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© ***G. U. Urazboev, A. K. Babadjanova, D. R. Saparbaeva***

INTEGRATION OF THE HARRY DYM EQUATION WITH AN INTEGRAL TYPE SOURCE

In the work, we deduce the evolution of scattering data for a spectral problem associated with the nonlinear evolutionary equation of Harry Dym with a self-consistent source of integral type. The obtained equalities completely determine the scattering data for any t, which makes it possible to apply the method of the inverse scattering problem to solve the Cauchy problem for the Harry Dym equation with an integral type source.

Keywords: nonlinear evolution equation, Harry Dym equation, integral source, inverse scattering method, Gelfand–Levitan–Marchenko equation.

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Introduction

The Harry Dym equation was firstly introduced by Harry Dym and Martin Kruskal as an evolution equation solvable by a spectral problem based on the string equation [1], and rediscovered in a more general form in the papers [2, 3].

The Harry Dym equation is a completely integrable equation [4, 5], which can be solved by inverse scattering transformation [6–8]. A parametric representation for a one-cusp soliton solution of Harry Dym equation was found in the paper [9], while the double-pole solution of the initial value problem for the Harry Dym equation was obtained by using the inverse scattering transform (IST) method [10].

In the work [11] the extended Harry Dym hierarchy which contains the Harry Dym hierarchy with self-consistent sources and the constrained Harry Dym hierarchy were constructed.

In this paper we have integrated the Harry Dym equation with the integral type source by the inverse scattering technique [12].

We consider the following system of equations

$$\begin{gathered} q(x,t)_t = 2\left(\frac{1}{\sqrt{1+q(x,t)}}\right)_{xxx} - 2\int_{-\infty}^{\infty}(1+q(x,t))(\phi^2(x,\xi))_x\,d\xi - \\ - q(x,t)_x\int_{-\infty}^{\infty}\phi^2(x,\xi)\,d\xi, \end{gathered} \tag{0.1}$$

$$L\phi(x,\xi) \equiv \phi''(x,\xi) + \xi^2 q(x,t)\phi(x,\xi) = -\xi^2\phi(x,\xi), \tag{0.2}$$

$$q(x,0) = q_0(x), \tag{0.3}$$

where $\phi(x,\xi)$ satisfies the following asymptotics

$$\phi(x,\xi) \to s(\xi)e^{i\zeta x} + t(\xi)e^{-i\zeta x}, \quad x \to +\infty. \tag{0.4}$$

Here $s(\xi)$ and $t(\xi)$ are given real continuous functions satisfying the Lipschitz condition and $s^2(\xi) + t^2(-\xi) = 0$.

The initial condition has the following properties

1.

$$\int_{-\infty}^{\infty}(1+x^2)\left(|q_0(x)|+\left|1-\frac{1}{1+q_0(x)}\right|\right)\,dx<\infty. \tag{0.5}$$

2. The operator $L(0)$ has no spectral singularities and has exactly N eigenvalues.

Our aim is to find the solution $\{q(x,t),\phi(x,\xi)\}$ assuming the existence in the sence of the following description: $q(x,t)$ is sufficiently smooth and sufficiently rapidly tend to zero as $|x|\to\infty$:

$$q(x,t)\to 0. \tag{0.6}$$

If we set

$$B=2\lambda^2\left[\frac{2}{\sqrt{1+q(x,t)}}\frac{\partial}{\partial x}-\left(\frac{1}{\sqrt{1+q(x,t)}}\right)_x\right],$$

then the first equation in the system can be rewritten in the following form

$$L_t=[B,L]+G, \tag{0.7}$$

$$G=-2\lambda^2\int_{-\infty}^{\infty}(1+q(x,t))(\phi^2(x,\xi))'\,d\xi-\lambda^2 q_x(x,t)\int_{-\infty}^{\infty}\phi^2(x,\xi)\,d\xi. \tag{0.8}$$

We will derive the time evolution equations with which we will be able to find the solution of the considering problem (0.1)–(0.6) via the inverse scattering transform method.

§ 1. Scattering problem

We consider the following eigenvalue problem [7]:

$$LX\equiv X''+\lambda^2 q(x)X=-\lambda^2 X. \tag{1.1}$$

Lemma 1. *Let $X(x,\lambda)$ and $Y(x,\mu)$ be solutions of the equation (1.1) corresponding to the parameters λ and μ, respectively. Then the following expression is hold*

$$\frac{dW(X(x,\lambda),Y(x,\mu))}{dx}=(1+q(x))(\mu^2-\lambda^2)X(x,\lambda),Y(x,\mu),$$

where $W(X(x,\lambda),Y(x,\mu))=X(x,\lambda)Y'(x,\mu)-X'(x,\lambda)Y(x,\mu)$.

We introduce the Jost solutions of (1.1) by

$$g(x,\lambda)\to e^{-i\lambda x},\ \ x\to-\infty,$$

$$f(x,\lambda)\to e^{i\lambda x},\ \ x\to\infty,$$

and for real λ parameters the pairs $\{g(x,\lambda),\,g(x,-\lambda)\}$ and $\{f(x,\lambda),\,f(x,-\lambda)\}$ are pairs of linearly independent solutions of (1.1). Therefore, following relation is hold

$$g(x,\lambda)=a(\lambda)f(x,-\lambda)+b(\lambda)f(x,\lambda), \tag{1.2}$$

where,

$$a(\lambda)=\frac{W(g(x,\lambda),f(x,\lambda))}{2i\lambda}, \tag{1.3}$$

$$b(\lambda) = \frac{W(f(x,-\lambda), g(x,\lambda))}{2i\lambda},$$

$$a(\lambda)a(-\lambda) - b(\lambda)b(-\lambda) = 1. \tag{1.4}$$

As $|x| \to \infty$, the following analytic properties of $a(\lambda)$ and the Jost solutions for large $|\lambda|$ are valid:

$$ae^{-i\lambda x} = 1 + O\left(\frac{1}{\lambda}\right),$$

$$f(x,\lambda)e^{i\lambda(x-\varepsilon_-)} = (1+q)^{-1/4} + O\left(\frac{1}{\lambda}\right),$$

$$f(x,\lambda)e^{-i\lambda(x+\varepsilon_+)} = (1+q)^{-1/4} + O\left(\frac{1}{\lambda}\right),$$

where

$$\varepsilon_+ = \int_x^\infty \sigma_{-1}\,dx, \quad \varepsilon_- = \int_{-\infty}^x \sigma_{-1}\,dx, \quad \sigma_{-1} = 1 - \sqrt{1+q}.$$

Therefore, $a(\lambda)$ has a finite number of zeros in the upper half plane $\mathbb{C}^+$, $\lambda_n = i\chi_n$ $(\chi_n > 0)$, $n = 1, 2, \ldots, N$, which are assumed simple, where $-\chi_n^2$, $n = \overline{1, N}$, are the eigenvalues of L. According to the representation (1.3)

$$g(x,\lambda_k) = c_k f(x,\lambda_k), \quad k = 1, 2, \ldots, N.$$

The following integral representation is valid for the Jost function $f(x,\lambda)$

$$f(x,\lambda) = e^{i\lambda(x+\varepsilon_+)} + e^{i\lambda\varepsilon_+}\int_x^\infty K(x,s)e^{i\lambda s}\,ds,$$

where the kernel K is assumed to satisfy

$$\lim_{s\to\infty} K(x,s) = 0$$

and have relation with $q(x)$ in this form

$$1 + q(x) = [1 - K(x,x)]^{-4}.$$

For $x \le y$, $K(x,y)$ kernel satisfies the following Gelfand–Levitan–Marchenko equation:

$$K(x,y) - F(x+y) - \int_x^\infty K(x,s)F'(s+y)\,ds = 0.$$

Here $F(z)$ is defined by

$$F(z) = \sum_{k=1}^N c_k \frac{e^{i\lambda_k(z+2\varepsilon_+(x))}}{i\lambda_k} + \frac{1}{2\pi}\int_{-\infty}^\infty R(\lambda)\frac{e^{i\lambda(z+2\varepsilon_+(x))}}{i\lambda}\,d\lambda.$$

Definition 1. The set of $\{R(\lambda), c_k, \lambda_k,\ k = \overline{1, N}\}$ is called *the scattering data* associated to the equation (1.1).

§ 2. Evolution of the scattering data

Let $\psi_n = \psi(x, \lambda_n)$ be the normalized eigenfunctions of the operator L corresponding to the eigenvalues λ_n^2, $n = \overline{1, N}$,

$$L\psi_n = \psi_n'' + \lambda_n^2 q(x,t)\psi_n = -\lambda_n^2\psi_n.$$

We differentiate this equation by t and do scalar multiplication by ψ_n. Considering the self-adjointness of L operator and using the equalities (0.7) and (0.8) we have

$$\frac{d\lambda_n^2}{dt} = \lambda_n^2 \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \left\{ 2(1+q)(\phi^2)(\psi_n^2) + q_x\phi^2(\psi_n^2) \right\} \, dx \right] d\xi.$$

If we do some calculations and using Lemma 1 on the right-hand side integral of the above equality, we get

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} ((1+q)\phi\phi\psi_n\psi_n)' \, dx + \int_{-\infty}^{\infty} 2(1+q)\phi\psi_n W\{\phi,\, \psi_n\} \, dx \right] d\xi =$$
$$= \int_{-\infty}^{\infty} \left[(1+q)\phi^2\psi_n\psi_n\} + \frac{W^2\{\phi(x,\xi),\, \psi_n(x,\lambda_n)\}}{\lambda_n^2 - \xi^2} \right] \Bigg|_{-\infty}^{\infty} d\xi = 0,$$

$$\frac{d\lambda_n^2}{dt} = 0.$$

This means that the eigenvalues of the operator L do not depend on t.

Let $F_0 = F_0(x, t, \lambda)$ be a solution of the following equation

$$LF_0 = F_0'' + \lambda^2 q(x,t)F_0 = -\lambda^2 F_0 \tag{2.1}$$

and let F^* satisfies the following equation

$$\frac{\partial F^*}{\partial x} = (1 + q(x,t))\phi(x,\xi)F_0. \tag{2.2}$$

Then, it is easy to show that for $\operatorname{Im}\lambda > 0$ the following function

$$H_0(\lambda) = \dot{F}_0 - BF_0 - \lambda^2 \int_{-\infty}^{\infty} \phi(x,\xi)F^* \, d\xi$$

satisfies the following equation

$$LH_0(\lambda) - \lambda H_0(\lambda) = \lambda^2 \int_{-\infty}^{\infty} \phi\hat{H} \, d\xi,$$

where

$$\hat{H} = (\xi^2 - \lambda^2)F^* + W\,\{\phi, F_0\}.$$

and dot means the derivative respecting to t.

Lemma 2. *Let $f(x,\lambda)$ and $g(x,\lambda)$ be the Jost solutions of the equation (2.1), then the following functions*

$$F^{*-}(\lambda) = \int_x^{-\infty} (1+q)\phi(x,\xi)g(x,\lambda) \, dx,$$

$$F^{*+}(\lambda) = \int_x^{\infty} (1+q)\phi(x,\xi)f(x,\lambda) \, dx$$

satisfy the equation (2.2)*, in result for* $\operatorname{Im}\lambda > 0$

$$\begin{aligned} H_0^-(\lambda) &= \dot g(x,\lambda) - Bg(x,\lambda) - \lambda^2\int_{-\infty}^{\infty}\phi F^{*-}(\lambda)\,d\xi, \\ H_0^+(\lambda) &= \dot f(x,\lambda) - Bf(x,\lambda) - \lambda^2\int_{-\infty}^{\infty}\phi F^{*+}(\lambda)\,d\xi, \end{aligned} \tag{2.3}$$

satisfy the equation (2.1)*. For real* λ *parameters the following expressions hold:*

$$\begin{aligned} H_0^-(\lambda) &= \dot g(x,\lambda) - Bg(x,\lambda) - \lambda^2 v.p.\int_{-\infty}^{\infty}\phi(x,\xi)\frac{W\{\phi(x,\xi),g(x,\lambda)\}}{\xi^2-\lambda^2}\,d\xi \\ &- \frac{\pi\lambda i}{2}\phi(x,\lambda)W\{\phi(x,\lambda),g(x,\lambda)\} - \frac{\pi\lambda i}{2}\phi(x,-\lambda)W\{\phi(x,-\lambda),g(x,\lambda)\}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} H_0^+(\lambda) &= \dot f(x,\lambda) - Bf(x,\lambda) - \lambda^2 v.p.\int_{-\infty}^{\infty}\phi(x,\xi)\frac{W\{\phi(x,\xi),f(x,\lambda)\}}{\xi^2-\lambda^2}\,d\xi \\ &- \frac{\pi\lambda i}{2}\phi(x,\lambda)W\{\phi(x,\lambda),f(x,\lambda)\} - \frac{\pi\lambda i}{2}\phi(x,-\lambda)W\{\phi(x,-\lambda),f(x,\lambda)\}. \end{aligned} \tag{2.5}$$

P r o o f.

$$\begin{aligned} LH_0 &= L\dot F_0 - LBF_0 - L\lambda^2\int_{-\infty}^{\infty}\phi F^*\,d\xi = (LF_0)_t - L_tF_0 - LBF_0 - L\lambda^2\int_{-\infty}^{\infty}\phi F^*\,d\xi \\ &= -\lambda^2\dot F_0 - BLF_0 + LBF_0 - GF_0 - LBF_0 - L\lambda^2\int_{-\infty}^{\infty}\phi F^*\,d\xi \\ &= -\lambda^2\dot F_0 + B\lambda^2F_0 - GF_0 - L\lambda^2\int_{-\infty}^{\infty}\phi F^*\,d\xi \\ &= -\lambda^2H_0 - \lambda^2\int_{-\infty}^{\infty}\lambda^2(1+q)\phi F^*\,d\xi - GF_0 + \lambda^2\int_{-\infty}^{\infty}\xi^2(1+q)\phi F^*\,d\xi \\ &- 2\int_{-\infty}^{\infty}\phi'(1+q)\phi F_0\,d\xi - \int_{-\infty}^{\infty}(1+q)\phi\phi' F_0\,d\xi - \int_{-\infty}^{\infty}(1+q)\phi\phi F_0'\,d\xi - \int_{-\infty}^{\infty}q_x\phi^2F_0\,d\xi. \\ LH_0 &= -\lambda^2(1+q)H_0 + \lambda^2\int_{-\infty}^{\infty}\left[(\xi^2-\lambda^2)(1+q)\phi F^* + (1+q)\phi W\left\{\phi,F_0\right\}\right]\,d\xi - GF_0 \\ &- 2\int_{-\infty}^{\infty}(1+q)(\phi^2)'F_0\,d\xi - \int_{-\infty}^{\infty}q_x\phi^2F_0\,d\xi. \end{aligned}$$

As $\hat H \underset{|x|\to\infty}{\to} 0$ and $\frac{\partial \hat H}{\partial x} = (\xi^2-\lambda^2)F_0\phi(1+q) + (\lambda^2-\xi^2)\phi F_0(1+q) = 0$, we have expression (2.3). If we use the following equalities

$$\begin{aligned} F^{*-}(\lambda) &= \frac{W\{\phi(x,\xi)g(x,\lambda)\}}{\xi^2-\lambda^2}, \\ F^{*+}(\lambda) &= \frac{W\{\phi(x,\xi)f(x,\lambda)\}}{\xi^2-\lambda^2}, \end{aligned}$$

and the Sokhotski–Plemelj formulas for $\lambda \in \mathbb{R}$ in (2.3), we find (2.4) and (2.5). □

According to (0.4), we can write the following equality

$$\phi(x,\lambda) = s(\lambda)f(x,-\lambda) + t(\lambda)f(x,\lambda).$$

By virtue of (1.2) and (1.4), we have

$$\phi(x,\lambda) = p(\lambda)g(x,-\lambda) + r(\lambda)g(x,\lambda),$$

where

$$r(\lambda) = s(\lambda)a(-\lambda) - t(\lambda)b(-\lambda),$$
$$p(\lambda) = t(\lambda)a(\lambda) - b(\lambda)s(\lambda).$$

Now, we will find the asymptotes of $H_0(\lambda)$ for $\lambda \in \mathbb{R}$, as $x \to \mp\infty$. When $x \to -\infty$, we have

$$H_0^- \to 4i\lambda^3 e^{-i\lambda x} - \lambda^2 v.p.\int_{-\infty}^{\infty} \frac{\phi W\{\phi,\, g\}}{2\lambda}\left[\frac{1}{\xi-\lambda} - \frac{1}{\xi+\lambda}\right] d\xi$$
$$- \frac{\pi i\lambda}{2}\phi(x,\lambda)W\{\phi(x,\lambda),\, g(x,\lambda)\} - \frac{\pi i\lambda}{2}\phi(x,-\lambda)W\{\phi(x,-\lambda),\, g(x,\lambda)\}.$$

As

$$W\{\varphi(x,\lambda),\, g(x,\lambda)\} = -2i\lambda p(\lambda),$$
$$W\{\varphi(x,-\lambda),\, g(x,\lambda)\} = -2i\lambda r(-\lambda),$$

we find

$$H_0^- \to 4i\lambda^3 e^{-i\lambda x} + i\lambda^2 v.p.\int_{-\infty}^{\infty} p(\xi)r(\xi)\left[\frac{1}{\xi-\lambda} - \frac{1}{\xi+\lambda}\right] d\xi e^{-i\lambda x}$$
$$- \lambda^2\pi[p(\lambda)r(\lambda) + p(-\lambda)r(-\lambda)]e^{-i\lambda x}.$$

In accordance with the uniqueness of the Jost solutions, we can write

$$\begin{aligned} H_0^-(\lambda) = 4i\lambda^3 g(x,\lambda) + i\lambda^2 v.p.\int_{-\infty}^{\infty} p(\xi)r(\xi)\left[\frac{1}{\xi-\lambda} - \frac{1}{\xi+\lambda}\right] d\xi g(x,\lambda) \\ - \lambda^2\pi[p(\lambda)r(\lambda) + p(-\lambda)r(-\lambda)]g(x,\lambda). \end{aligned} \tag{2.6}$$

Similarly for $x \to +\infty$, we have

$$H_0^+ \to -4i\lambda^3 e^{i\lambda x} - \lambda^2 v.p.\int_{-\infty}^{\infty} \frac{\phi W\{\phi,\, f\}}{2\lambda}\left[\frac{1}{\xi-\lambda} - \frac{1}{\xi+\lambda}\right] d\xi$$
$$- \frac{\pi i\lambda^2}{2\lambda}\phi(\lambda)W\{\phi(\lambda),\, f(\lambda)\} - \frac{\pi i\lambda^2}{2\lambda}\phi(-\lambda)W\{\phi(-\lambda),\, f(\lambda)\}.$$

Considering that,

$$W\{\phi(\lambda),\, f(\lambda)\} = 2i\lambda t(\lambda),$$
$$W\{\phi(-\lambda),\, f(\lambda)\} = 2i\lambda s(-\lambda),$$

we obtain

$$H_0^+ \to -4i\lambda^3 e^{i\lambda x} + i\lambda^2 v.p.\int_{-\infty}^{\infty} s(\xi)t(\xi)\left[\frac{1}{\xi-\lambda} - \frac{1}{\xi+\lambda}\right] d\xi e^{i\lambda x}$$
$$+ \lambda^2\pi[s(\lambda)t(\lambda) + s(-\lambda)t(-\lambda)]e^{i\lambda x}.$$

Hence,

$$\begin{aligned} H_0^+(\lambda) = -4i\lambda^3 f(x,\lambda) - i\lambda^2 v.p.\int_{-\infty}^{\infty} s(\xi)t(\xi)\left[\frac{1}{\xi-\lambda} - \frac{1}{\xi+\lambda}\right] d\xi f(x,\lambda) \\ + \lambda^2\pi[s(\lambda)t(\lambda) + s(-\lambda)t(-\lambda)]f(x,\lambda). \end{aligned} \tag{2.7}$$

Analogously, we find

$$H_0^+(-\lambda) = 4i\lambda^3 f(x,-\lambda) + i\lambda^2 v.p.\int_{-\infty}^{\infty} s(\xi)t(\xi)\left[\frac{1}{\xi-\lambda} - \frac{1}{\xi+\lambda}\right] d\xi f(x,-\lambda) \\ - \lambda^2\pi[s(\lambda)t(\lambda) + s(-\lambda)t(-\lambda)]f(x,-\lambda). \tag{2.8}$$

Lemma 3. *For real λ parameter it is hold*

$$\dot{R}(\lambda) = 8i\lambda^3 R(\lambda) + 2i\lambda^2 v.p.\int_{-\infty}^{\infty} s(\xi)t(\xi)\left[\frac{1}{\xi-\lambda} - \frac{1}{\xi+\lambda}\right] d\xi R(\lambda) \\ + 2\lambda^2\pi[s(\lambda)t(\lambda) + s(-\lambda)t(-\lambda)]R(\lambda). \tag{2.9}$$

P r o o f. For real λ parameter we introduce the following function

$$\tilde{H} = H_0^- - a(\lambda)H_0^+(-\lambda) - b(\lambda)H_0^+(\lambda). \tag{2.10}$$

Substituting the asymptotes (2.6), (2.7) and (2.8) into the expression (2.10), we find

$$H_0^- - a(\lambda)H_0^+(-\lambda) - b(\lambda)H_0^+(\lambda) = f(x,\lambda)\{8i\lambda^3\dot{b}(\lambda) \\ + i\lambda^2 b(\lambda)v.p.\int_{-\infty}^{\infty} p(\xi)r(\xi)\left[\frac{1}{\xi-\lambda} - \frac{1}{\xi+\lambda}\right] d\xi - \lambda^2\pi b(\lambda)[p(\lambda)r(\lambda) + p(-\lambda)r(-\lambda)] \\ + i\lambda^2 b(\lambda)v.p.\int_{-\infty}^{\infty} s(\xi)t(\xi)\left[\frac{1}{\xi-\lambda} - \frac{1}{\xi+\lambda}\right] d\xi - \lambda^2\pi b(\lambda)[s(\lambda)t(\lambda) + s(-\lambda)t(-\lambda)]\} \\ + f(x,-\lambda)\{i\lambda^2 a(\lambda)v.p.\int_{-\infty}^{\infty} p(\xi)r(\xi)\left[\frac{1}{\xi-\lambda} - \frac{1}{\xi+\lambda}\right] d\xi - \lambda^2\pi a(\lambda)[p(\lambda)r(\lambda) \\ + p(-\lambda)r(-\lambda)] - i\lambda^2 a(\lambda)v.p.\int_{-\infty}^{\infty} s(\xi)t(\xi)\left[\frac{1}{\xi-\lambda} - \frac{1}{\xi+\lambda}\right] d\xi + \\ +\lambda^2\pi a(\lambda)[s(\lambda)t(\lambda) + s(-\lambda)t(-\lambda)]\}$$

In other hand,

$$H_0^- - a(\lambda)H_0^+(-\lambda) - b(\lambda)H_0^+(\lambda) = f(x,-\lambda)\{\dot{a}(\lambda) - \lambda^2\pi b(-\lambda)[p^2(\lambda) + r^2(-\lambda)] \\ - \lambda^2\pi a(\lambda)[p(\lambda)r(\lambda) + p(-\lambda)r(-\lambda)] - \lambda^2\pi b(\lambda)[s^2(\lambda) + t^2(-\lambda)] \\ + \lambda^2\pi a(\lambda)[s(\lambda)t(\lambda) + s(-\lambda)t(-\lambda)]\} + +f(x,\lambda)\{\dot{b}(\lambda)- \\ -\lambda^2\pi a(-\lambda)[p^2(\lambda) + r^2(-\lambda)] - \lambda^2\pi b(\lambda)[p(\lambda)r(\lambda) + p(-\lambda)r(-\lambda)] \\ - \lambda^2\pi b(\lambda)[s(\lambda)t(\lambda) + s(-\lambda)t(-\lambda)] + \lambda^2\pi a(\lambda)[s^2(\lambda) + t^2(-\lambda)]\}.$$

Comparing last two equalities, we get

$$\dot{a}(\lambda) = \lambda^2\pi b(-\lambda)[p^2(\lambda) + r^2(-\lambda)] + \lambda^2\pi b(\lambda)[s^2(\lambda) + t^2(-\lambda)] \\ + i\lambda^2 a(\lambda)v.p.\int_{-\infty}^{\infty} p(\xi)r(\xi)\left[\frac{1}{\xi-\lambda} - \frac{1}{\xi+\lambda}\right] d\xi \\ - i\lambda^2 a(\lambda)v.p.\int_{-\infty}^{\infty} s(\xi)t(\xi)\left[\frac{1}{\xi-\lambda} - \frac{1}{\xi+\lambda}\right] d\xi.$$

$$\dot{b}(\lambda) = \lambda^2\pi[p^2(\lambda) + r^2(\lambda)]a(-\lambda) - \lambda^2\pi a(\lambda)[s^2(\lambda) + t^2(-\lambda)] + 8i\lambda^3\dot{b}(\lambda) \\ + i\lambda^2 b(\lambda)v.p.\int_{-\infty}^{\infty} p(\xi)r(\xi)\left[\frac{1}{\xi-\lambda} - \frac{1}{\xi+\lambda}\right] d\xi \\ + i\lambda^2 b(\lambda)v.p.\int_{-\infty}^{\infty} s(\xi)t(\xi)\left[\frac{1}{\xi-\lambda} - \frac{1}{\xi+\lambda}\right] d\xi.$$

As $\dot{R}(\lambda)a(\lambda) = \dot{b}(\lambda) - R(\lambda)\dot{a}(\lambda)$ and $s^2(\lambda) + t^2(-\lambda) = 0$, we get (2.9). □

Lemma 4. *$c_k(t)$ function satisfy the following differential equation*

$$\dot{c}_k(t) = 8i\lambda_k^3 c_k(t). \tag{2.11}$$

P r o o f. We introduce the following function

$$h(\lambda_k) = h^-(\lambda_k) - c_k(t)h^+(\lambda_k).$$

We define the functions as $h^-(\lambda_k)$ and $h^+(\lambda_k)$

$$h^-(\lambda_k) = \dot{g}(x,\lambda_k) - Bg(x,\lambda_k) - \int_{-\infty}^{\infty} \phi F^{*-}(x,\xi,\lambda_k)\, d\xi,$$

$$h^+(\lambda_k) = \dot{f}(x,\lambda_k) - Bf(x,\lambda_k) - \int_{-\infty}^{\infty} \phi F^{*+}(x,\xi,\ \lambda_k)\, d\xi,$$

and

$$F^{*-}(\lambda_k) = -\int_{-\infty}^{x} (1+q)\phi(x,\xi)g(x,\lambda_k)\, dx,$$

$$F^{*+}(\lambda_k) = \int_{x}^{\infty} (1+q)\phi(x,\xi)f(x,\lambda_k)\, dx,$$

From the asymptotes for the expression (2.3) we have

$$h^-(\lambda_k) = 4i\lambda_k^3 g(x,\lambda_k),$$

$$h^+(\lambda_k) = -4i\lambda_k^3 f(x,\lambda_k),$$

As $g(x,\lambda_k) = c_k f(x,\lambda_k)$, we get

$$h(\lambda_k) = 8i\lambda_k^3 c_k(t) f(x,\lambda_k), \tag{2.12}$$

In other hand,

$$h(\lambda_k) = \dot{c}_k(t)f(x,\lambda_k) \ + c_k(t)\lambda_k^2 \int_{-\infty}^{\infty} \phi(x,\xi) \int_{-\infty}^{\infty} (1+q)\phi(x,\xi)f(x,\lambda_k)\, dx\, d\xi.$$

As we know that

$$\int_{-\infty}^{\infty} (1+q)\phi(x,\xi)f(x,\lambda_k)\, dx = \int_{-\infty}^{\infty} \frac{W'\{\phi(x,\xi)f(x,\lambda_k)\}}{\lambda_k^2 - \xi^2}\, dx = 0,$$

we obtain

$$h = \dot{c}_k(t)f(x,\lambda_k). \tag{2.13}$$

Comparing (2.12) and (2.13), we have

$$\dot{c}_k(t)f(x,\lambda_k) = 8i\lambda_k^3 c_k(t)f(x,\lambda_k).$$

Thus, we find the relation (2.11). □

Theorem 1. *Let $\{q(x,t),\phi(x,\xi)\}$ be a solution of the problem* (0.1)–(0.6)*, then the scattering data associated to the equation* (1.1) *fulfill the following relations*

$$\dot{R}(\lambda) = 8i\lambda^3 R(\lambda) + 2i\lambda^2 v.p. \int_{-\infty}^{\infty} s(\xi)t(\xi) \left[\frac{1}{\xi-\lambda} - \frac{1}{\xi+\lambda}\right] d\xi R(\lambda)$$
$$+ 2\lambda^2\pi[s(\lambda)t(\lambda) + s(-\lambda)t(-\lambda)]R(\lambda), \quad \lambda \in \mathbb{R},$$

$$\frac{d\lambda_n^2}{dt} = 0,$$

$$\dot{c}_k(t) = 8i\lambda_k^3 c_k(t).$$

Remark 1. The obtained results completely specify the time evolution of the scattering data, which allows us to find the solution of the considered problem (0.1)–(0.6) via the inverse scattering method.

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Urazboev Gayrat Urazalievich, Doctor of Physics and Mathematics, Professor, Department of Applied Mathematics, Urgench State University, ul. Khamida Alimdjana, 14, Urgench, 220100, Uzbekistan;
Main Researcher, V.I. Romanovskiy Institute of Mathematics, Khorezm branch of Uzbekistan Academy of Sciences, ul. Khamida Alimdjana, 14, Urgench, 220100, Uzbekistan.
ORCID: https://orcid.org/0000-0002-7420-2516
E-mail: gayrat71@mail.ru

Babadjanova Aygul Kamildjanovna, PhD of Physics and Mathematics, Senior Researcher, Department of Mathematics, V.I. Romanovskiy Institute of Mathematics, Khorezm branch of Uzbekistan Academy of Sciences, ul. Khamida Alimdjana, 14, Urgench, 220100, Uzbekistan.
ORCID: https://orcid.org/0000-0003-3458-3913
E-mail: oygul@bk.ru

Saparbaeva Dilfuza Ruzimbaevna, Senior lecturer, Department of Applied Mathematics, Urgench State University, ul. Khamida Alimdjana, 14, Urgench, 220100, Uzbekistan.
ORCID: https://orcid.org/0000-0001-9530-9243
E-mail: sdilfuza76@mail.ru

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Г. У. Уразбоев, А. К. Бабаджанова, Д. Р. Сапарбаева

Интегрирование уравнения Гарри Дима с источником интегрального типа

Ключевые слова: нелинейное эволюционное уравнение, уравнение Гарри Дима, интегральный источник, метод обратной задачи рассеяния, уравнение Гельфанда–Левитана–Марченко.

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В работе выводится эволюция данных рассеяния спектральной задачи, связанной с нелинейным эволюционным уравнением Гарри Дима с самосогласованным источником интегрального типа. Полученные равенства полностью определяют данные рассеяние при любом t, что позволяет применить метод обратной задачи рассеяния для решения задачи Коши для уравнения Гарри Дима с источником интегрального типа.

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Уразбоев Гайрат Уразалиевич, д. ф.-м. н., профессор кафедры прикладной математики, Ургенчский государственный университет, ул. Хамида Алимджана, 14, Ургенч, 220100, Узбекистан;
главный научный сотрудник, Хорезмский отдел Института математики им. В. И. Романовского АН РУз, ул. Хамида Алимджана, 14, Ургенч, 220100, Узбекистан.
ORCID: https://orcid.org/0000-0002-7420-2516
E-mail: gayrat71@mail.ru

Бабаджанова Айгул Камилджановна, к. ф.-м. н., старший научный сотрудник, Хорезмский отдел Института математики им. В. И. Романовского АН РУз, ул. Хамида Алимджана, 14, Ургенч, 220100, Узбекистан.
ORCID: https://orcid.org/0000-0003-3458-3913
E-mail: oygul@bk.ru

Сапарбаева Дилфуза Рузимбаевна, старший преподаватель, кафедра прикладной математики, Ургенчский государственный университет, ул. Хамида Алимджана, 14, Ургенч, 220100, Узбекистан.
ORCID: https://orcid.org/0000-0001-9530-9243
E-mail: sdilfuza76@mail.ru

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