-dimensional Schrödinger operator with an energy-dependent potential. I, *Annales de l'I. H. P. Physique théorique*, 1976, vol. 25, no. 2, pp. 105–118. http://eudml.org/doc/75912
44. Jaulent M., Jean C. The inverse problem for the one-dimensional Schrödinger operator with an energy-dependent potential. II, *Annales de l'I. H. P. Physique théorique*, 1976, vol. 25, no. 2, pp. 119–137. http://eudml.org/doc/75913
45. Maksudov F. G., Gusejnov G. Sh. On solution of the inverse scattering problem for a quadratic pencil of one-dimensional Schrödinger operators on the whole axis, *Sov. Math., Dokl.*, 1986, vol. 34, pp. 34–38. https://zbmath.org/?q=an:0629.34013
46. Laptev A., Shterenberg R., Sukhanov V. Inverse spectral problems for Schrödinger operators with energy depending potentials, *Probability and Mathematical Physics: A Volume in Honor of Stanislav Molchanov*, AMS, 2007, pp. 341–351. https://doi.org/10.1090/crmp/042/19
47. Aktosun T., Klaus M., van der Mee C. Wave scattering in one dimension with absorption, *Journal of Mathematical Physics*, 1998, vol. 39, issue 4, pp. 1957–1992. https://doi.org/10.1063/1.532271
48. Babazhanov B. A., Khasanov A. B. Inverse problem for a quadratic pencil of Sturm–Liouville operators with finite-gap periodic potential on the half-line, *Differential Equations*, 2007, vol. 43, issue 6, pp. 737–744. https://doi.org/10.1134/S0012266107060018
49. Babadzhanov B. A., Khasanov A. B., Yakhshimuratov A. B. On the inverse problem for a quadratic pencil of Sturm–Liouville operators with periodic potential, *Differential Equations*, 2005, vol. 41, issue 3, pp. 310–318. https://doi.org/10.1007/s10625-005-0163-9

Received 11.01.2022

Accepted 06.05.2022

Bazar Atajanovich Babajanov, Doctor of Physics and Mathematics, Professor, Department of Applied Mathematics and Mathematical Physics, Urgench State University, ul. Khamida Alimdjana, 14, Urgench, 220100, Uzbekistan;
Main Researcher, V. I. Romanovskiy Institute of Mathematics, Khorezm Branch of Uzbekistan Academy of Sciences, ul. Khamida Alimdjana, 14, Urgench, 220100, Uzbekistan.
ORCID: https://orcid.org/0000-0001-6878-791X
E-mail: a.murod@mail.ru

Azizbek Shavkatovich Azamatov, PhD Student on Mathematics, Department of Applied Mathematics and Mathematical Physics, Urgench State University, ul. Khamida Alimdjana, 14, Urgench, 220100, Uzbekistan.
E-mail: azizbek.shavkatovich@gmail.com

Citation: B. A. Babajanov, A. Sh. Azamatov. Integration of the Kaup–Boussinesq system with a self-consistent source via inverse scattering method, *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki*, 2022, vol. 32, issue 2, pp. 153–170.

МАТЕМАТИКА 2022. Т. 32. Вып. 2. С. 153–170.

Б. А. Бабажанов, А. Ш. Азаматов

Интегрирование системы Каупа–Буссинеска с самосогласованным источником с помощью метода обратного рассеяния

Ключевые слова: нелинейное уравнение солитона, система Каупа–Буссинеска, самосогласованный источник, метод обратного рассеяния, квадратичный пучок уравнений Штурма–Лиувилля.

УДК 517.957

DOI: 10.35634/vm220201

В данной работе рассматривается система Каупа–Буссинеска с самосогласованным источником. Показано, что система Каупа–Буссинеска с самосогласованным источником может быть проинтегрирована методом обратной задачи рассеяния. Для решения рассматриваемой задачи используются прямая и обратная задачи рассеяния уравнения Штурма–Лиувилля с потенциалом, зависящим от энергии. Определена временная эволюция данных рассеяния для уравнения Штурма–Лиувилля с энергозависимыми потенциалами, связанными с решением системы Каупа–Буссинеска с самосогласованным источником. Полученные равенства полностью определяют данные рассеяния при любом t, что позволяет применить метод обратной задачи рассеяния для решения задачи Коши для системы Каупа–Буссинеска с самосогласованным источником.

СПИСОК ЛИТЕРАТУРЫ

1. Boussinesq J. Théorie de l'intumescence liquide appelée onde solitaire ou de translation, se propageant dans un canal rectangulaire // Comptes rendus hebdomadaires des séances de l'Académie des Sciences. 1871. Vol. 72. P. 755–759.
2. Kaup D. J. A higher-order water-wave equation and the method for solving it // Progress of Theoretical Physics. 1975. Vol. 54. Issue 2. P. 396–408. https://doi.org/10.1143/PTP.54.396
3. Matveev V. B., Yavor M. I. Solutions presque périodiques et a N-solitons de l'equation hydrodynamique non linéaire de Kaup // Annales de l'I. H. P. Physique théorique. 1979. Vol. 31. No. 1. P. 25–41. http://eudml.org/doc/76040
4. Смирнов А. О. Вещественные конечнозонные регулярные решения уравнения Каупа–Буссинеска // Теоретическая и математическая физика. 1986. Т. 66. № 1. С. 30–46. http://mi.mathnet.ru/tmf4592
5. Митропольский Ю. А., Боголюбов Н. Н. (мл.), Прикарпатский А. К., Самойленко В. Г. Интегрируемая динамическая система: спектральные и дифференциально-геометрические аспекты // Киев: Наукова думка, 1987.
6. Jaulent M., Miodek I. Nonlinear evolution equation associated with 'energy-dependent Schrödinger potentials' // Letters in Mathematical Physics. 1976. Vol. 1. No. 3. P. 243–250. https://doi.org/10.1007/BF00417611
7. Sattinger D. H., Szmigielski J. Energy dependent scattering theory // Differential and Integral Equations. 1995. Vol. 8. No. 5. P. 945–959.
8. Ivanov R., Lyons T. Integrable models for shallow water with energy dependent spectral problems // Journal of Nonlinear Mathematical Physics. 2012. Vol. 19. Suppl. 1. P. 72–88. https://doi.org/10.1142/S1402925112400086
9. Haberlin J., Lyons T. Solitons of shallow-water models from energy-dependent spectral problems // The European Physical Journal Plus. 2018. Vol. 133. Article number: 16. https://doi.org/10.1140/epjp/i2018-11848-8
10. Wazwaz A.-M. The generalized Kaup–Boussinesq equation: multiple soliton solutions // Waves in Random and Complex Media. 2015. Vol. 25. Issue 4. P. 473–481. https://doi.org/10.1080/17455030.2015.1016474
11. Chen Ch., Jiang Y.-L. Invariant solutions and conservation laws of the generalized Kaup–Boussinesq equation // Waves in Random and Complex Media. 2019. Vol. 29. Issue 1. P. 138–152. https://doi.org/10.1080/17455030.2017.1418098

12. Li W.-H., Wang Y. Exact dynamical behavior for a dual Kaup–Boussinesq system by symmetry reduction and coupled trial equations method // Advances in Difference Equations. 2019. Vol. 2019. Article number: 451. https://doi.org/10.1186/s13662-019-2357-0
13. Hosseini K., Ansari R., Gholamin P. Exact solutions of some nonlinear systems of partial differential equations by using the first integral method // Journal of Mathematical Analysis and Applications. 2012. Vol. 387. Issue 2. P. 807–814. https://doi.org/10.1016/j.jmaa.2011.09.044
14. Zhou J., Tian L., Fan X. Solitary-wave solutions to a dual equation of the Kaup–Boussinesq system // Nonlinear Analysis: Real World Applications. 2010. Vol. 11. Issue 4. P. 3229–3235. https://doi.org/10.1016/j.nonrwa.2009.11.017
15. Mel'nikov V. K. Integration method of the Korteweg–de Vries equation with a self-consistent source // Physics Letters A. 1988. Vol. 133. Issue 9. P. 493–496. https://doi.org/10.1016/0375-9601(88)90522-1
16. Mel'nikov V. K. Integration of the Korteweg–de Vries equation with a source // Inverse Problems. 1990. Vol. 6. No. 2. P. 233–246. https://doi.org/10.1088/0266-5611/6/2/007
17. Leon J., Latifi A. Solution of an initial-boundary value problem for coupled nonlinear waves // Journal of Physics A: Mathematical and General. 1990. Vol. 23. Issue 8. P. 1385–1403. https://doi.org/10.1088/0305-4470/23/8/013
18. Claude C., Latifi A., Leon J. Nonlinear resonant scattering and plasma instability: an integrable model // Journal of Mathematical Physics. 1991. Vol. 32. Issue 12. P. 3321–3330. https://doi.org/10.1063/1.529443
19. Mel'nikov V. K. Integration of the nonlinear Schrödinger equation with a self-consistent source // Communications in Mathematical Physics. 1991. Vol. 137. Issue 2. P. 359–381. https://doi.org/10.1007/BF02431884
20. Gerdjikov V. S., Grahovski G., Ivanov R. On the integrability of KdV hierarchy with self-consistent sources // Communications on Pure and Applied Analysis. 2012. Vol. 11. Issue 4. P. 1439–1452. https://doi.org/10.3934/cpaa.2012.11.1439
21. Li Q., Zhang J. B., Chen D. Y. The eigenfunctions and exact solutions of discrete mKdV hierarchy with self-consistent sources via the inverse scattering transform // Advances in Applied Mathematics and Mechanics. 2015. Vol. 7. Issue 5. P. 663–674. https://doi.org/10.4208/aamm.2013.m450
22. Bondarenko N., Freiling G., Urazboev G. Integration of the matrix KdV equation with self-consistent sources // Chaos, Solitons and Fractals. 2013. Vol. 49. P. 21–27. https://doi.org/10.1016/j.chaos.2013.02.010
23. Babadjanova A., Kriecherbauer T., Urazboev G. The periodic solutions of the discrete modified KdV equation with a self-consistent source // Applied Mathematics and Computation. 2020. Vol. 376. 125136. https://doi.org/10.1016/j.amc.2020.125136
24. Urazboev G. U., Babadjanova A. K., Saparbaeva D. R. Integration of the Harry Dym equation with an integral type source // Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki. 2021. Vol. 31. Issue 2. P. 285–295. https://doi.org/10.35634/vm210209
25. Grinevich P. G., Taimanov I. A. Spectral conservation laws for periodic nonlinear equations of the Melnikov type // Geometry, Topology, and Mathematical Physics: S. P. Novikov's Seminar: 2006–2007. AMS, 2008. P. 125–138. https://doi.org/10.48550/arXiv.0801.4143
26. Urazboev G., Babadjanova A. On the integration of the matrix modified Korteweg–de Vries equation with self-consistent source // Tamkang Journal of Mathematics. 2019. Vol. 50. No. 3. P. 281–291. https://doi.org/10.5556/j.tkjm.50.2019.3355
27. Yakhshimuratov A. The nonlinear Schrödinger equation with a self-consistent source in the class of periodic functions // Mathematical Physics, Analysis and Geometry. 2011. Vol. 14. Issue 2. P. 153–169. https://doi.org/10.1007/s11040-011-9091-5
28. Babajanov B., Fečkan M., Urazbaev G. On the periodic Toda lattice hierarchy with an integral source // Communications in Nonlinear Science and Numerical Simulation. 2017. Vol. 52. P. 110–123. https://doi.org/10.1016/j.cnsns.2017.04.023
29. Бабажанов Б. А., Хасанов А. Б. О периодической цепочке Тоды с интегральным источником //

Теоретическая и математическая физика. 2015. Т. 184. № 2. С. 253–268.
https://doi.org/10.4213/tmf8859

30. Huang Y., Zeng Y., Ragnisco O. The Degasperis–Procesi equation with self-consistent sources // Journal of Physics A: Mathematical and Theoretical. 2008. Vol. 41. Issue 35. 355203.
https://doi.org/10.1088/1751-8113/41/35/355203
31. Zeng Y.-B., Ma W.-X., Shao Y.-J. Two binary Darboux transformations for the KdV hierarchy with self-consistent sources // Journal of Mathematical Physics. 2001. Vol. 42. Issue 5. P. 2113–2128.
https://doi.org/10.1063/1.1357826
32. Zeng Y.-B., Shao Y.-J., Ma W.-X. Integral-type Darboux transformations for the mKdV hierarchy with self-consistent sources // Communications in Theoretical Physics. 2002. Vol. 38. No. 6. P. 641–648.
https://doi.org/10.1088/0253-6102/38/6/641
33. Zeng Y.-B., Shao Y.-J., Ma W.-X. Positon solutions of the KdV equation with self-consistent sources // Theoretical and Mathematical Physics. 2003. Vol. 137. Issue 2. P. 1622–1631.
https://doi.org/10.1023/A:1027378306253
34. Xiao T., Zeng Y.-B. Generalized Darboux transformations for the KP equation with self-consistent sources // Journal of Physics A: Mathematical and General. 2004. Vol. 37. No. 28. P. 7143–7162.
https://doi.org/10.1088/0305-4470/37/28/006
35. Gegenhasi, Xiaorong Bai. On the modified discrete KP equation with self-consistent sources // Journal of Nonlinear Mathematical Physics. 2017. Vol. 24. Issue 2. P. 224–238.
https://doi.org/10.1080/14029251.2017.1313476
36. Zhang D. The N-soliton solutions of some soliton equations with self-consistent sources // Chaos, Solitons and Fractals. 2003. Vol. 18. Issue 1. P. 31–43.
https://doi.org/10.1016/S0960-0779(02)00636-7
37. Matsuno Y. Bilinear Backlund transformation for the KdV equation with a source // Journal of Physics A: Mathematical and General. 1991. Vol. 24. No. 6. P. 273–277.
https://doi.org/10.1088/0305-4470/24/6/005
38. Deng S.-F., Chen D.-Y., Zhang D.-J. The multisoliton solutions of the KP equation with self-consistent sources // Journal of the Physical Society of Japan. 2003. Vol. 72. Issue 9. P. 2184–2192.
https://doi.org/10.1143/JPSJ.72.2184
39. Zhang D.-J., Chen D.-Y. The N-soliton solutions of the sine-Gordon equation with self-consistent sources // Physica A: Statistical Mechanics and its Applications. 2003. Vol. 321. Issues 3–4. P. 467–481. https://doi.org/10.1016/S0378-4371(02)01742-9
40. Cabada A., Yakhshimuratov A. The system of Kaup equations with a self-consistent source in the class of periodic functions // Journal of Mathematical Physics, Analysis, Geometry. 2013. Vol. 9. No. 3. P. 287–303.
41. Yakhshimuratov A., Kriecherbauer T., Babajanov B. On the construction and integration of a hierarchy for the Kaup system with a self-consistent source in the class of periodic functions // Journal of Mathematical Physics, Analysis, Geometry. 2021. Vol. 17. No. 2. P. 233–257.
https://doi.org/10.15407/mag17.02.233
42. Jaulent M. On an inverse scattering problem with an energy-dependent potential // Annales de l'I. H. P. Physique théorique. 1972. Vol. 17. No. 4. P. 363–378. http://eudml.org/doc/75763
43. Jaulent M., Jean C. The inverse problem for the one-dimensional Schrödinger operator with an energy-dependent potential. I // Annales de l'I. H. P. Physique théorique. 1976. Vol. 25. No. 2. P. 105–118.
http://eudml.org/doc/75912
44. Jaulent M., Jean C. The inverse problem for the one-dimensional Schrödinger operator with an energy-dependent potential. II // Annales de l'I. H. P. Physique théorique. 1976. Vol. 25. No. 2. P. 119–137.
http://eudml.org/doc/75913
45. Максудов Ф. Г., Гусейнов Г. Ш. О решении обратной задачи рассеяния для квадратичного пучка одномерных операторов Шрёдингера на всей оси // Докл. АН СССР. 1986. Т. 289. № 1. С. 42–46.
http://mi.mathnet.ru/dan47574
46. Laptev A., Shterenberg R., Sukhanov V. Inverse spectral problems for Schrödinger operators with energy depending potentials // Probability and mathematical physics: A volume in honor of Stanislav

Molchanov. AMS, 2007. P. 341–351. https://doi.org/10.1090/crmp/042/19
47. Aktosun T., Klaus M., van der Mee C. Wave scattering in one dimension with absorption // Journal of Mathematical Physics. 1998. Vol. 39. Issue 4. P. 1957–1992. https://doi.org/10.1063/1.532271
48. Бабажанов Б. А., Хасанов А. Б. Обратная задача для квадратичного пучка операторов Штурма–Лиувилля с конечнозонным периодическим потенциалом на полуоси // Дифференциальные уравнения. 2007. Т. 43. № 6. С. 723–730. https://www.elibrary.ru/item.asp?id=9550545
49. Бабажанов Б. А., Хасанов А. Б., Яхшимуратов А. Б. Об обратной задаче для квадратичного пучка операторов Штурма–Лиувилля с периодическим потенциалом // Дифференциальные уравнения. 2005. Т. 41. № 3. С. 298–305. http://mi.mathnet.ru/de11237

Поступила в редакцию 11.01.2022

Принята к публикации 06.05.2022

Бабажанов Базар Атажанович, д. ф.-м. н., профессор, кафедра прикладной математики и математической физики, Ургенчский государственный университет, 220100, Узбекистан, г. Ургенч, ул. Хамида Алимджана, 14;
ведущий научный сотрудник, Институт математики им. В. И. Романовского Хорезмского филиала Академии наук Узбекистана, 220100, Узбекистан, г. Ургенч, ул. Хамида Алимджана, 14.
ORCID: https://orcid.org/0000-0001-6878-791X
E-mail: a.murod@mail.ru

Азаматов Азизбек Шавкатович, аспирант по математике, кафедра прикладной математики и математической физики, Ургенчский государственный университет, 220100, Узбекистан, г. Ургенч, ул. Хамида Алимджана, 14.
E-mail: azizbek.shavkatovich@gmail.com

Цитирование: Б. А. Бабажанов, А. Ш. Азаматов. Интегрирование системы Каупа–Буссинеска с самосогласованным источником с помощью метода обратного рассеяния // Вестник Удмуртского университета. Математика. Механика. Компьютерные науки. 2022. Т. 32. Вып. 2. С. 153–170.