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© *V. D. Kryakvin, G. P. Omarova*

THE BOUTET DE MONVEL OPERATORS IN VARIABLE HÖLDER–ZYGMUND SPACES ON $\mathbb{R}^n_+$

We consider Green operators from the Boutet de Monvel algebra in the Hölder–Zygmund spaces of variable smoothness on $\overline{\mathbb{R}}^n_+$. The order of smoothness depends on a point in the domain and may take negative values. The sufficient conditions of boundedness of the Boutet de Monvel operators are obtained.

Keywords: the Boutet de Monvel calculus, Green operator, Hölder–Zygmund space, variable smoothness.

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Introduction

The main aim of the paper is the study of the operators from the L. Boutet de Monvel algebra on $\mathbb{R}^n_+$ acting in Hölder–Zygmund spaces of variable smoothness. Pioneering work is [12], in which a symbolic calculus of operators of the form

$$\mathcal{A} = \begin{pmatrix} P_+ + G & K \\ T & Q \end{pmatrix} : \begin{matrix} C_c^\infty(\Omega'\times\overline{\mathbb{R}}_+)\otimes\mathbb{C}^M \\ \oplus \\ C_0^\infty(\Omega')\otimes\mathbb{C}^N \end{matrix} \to \begin{matrix} C^\infty(\Omega'\times\overline{\mathbb{R}}_+)\otimes\mathbb{C}^{M'} \\ \oplus \\ C^\infty(\Omega')\otimes\mathbb{C}^{N'} \end{matrix},$$

(where $\Omega' \subset \mathbb{R}^{n-1}$, $P_+ = r^+Pe^+$ is a pseudodifferential operator in a half-space with a symbol from the L. Hörmander class $S^m_{1,0}$, satisfying the transmission property with respect to $x_n = 0$, G is a singular Green operator, K is a Poisson (or potential) operator, T is a trace operator, Q — pseudodifferential operator on the boundary), which are called Green operators, was constructed, as well as conditions for the solvability of an elliptic equation on a compact manifold with boundary were given and the index of the problem was found. An alternative approach to the solvability of such equations in the scalar case had been proposed by Vishik and Eskin (see [4] and references there), which had been based on factorization of the symbol of the pseudodifferential operator. The investigation of L. Boutet de Monvel was continued by S. Rempel and B.-W. Schulze (see [13] and references there), G. Grubb [5, 6], J. Johnsen [8], E. M. Shargorodsky [3] and others. It should be noted that Green operators were considered in L_p spaces, in Besov–Lizorkin–Tribel spaces, and Hölder–Zygmund spaces in these papers. But the exponents and parameters of the functional spaces are constant and do not depend on a point of the domain in all these cases. This leads to the fact that local properties of solutions were not appreciated. In this article, our task is to take into account, in the spirit of the works [1, 2, 9–11], the local smoothness of functions and distributions under the action of the Green operator.

The paper is organized as follows. In § 1 and § 2 we give the necessary definitions and statements about the Hölder–Zygmund spaces of variable smoothness and the pseudodifferential operators, respectively. The classes of Poisson operators, trace operators and singular Green operators are introduced in § 3, § 4 and § 5, respectively. The theorems on the boundedness of these operators are also proved there. The boundedness of a pseudodifferential operator with the transmission property in a half-space is proved in § 6. In the final section we present a theorem on the boundedness of the Green operator in Hölder–Zygmund spaces with a variable smoothness in a half-space. The results obtained are important for studying the solvability of general elliptic boundary value problems on smooth manifolds and for analyzing the local smoothness of solutions. The authors hope to present the results on this topic in the near future.

Throughout the paper we use the standard notations: for elements $x = (x_1, x_2, \dots, x_n)$ and $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ of the real n-dimensional Euclidean space $\mathbb{R}^n$ their scalar product is defined by $x \cdot \xi = x_1\xi_1 + \dots + x_n\xi_n$, the norm is defined as $|x| = \sqrt{x \cdot x}$, $dx = dx_1 \dots dx_n$ and $\langle x \rangle^{2t} = (1 + |x|^2)^t$. Let us denote by $\mathbb{N} = \{1, 2, 3, \dots\}$ the set of positive integers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n_+$ be a half-space in $\mathbb{R}^n$ specified by unequality $x_n > 0$, and $\overline{\mathbb{R}}^n_+ = \{x \in \mathbb{R}^n \Big| x_n \geq 0\}$ is its closure. The extension of a function (or distribution) by zero from $\mathbb{R}^n_+$ to $\mathbb{R}^n$ is denoted by e^+, and r^+ is the restriction from $\mathbb{R}^n$ to $\overline{\mathbb{R}}^n_+$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_+$ as usual we denote $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\partial^{\alpha_j}_{x_j} = \partial^{\alpha_j}/\partial_{x_j}$, $\partial^\alpha_x = \partial^{\alpha_1}_{x_1} \dots \partial^{\alpha_n}_{x_n}$, and $D_j = -i\partial/\partial_{x_j}$, $D^\alpha = D^{\alpha_1}_1 \dots D^{\alpha_n}_n$. $C^\infty(\mathbb{R}^n)$, $C_b(\mathbb{R}^n)$, $C^\infty_0(\mathbb{R}^n)$ and $S(\mathbb{R}^n)$ is the set of all infinitely differentiable functions on $\mathbb{R}^n$, the set of all continuous and bounded functions on $\mathbb{R}^n$, the space of compactly supported smooth functions and the Schwartz space, respectively. Let $S'(\mathbb{R}^n)$ be the space of the distributions under $S(\mathbb{R}^n)$, the spaces $C^\infty(\overline{\mathbb{R}}^n_+)$ and $S(\overline{\mathbb{R}}^n_+)$ are specified by the restriction on $\overline{\mathbb{R}}^n_+$, $S'(\mathbb{R}^n_+)$ is the set of all distributions under $S(\mathbb{R}^n_+) \subset S(\mathbb{R}^n)$. The subset of $C^\infty(\overline{\mathbb{R}}^n_+)$ of all functions with compact supports in $\overline{\mathbb{R}}^n_\pm$ is denoted by $C^\infty_c(\overline{\mathbb{R}}^n_\pm)$. The Fourier transform for $u \in S(\mathbb{R}^n)$ is denoted by

$$Fu(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx,$$

and for its inverse the notation

$$F^{-1}v(x) = \check{v}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi\cdot x} v(\xi)\, d\xi$$

is used. For $u(x', x_n) \in S(\mathbb{R}^n)$, where $x' = (x_1, x_2, \dots, x_{n-1})$, a partial transformation in x' is specified by

$$\acute{u}(\xi', x_n) = F_{x'\to\xi'} u(x', x_n) = \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} u(x', x_n)\, dx'.$$

Let $\gamma_0 u(x') = \lim\limits_{x_n\to+0} u(x', x_n)$ be the restriction to the hyperplane $x_n = 0$ and $\gamma_j = \gamma_0 D^j_{x_n}$. The function $\operatorname{cap}(\cdot) \in C^\infty_0(\mathbb{R})$ such that $0 \leq \operatorname{cap}(t) \leq 1$ and $\operatorname{cap}(t) = 1$ for $|t| \leq 1$ and $\operatorname{cap}(t) = 0$ for $|t| \geq 2$ will be useful below. It should be noted that a symbol C without indices will be used to denote various positive constants in the estimates given below.

§ 1. Hölder–Zygmund spaces of variable smoothness

For Hölder–Zygmund spaces of variable smoothness to be defined we introduce a bounded continuous real-valued function $s(\cdot)$ on $\mathbb{R}^n$ which satisfies for any $x \in \mathbb{R}^n$ and $0 < |y| < 1$ the condition

$$|s(x+y) - s(x)| \leq \frac{S_1}{|\log_2 |y||} \tag{1.1}$$

with not depending on x and y constant $S_1 > 0$.

In order to define the Hölder–Zygmund spaces of the variable smoothness we use the Littlewood–Paley partition of unity (see, e.g., [15, p. 57], [14, p. 241–243]). Let $\lambda_0(\xi) = \operatorname{cap}(|\xi|)$. For integer $j \geq 1$ we denote

$$\lambda_j(\xi) = \lambda_0(2^{-j}\xi) - \lambda_0(2^{-j+1}\xi).$$

Then $\operatorname{supp} \lambda_0 \subset \{\xi \in \mathbb{R}^n \mid |\xi| \leq 2\}$ and $\operatorname{supp} \lambda_j \subset \{\xi \in \mathbb{R}^n \mid 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$ for $j \in \mathbb{N}$. It is obviously that $\sum\limits_{j=0}^{\infty} \lambda_j = 1$.

Definition 1 (see [9]). It is said that the distribution $f \in S'(\mathbb{R}^n)$ belongs to the Hölder–Zygmund space $\Lambda^{s(\cdot)}(\mathbb{R}^n)$ of variable smoothness $s(\cdot)$ if

$$\|f\|_{\Lambda^{s(\cdot)}(\mathbb{R}^n)} = \sup_{j\in\mathbb{Z}_+} \|2^{js(\cdot)}\lambda_j(D)f\|_{L_\infty(\mathbb{R}^n)} < \infty. \tag{1.2}$$

If $s(\cdot) = s$ is constant then it is clear from the definition that the space $\Lambda^{s(\cdot)}(\mathbb{R}^n)$ coincides with the usual Hölder–Zygmund space $\Lambda^s(\mathbb{R}^n)$ (see e.g. [14,15]) and $\Lambda^{s_2(\cdot)}(\mathbb{R}^n) \subset \Lambda^{s_1(\cdot)}(\mathbb{R}^n)$ provided $s_1(x) \le s_2(x)$ for any x. Let

$$s_- = \inf_{x\in\mathbb{R}^n} s(x) \quad \text{and} \quad s_+ = \sup_{x\in\mathbb{R}^n} s(x),$$

then the following estimates for the norms hold:

$$\|u\|_{\Lambda^{s_-}(\mathbb{R}^n)} \le \|u\|_{\Lambda^{s(\cdot)}(\mathbb{R}^n)} \le \|u\|_{\Lambda^{s_+}(\mathbb{R}^n)}.$$

The norm (1.2) is equivalent to the norm

$$\|u\|_{L_\infty(\mathbb{R}^n)} + \sup_{0<|h|<1}\sup_{x\in\mathbb{R}^n} \frac{|u(x+2h) - 2u(x+h) + u(x)|}{|h|^{s(x)}}$$

if $0 < s_- < s(x) < s_+ < 2$ for all $x \in \mathbb{R}^n$ (see [9]). Let us introduce the Banach space $\Lambda^{s(\cdot)}(\overline{\mathbb{R}}^n_+)$ of all distributions from $S'(\overline{\mathbb{R}}^n_+)$, admitting extension to $\mathbb{R}^n$, which belongs to $\Lambda^{s(\cdot)}(\mathbb{R}^n)$. Norm in $\Lambda^{s(\cdot)}(\overline{\mathbb{R}}^n_+)$ is defined by formula

$$\|u\|_{\Lambda^{s(\cdot)}(\overline{\mathbb{R}}^n_+)} = \inf_{lu\in\Lambda^{s(\cdot)}(\mathbb{R}^n)} \|lu\|_{\Lambda^{s(\cdot)}(\mathbb{R}^n)},$$

where infimum is taken over all extensions of distribution u to $\mathbb{R}^n$ belonging to $\Lambda^{s(\cdot)}(\mathbb{R}^n)$.

§ 2. Pseudodifferential operator

Definition 2 (see [7]). Assuming $m \in \mathbb{R}$ we define Hörmander class $S^m(\mathbb{R}^{2n})$ $(= S^m_{1,0}(\mathbb{R}^{2n}))$ to consist of all functions $a \in C^\infty(\mathbb{R}^{2n})$ such that

$$|a|^m_{p,q} = \max_{|\alpha|\le p,\,|\beta|\le q}\ \sup_{(x,\xi)\in\mathbb{R}^{2n}} \left|\partial_x^\beta\partial_\xi^\alpha a(x,\xi)\right|(1+|\xi|)^{|\alpha|-m} < \infty$$

for every $p,\ q \in \mathbb{Z}_+$.

For every symbol $a \in S^m(\mathbb{R}^{2n})$ we associate a pseudodifferential operator $A = a(x,D)$ acting upon a function $f \in S(\mathbb{R}^n)$ by the formula

$$Af(x) = \frac{1}{(2\pi)^n}\int a(x,\xi)e^{ix\cdot\xi}\hat{f}(\xi)\,d\xi.$$

The class of corresponding operators will be denoted by OPS^m. The pseudodifferential operators from the class OPS^m are well coordinated to the Hölder–Zygmund spaces of the variable smoothness in the sense that they act continuously in a suitable pair of the spaces. Namely, the following theorem holds.

Theorem 1 (see [9]). *Suppose a is a symbol in $S^m(\mathbb{R}^{2n})$. Then the operator $A = a(x,D)$ is a bounded mapping from $\Lambda^{s(\cdot)}(\mathbb{R}^n)$ to $\Lambda^{s(\cdot)-m}(\mathbb{R}^n)$, and there exist a constant $C > 0$ and non-negative integers p and q not depending on the symbol a such that*

$$\|A\| \le C|a|^m_{p,q}.$$

In order for the function r^+Ae^+u to belong to $C^\infty(\overline{\mathbb{R}}^n_+)$ for any $u \in C_c^\infty(\overline{\mathbb{R}}^n_+)$ we will require the following additional condition for the symbol of $A \in OPS^m$ to be hold:

Definition 3 (see [4,12], [13, p. 170]). The pseudodifferential operator $a(x, D) \in OPS^m$, $m \in \mathbb{Z}$, satisfies *the transmission property* with respect to the boundary $\mathbb{R}^{n-1}$ (when $x_n = 0$), if each derivative of its symbol admits the series expansion

$$\partial^\gamma_{x_n} a(x,\xi)\Big|_{x_n=+0} = \sum_{s=0}^{m} \tilde{a}_s(x',\xi')\xi_n^s + \sum_{k=-\infty}^{\infty} a_k(x',\xi')\frac{(\langle\xi'\rangle - i\xi_n)^k}{(\langle\xi'\rangle + i\xi_n)^{k+1}},$$

where $\tilde{a}_s(x',\xi') \in S^{m-s}(\mathbb{R}^{2(n-1)})$ and $a_k(x',\xi')$ is a rapidly decreasing sequence in $S^{m+1}(\mathbb{R}^{2(n-1)})$, that is, for any seminorm $|\cdot|^{m+1}_{p,q}$ on $S^{m+1}(\mathbb{R}^{2(n-1)})$ and any number $N \in \mathbb{Z}_+$ there is a constant $C = C(p,q,N)$ such that

$$|a_k|^{m+1}_{p,q} \le C(1+|k|)^{-N}.$$

The following system of seminorms can be introduced in this class

$$|a|^m_\omega = \sum_{s=0}^{m} |\tilde{a}_s|^{m-s}_{p,q} + \sup_{k\in\mathbb{Z}} |a_k|^{m+1}_{p,q}(1+|k|)^N$$

for any $\omega = (\gamma, p, q, N) \in \mathbb{Z}^4_+$.

§ 3. Poisson operator

Definition 4 (see [12,13]). Let $k(x',\xi)$ be infinitely smooth function on $\mathbb{R}^{n-1} \times \mathbb{R}^n$, admitting the series expansion

$$k(x',\xi) = \sum_{j=0}^{\infty} k_j(x',\xi')\frac{(\langle\xi'\rangle - i\xi_n)^j}{(\langle\xi'\rangle + i\xi_n)^{j+1}},$$

where k_j is a rapidly decreasing sequence in $S^{m+1}(\mathbb{R}^{2(n-1)})$, i.e., for any seminorm $|\cdot|^{m+1}_{p,q}$ on $S^{m+1}(\mathbb{R}^{2(n-1)})$ and for any $N \in \mathbb{Z}_+$ there is a constant $C = C(p,q,N)$ such that

$$|k_j|^{m+1}_{p,q} \le C(1+j)^{-N}.$$

The function $k(x',\xi)$ is called a *potential symbol* of order $m \in \mathbb{R}$.

The space of all potential symbols of order m is denoted by $\mathcal{K}^m(\mathbb{R}^{n-1},\mathbb{R}^n)$, $\mathcal{K}^{-\infty}(\mathbb{R}^{n-1},\mathbb{R}^n) = \bigcap_m \mathcal{K}^m(\mathbb{R}^{n-1},\mathbb{R}^n)$. Let us define the following system of seminorms on $\mathcal{K}^m(\mathbb{R}^{n-1},\mathbb{R}^n)$:

$$|k|^m_\omega = \sup_{j\in\mathbb{Z}_+} |k_j|^{m+1}_{p,q}(1+j)^N,$$

for any $\omega = (p,q,N) \in \mathbb{Z}^3_+$.

The Poisson operator was introduced in the works of Vishik and Eskin (see [4, p. 76, 198]) and used to be called a potential type (or coboundary) operator. The Poisson operator K arises in the description of the solution of the boundary value problem for the elliptic (pseudo)differential operator.

Definition 5 (see [4,12,13]). The Poisson operator with the symbol $k(x',\xi) \in \mathcal{K}^m(\mathbb{R}^{n-1},\mathbb{R}^n)$ of order m acts on the function $v \in S(\mathbb{R}^{n-1})$ by the following formula:

$$Kv(x) = (2\pi)^{-n}\int_{\mathbb{R}_+} d\xi_n \int_{\mathbb{R}^{n-1}} e^{ix\cdot\xi}k(x',\xi)\acute{v}(\xi')\,d\xi'.$$

The class of all such operators is denoted by $OP(\mathcal{K}^m)$.

The Poisson operators can be described using the following symbol-kernel, which is determined by the formula

$$\tilde{k}(x', x_n, \xi') = F^{-1}_{\xi_n \to x_n} k(x', \xi', \xi_n),$$

where symbol-kernel $\tilde{k} \in S^m(\mathbb{R}^{2(n-1)}, S(\overline{\mathbb{R}}_+))$. The symbol-kernel allows the Poisson operator K to be written according to the formula

$$Kv(x) = (2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} \tilde{k}(x', x_n, \xi') \acute{v}(\xi')\, d\xi',$$

for $v \in S(\mathbb{R}^{n-1})$.

Let P be a pseudodifferential operator in a half-space satisfying the transmission property with respect to the boundary $x_n = 0$, then the operator $K_P : S(\mathbb{R}^{n-1}) \to S(\overline{\mathbb{R}}^n_+)$ defined by the formula

$$K_P v(x) = r^+ P\big(v(x') \otimes \delta(x_n)\big)$$

is a Poisson operator (see, for instance, [4, p. 199], [12], [13, p. 212]). It was proved that any Poisson operator can be represented in this form, namely, the following lemma is true.

Lemma 1 (see [4, 6, 12, 13]). *For any $k(x', \xi) \in \mathcal{K}^m(\mathbb{R}^{n-1}, \mathbb{R}^n)$ there is a symbol $p(x, \xi) \in$ $\in S^m(\mathbb{R}^{2n})$ with the transmission property such that*

$$Kv = r^+ p(x, D)\big(v(x') \otimes \delta(x_n)\big), \quad v \in S(\mathbb{R}^{n-1}).$$

Any seminorm of the symbol of the pseudodifferential operator P is estimated by a finite number of seminorms of the symbol of the Poisson operator K.

Lemma 2. *Suppose $s(x', 0) + 1 \le s_1 < 0$ holds for any $x' \in \mathbb{R}^{n-1}$. Then there exists a constant $C > 0$ such that for any distribution $v \in \Lambda^{s(\cdot', 0)+1}(\mathbb{R}^{n-1})$ the following inequality is satisfied*

$$\|v(\cdot') \otimes \delta(\cdot_n)\|_{\Lambda^{s(\cdot)}(\mathbb{R}^n)} \le C \|v\|_{\Lambda^{s(\cdot', 0)+1}(\mathbb{R}^{n-1})}. \tag{3.1}$$

P r o o f. Let $v \in \Lambda^{s(\cdot', 0)+1}(\mathbb{R}^{n-1})$. We define $\tilde{\lambda}_0$ by the formula

$$\tilde{\lambda}_0(\xi) = \mathrm{cap}(\xi_1)\mathrm{cap}(\xi_2)\ldots\mathrm{cap}(\xi_n).$$

For integer $j > 0$ we denote

$$\tilde{\lambda}_j(\xi) = \tilde{\lambda}_0(2^{-j}\xi) - \tilde{\lambda}_0(2^{-j+1}\xi).$$

The system of functions $\tilde{\lambda}_j$ for $j \ge 0$ gives the equivalent norm in $\Lambda^{s(\cdot)}(\mathbb{R}^n)$ (see [10]). Then let us consider the expression

$$\left|2^{js(x)}\tilde{\lambda}_j(D)v(x') \otimes \delta(x_n)\right| = \left|2^{js(x)} \int_{\mathbb{R}^n} \tilde{\lambda}_j(\xi)\acute{v}(\xi') e^{ix'\cdot\xi'} e^{ix_n\cdot\xi_n}\, \bar{d}\xi'\, \bar{d}\xi_n\right|.$$

By replacing the function $\tilde{\lambda}_j(\xi)$ under the integral with $\tilde{\lambda}_0(2^{-j}\xi) - \tilde{\lambda}_0(2^{-j+1}\xi)$, we obtain

$$\left|2^{js(x)}\tilde{\lambda}_j(D)v(x') \otimes \delta(x_n)\right| = (2\pi)^{-n}\bigg|2^{js(x)} \int_{\mathbb{R}^n} \Big(\tilde{\lambda}_0(2^{-j}\xi', 0)\tilde{\lambda}_0(0, 2^{-j}\xi_n) -$$
$$- \tilde{\lambda}_0(2^{-j+1}\xi', 0)\tilde{\lambda}_0(0, 2^{-j+1}\xi_n)\Big)\, \acute{v}(\xi') e^{ix'\cdot\xi'} e^{ix_n\cdot\xi_n}\, d\xi'\, d\xi_n\bigg| =$$

$$= (2\pi)^{-n+1}\left|2^{js(x)}\int_{\mathbb{R}^{n-1}}\Big(\tilde{\lambda}_0(2^{-j}\xi')l_j(x_n) - \tilde{\lambda}_0(2^{-j+1}\xi')l_{j-1}(x_n)\Big)\acute{v}(\xi')e^{ix'\cdot\xi'}d\xi'\right|,$$

where $\tilde{\lambda}_0(\eta') = \tilde{\lambda}_0(\eta', 0)$ and l_j denotes the inverse Fourier transform of the function $\tilde{\lambda}_0(0, 2^{-j}\xi_n)$ for $j \geq 0$:

$$l_j(x_n) = (2\pi)^{-1}\int_{\mathbb{R}}\tilde{\lambda}_0(0, 2^{-j}\xi_n)e^{ix_n\cdot\xi_n}d\xi_n = 2^j l_0(2^j x_n).$$

Therefore

$$\left|2^{js(x)}\tilde{\lambda}_j(D)v(x') \otimes \delta(x_n)\right| =$$

$$= (2\pi)^{-n+1}\left|2^{js(x)+j}\int_{\mathbb{R}^{n-1}}\left(\tilde{\lambda}_0(2^{-j}\xi')l_0(2^j x_n) - \frac{1}{2}\tilde{\lambda}_0(2^{-j+1}\xi')l_0(2^{j-1}x_n)\right)\acute{v}(\xi')e^{ix'\cdot\xi'}\,d\xi'\right| \leq$$

$$\leq (2\pi)^{-n+1}\left|2^{js(x)+j}\int_{\mathbb{R}^{n-1}}\left(\tilde{\lambda}_0(2^{-j}\xi') - \tilde{\lambda}_0(2^{1-j}\xi')\right)l_0(2^j x_n)\acute{v}(\xi')e^{ix'\cdot\xi'}\,d\xi'\right| +$$

$$+ (2\pi)^{-n+1}\left|2^{js(x)+j}\int_{\mathbb{R}^{n-1}}\tilde{\lambda}_0(2^{1-j}\xi')\left(l_0(2^j x_n) - \frac{1}{2}l_0(2^{j-1}x_n)\right)\acute{v}(\xi')e^{ix'\cdot\xi'}\,d\xi'\right| = I_1 + I_2,$$

where

$$I_1 = (2\pi)^{-n+1}\left|2^{js(x',0)+j}\int_{\mathbb{R}^{n-1}}\left(\tilde{\lambda}_0(2^{-j}\xi') - \right.\right.$$

$$\left.\left.- \tilde{\lambda}_0(2^{1-j}\xi')\right)\acute{v}(\xi')e^{ix'\cdot\xi'}\,d\xi'\right| \cdot \left|2^{j(s(x',x_n)-s(x',0))}l_0(2^j x_n)\right|$$

and

$$I_2 = (2\pi)^{-n+1}\left|2^{js(x',0)+j}\int_{\mathbb{R}^{n-1}}\tilde{\lambda}_0(2^{1-j}\xi')\acute{v}(\xi')e^{ix'\cdot\xi'}\,d\xi'\right| \cdot$$

$$\cdot\left|2^{j(s(x',x_n)-s(x',0))}\left(l_0(2^j x_n) - \frac{1}{2}l_0(2^{j-1}x_n)\right)\right|.$$

Considering that $\tilde{\lambda}_0(2^{-j}\xi') - \tilde{\lambda}_0(2^{1-j}\xi') = \tilde{\lambda}_j(\xi')$ we obtain for the first factor in I_1:

$$(2\pi)^{-n+1}\left|2^{js(x',0)+j}\int_{\mathbb{R}^{n-1}}\left(\tilde{\lambda}_0(2^{-j}\xi') - \tilde{\lambda}_0(2^{1-j}\xi')\right)\acute{v}(\xi')e^{ix'\cdot\xi'}\,d\xi'\right| =$$

$$= \left|2^{j(s(x',0)+1)}\tilde{\lambda}_j(D', 0)v(x')\right| \leq C\|v\|_{\Lambda^{s(\cdot',0)+1}(\mathbb{R}^{n-1})}.$$

It should be noted that the function $\tilde{\lambda}_0(2^{1-j}\xi')$ in the first factor of I_2 can be written as:

$$\begin{aligned}\tilde{\lambda}_0(2^{1-j}\xi') = \left(\tilde{\lambda}_0(2^{-j+1}\xi') - \tilde{\lambda}_0(2^{-j+2}\xi')\right) + \left(\tilde{\lambda}_0(2^{-j+2}\xi') - \tilde{\lambda}_0(2^{-j+3}\xi')\right) +\\ \ldots + \left(\tilde{\lambda}_0(2^{-1}\xi') - \tilde{\lambda}_0(\xi')\right) + \tilde{\lambda}_0(\xi') = \sum_{k=0}^{j-1}\tilde{\lambda}_k(\xi').\end{aligned} \tag{3.2}$$

Thus, applying the formula (3.2) we have for the first factor of I_2

$$(2\pi)^{-n+1}\left|2^{j(s(x',0)+1)}\int_{\mathbb{R}^{n-1}}\tilde{\lambda}_0(2^{-j+1}\xi')\acute{v}(\xi')e^{ix'\cdot\xi'}\,d\xi'\right| = \left|2^{j(s(x',0)+1)}\sum_{k=0}^{j-1}\tilde{\lambda}_k(D', 0)v(x')\right| \leq$$

$$\leq \sum_{k=0}^{j-1}\left|2^{k(s(x',0)+1)}\tilde{\lambda}_k(D', 0)v(x')\right|2^{j(s(x',0)+1)}2^{-k(s(x',0)+1)} \leq$$

$$\leq \sup_{r\in\mathbb{Z}_+}\|2^{r(s(\cdot',0)+1)}\tilde{\lambda}_r(D', 0)v\|_{L_\infty(\mathbb{R}^{n-1})}\sum_{k=0}^{j-1}2^{(j-k)(s(x',0)+1)} = \|v\|_{\Lambda^{s(\cdot',0)+1}(\mathbb{R}^{n-1})}\frac{1 - 2^{j(s(x',0)+1)}}{2^{-s(x',0)-1} - 1}.$$

Since $s(x',0)+1 \le s_1 < 0$ holds for any $x' \in \mathbb{R}^{n-1}$, then

$$0 < 1 - 2^{j(s(x',0)+1)} < 1 \quad \text{and} \quad 2^{-s(x',0)-1} - 1 \ge 2^{-s_1} - 1 = c > 0.$$

Therefore

$$\left| 2^{j(s(x',0)+1)} \tilde{\lambda}_0(2^{-j+1}D',0)v(x') \right| \le C\|v\|_{\Lambda^{s(\cdot',0)+1}(\mathbb{R}^{n-1})}.$$

Let us consider the kernels l_j of pseudodifferential operators $\tilde{\lambda}_0(0,D_n)$. Then for any $M \in \mathbb{Z}_+$ there is a constant $A_M > 0$ such that

$$|l_0(z)| \le A_M |z|^{-M}.$$

This estimate is a particular case of the estimate in [14, p. 244]. Then for $z = 2^j x_n$

$$|l_0(2^j x_n)| \le A_M 2^{-jM} |x_n|^{-M}. \tag{3.3}$$

The boundedness of the second factor in I_1 will be proved by examining it on two sets $|x_n| \ge 2^{-\frac{j}{2}}$ and $|x_n| < 2^{-\frac{j}{2}}$. Let us consider first the case $|x_n| \ge 2^{-\frac{j}{2}}$, then $|x_n|^{-M} \le 2^{\frac{jM}{2}}$. Using (3.3), we obtain

$$\left| 2^{j(s(x',x_n)-s(x',0))} l_0(2^j x_n) \right| \le 2^{j(s(x',x_n)-s(x',0))} A_M 2^{-jM} |x_n|^{-M} \le$$
$$\le A_M 2^{j(s(x',x_n)-s(x',0))} 2^{-jM} 2^{\frac{jM}{2}} = A_M 2^{j(s(x',x_n)-s(x',0)-\frac{M}{2})} \le A_M < \infty$$

for $M > 2(s_+ - s_-)$. In the case $|x_n| < 2^{\frac{-j}{2}}$ we have $\log_2|x_n| < \frac{-j}{2}$ and $|\log_2|x_n|| > \frac{j}{2}$, therefore, from the inequality (1.1) it follows that

$$s(x',x_n) - s(x',0) \le \frac{S_1}{|\log_2|x_n||} < \frac{2S_1}{j}$$

and $j(s(x',x_n)-s(x',0)) < 2S_1$.
Choosing in (3.3) $M = 0$ we get

$$\left| 2^{j(s(x',x_n)-s(x',0)} l_0(2^j x_n) \right| \le A_0 4^{S_1} < \infty.$$

Hence the second factor in I_1 does not exceed the largest of two numbers A_M (where $M > > 2(s_+ - s_-)$) and $4^{S_1}A_0$. The second factor in I_2 is estimated in the same way as in I_1. Thus, collecting estimates for I_1 and I_2 together, we obtain the inequality (3.1). □

Theorem 2. *Let $k(x',\xi) \in \mathcal{K}^m(\mathbb{R}^{n-1},\mathbb{R}^n)$, then the Poisson operator K is bounded from $\Lambda^{s(\cdot',0)}(\mathbb{R}^{n-1})$ into $\Lambda^{s(\cdot)-m-1}(\overline{\mathbb{R}}^n_+)$. Moreover, there exist a constant $C > 0$ and $\omega \in \mathbb{Z}^3_+$ such that for all $v \in \Lambda^{s(\cdot',0)}(\mathbb{R}^{n-1})$ the following estimate holds*

$$\|Kv\|_{\Lambda^{s(\cdot)-m-1}(\overline{\mathbb{R}}^n_+)} \le C|k|^m_\omega \|v\|_{\Lambda^{s(\cdot',0)}(\mathbb{R}^{n-1})}.$$

P r o o f. Let $k(x',\xi) \in \mathcal{K}^m(\mathbb{R}^{n-1},\mathbb{R}^n)$. By lemma 1 there is a pseudodifferential operator P with the symbol $p(x,\xi) \in S^m$ with the transmission property such that operator K can be written

$$Kv = r^+ P(v \otimes \delta).$$

First assume that $s(x',0) \le s_+ < 0$ for any $x' \in \mathbb{R}^{n-1}$. Then according to Lemma 2 and Theorem 1 for $v \in \Lambda^{s(\cdot',0)}(\mathbb{R}^{n-1})$ there exist a constant C and non-negative integer p, q not depending on the symbol such that

$$\|Kv\|_{\Lambda^{s(\cdot)-m-1}(\overline{\mathbb{R}}^n_+)} = \left\| r^+ P(v \otimes \delta) \right\|_{\Lambda^{s(\cdot)-m-1}(\overline{\mathbb{R}}^n_+)} \le$$
$$\le \|P(v \otimes \delta)\|_{\Lambda^{s(\cdot)-m-1}(\mathbb{R}^n)} \le C|p|^m_{p,q} \|v \otimes \delta\|_{\Lambda^{s(\cdot)-1}(\mathbb{R}^n)} \le C|p|^m_{p,q} \|v\|_{\Lambda^{s(\cdot',0)}(\mathbb{R}^{n-1})}.$$

By Lemma 1 for any p, q there exists ω such that the inequality $|p|^m_{p,q} \le |k|^m_\omega$ is valid. Therefore, the operator norm of the operator K is estimated through the seminorm of its symbol. Now we allow s_+ to be arbitrary. Then for Poisson operator K there is a condition

$$Kv = r^+ P\left(\left(\langle D'\rangle^{-d} \cdot\right) \otimes \delta\right) \circ \langle D'\rangle^d v,\ v \in \Lambda^{s(\cdot',0)}(\mathbb{R}^{n-1}),$$

where $d \in \mathbb{R}$, $\langle D'\rangle^d$ is bounded from the space $\Lambda^{s(\cdot',0)}(\mathbb{R}^{n-1})$ to $\Lambda^{s(\cdot',0)-d}(\mathbb{R}^{n-1})$. Select $d > s_+$, for proving the theorem it suffices to show that the operator $K' = r^+ P\left(\left(\langle D'\rangle^{-d} \cdot\right) \otimes \delta\right)$ is bounded from $\Lambda^{s(\cdot',0)-d}(\mathbb{R}^{n-1})$ into $\Lambda^{s(\cdot)-m-1}(\overline{\mathbb{R}}^n_+)$, which is true since $s(x',0) - d < s_+ - d < 0$ and the operator K' is a Poisson operator with the symbol $k(x',\xi',\xi_n)\langle\xi'\rangle^{-d} \in \mathcal{K}^{m-d}(\mathbb{R}^{n-1},\mathbb{R}^n)$. □

§ 4. Trace operator

Definition 6 (see [4, 12, 13]). Let $t(x',\xi)$ be C^∞-function on $\mathbb{R}^{n-1} \times \mathbb{R}^n$ admitting the following series expansion:

$$t(x',\xi) = \sum_{j=0}^{r-1} \underline{t}_j(x',\xi')\xi_n^j + \sum_{k=0}^{\infty} t_k(x',\xi') \frac{\left(\langle\xi'\rangle + i\xi_n\right)^k}{\left(\langle\xi'\rangle - i\xi_n\right)^{k+1}} = t_1(x',\xi) + t_2(x',\xi),$$

where $\underline{t}_j \in S^{m-j}(\mathbb{R}^{2(n-1)})$ and t_k form a rapidly decreasing sequence in $S^{m+1}(\mathbb{R}^{2(n-1)})$, that is, for any seminorm $|\cdot|^{m+1}_{p,q}$ on $S^{m+1}(\mathbb{R}^{2(n-1)})$ and $N \in \mathbb{Z}_+$ there is a constant $C = C(p,q,N)$ such that

$$|t_k|^{m+1}_{p,q} \le C(1+k)^{-N}.$$

Function $t(x',\xi)$ is called a *trace* (boundary) symbol of order $m \in \mathbb{R}$ and of class $r \in \mathbb{Z}_+$. The space of all trace symbols of order m and class r is denoted by $\mathcal{L}^{m,r}(\mathbb{R}^{n-1},\mathbb{R}^n)$, the following system of seminorms is introduced

$$|t|^{m,r}_\omega = \sum_{j=0}^{r-1} |\underline{t}_j|^{m-j}_{p,q} + \sup_{k\in\mathbb{Z}_+} |t_k|^{m+1}_{p,q}(1+k)^N,$$

where $\omega = (p,q,N) \in \mathbb{Z}^3_+$. The intersection over m of all classes $\mathcal{L}^{m,r}(\mathbb{R}^{n-1},\mathbb{R}^n)$ is denoted by $\mathcal{L}^{-\infty,r}(\mathbb{R}^{n-1},\mathbb{R}^n)$.

Definition 7 (see [4, 12, 13]). The *trace* operator T with the symbol $t(x',\xi) \in \mathcal{L}^{m,r}(\mathbb{R}^{n-1},\mathbb{R}^n)$ is defined by the formula

$$Tu(x') = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix'\cdot\xi'} t(x',\xi)\widehat{e^+u}(\xi)\, d\xi_n\, d\xi', \quad u \in S(\overline{\mathbb{R}}^n_+).$$

The space of trace operators with symbols from $\mathcal{L}^{m,r}(\mathbb{R}^{n-1},\mathbb{R}^n)$ is denoted by $OP(\mathcal{L}^{m,r})$. We need the following statement:

Lemma 3 (see [6, 8]). *The trace operator T of order m and class r can be written as a sum*

$$Tu(x') = \sum_{j=0}^{r-1} T_j\gamma_j u(x') + T_0 u(x'), \quad u \in S(\overline{\mathbb{R}}^n_+),$$

where each $T_j = \underline{t}_j(x',D')$ is a pseudodifferential operator on $\mathbb{R}^{n-1}$ with symbol $\underline{t}_j(x',\xi') \in S^{m-j}(\mathbb{R}^{2(n-1)})$. Operator T_0 can be defined by the symbol-kernel $\tilde{t}_0 \in S^m(\mathbb{R}^{2(n-1)}, S(\overline{\mathbb{R}}_+))$ according to the formula

$$T_0 u(x') = (2\pi)^{-n-1} \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} \int_0^\infty \tilde{t}_0(x',x_n,\xi')\acute{u}(\xi',x_n)\, dx_n\, d\xi',$$

for $\tilde{t}_0(x,\xi') = F^{-1}_{\xi_n\to x_n} t_0(x',\xi)$.

It is easy to see that the restriction γ_0 to the hyperplane $x_n = 0$ acts continuously from $\Lambda^s(\overline{\mathbb{R}}^n_+)$ to $\Lambda^s(\mathbb{R}^{n-1})$ for arbitrary $s > 0$ (see, e.g., [13, p. 247], [15, p. 192], [14]). Then the trace γ_j is bounded from $\Lambda^s(\overline{\mathbb{R}}^n_+)$ to $\Lambda^{s-j}(\mathbb{R}^{n-1})$ if $s > j$ for any $j \in \mathbb{Z}_+$. It is not difficult to prove that the similar result is valid in the case of a variable order of smoothness:

Lemma 4. *The trace γ_j is a bounded mapping from $\Lambda^{s(\cdot)}(\overline{\mathbb{R}}^n_+)$ to $\Lambda^{s(\cdot',0)-j}(\mathbb{R}^{n-1})$, $j \in \mathbb{Z}_+$, if there exists $\varepsilon > 0$ such that $s(x', 0) - j \geq \varepsilon$ is fulfilled for all $x' \in \mathbb{R}^{n-1}$.*

In order to prove boundedness of the trace operator we need some facts about pseudodifferential operators with operator-valued symbols (see, e.g., [6], [13, p. 202]). Let B_1 and B_2 be Banach spaces, $B = \mathcal{B}(B_1, B_2)$ be the Banach space of bounded linear operators acting from B_1 to B_2. In the usual way we introduce the Banach-valued analogues: the Schwarts space $S(\mathbb{R}^{n-1}, B_i)$, the L. Hörmander class $S^m(\mathbb{R}^{2(n-1)}; B)$, the Hölder–Zygmund space $\Lambda^{s(\cdot')}(\mathbb{R}^{n-1}, B_i)$ with variable smoothness $s(x')$ and others. The theorem on the boundedness of a pseudodifferential operator in Hölder–Zygmund spaces can be almost literally carried over to the Banach-valued case. This means that there are non-negative integers p, q and constant $C > 0$ independent of u such that

$$\|a(x', D')u\|_{\Lambda^{s(\cdot')-m}(\mathbb{R}^{n-1}, B_2)} \leq C|a|^m_{p,q}\|u\|_{\Lambda^{s(\cdot')}(\mathbb{R}^{n-1}, B_1)}$$

where $a \in S^m(\mathbb{R}^{2(n-1)}; B)$.

Theorem 3. *The trace operator $T \in OP(\mathcal{L}^{m,r})$ is bounded from $\Lambda^{s(\cdot)}(\overline{\mathbb{R}}^n_+)$ into $\Lambda^{s(\cdot',0)-m}(\mathbb{R}^{n-1})$, if the following conditions are satisfied:*
(i) there exists $\tau > 0$ such that $s(x', x_n) \geq s(x', 0)$ when $x_n \in (0; \tau)$;
(ii) there exists $\varepsilon > 0$ such that $s(x', 0) \geq \max\{r - 1 + \varepsilon, \varepsilon\}$ for any $x' \in \mathbb{R}^{n-1}$.

P r o o f. Let $u(x) \in \Lambda^{s(\cdot)}(\overline{\mathbb{R}}^n_+)$. By Lemma 3 the trace operator T acts by the formula

$$Tu(x') = \sum_{j=0}^{r-1} T_j \gamma_j u(x') + T_0 u(x'),$$

where $T_j = \underline{t}_j(x', D')$ are pseudodifferential operators on $\mathbb{R}^{n-1}$ with symbols $\underline{t}_j \in S^{m-j}(\mathbb{R}^{2(n-1)})$, and T_0 is the trace operator of the zero class. The traces γ_j with $j = 0, 1, \ldots, r-1$ are continuous $\Lambda^{s(\cdot)}(\overline{\mathbb{R}}^n_+) \xrightarrow{\gamma_j} \Lambda^{s(\cdot',0)-j}(\mathbb{R}^{n-1})$ if there exists $\varepsilon > 0$ such that $s(x', 0) - j \geq \varepsilon$ holds for any $x' \in \in \mathbb{R}^{n-1}$. Operators T_j of the order $m - j$ are bounded from $\Lambda^{s(\cdot',0)-j}(\mathbb{R}^{n-1})$ into $\Lambda^{s(\cdot',0)-m}(\mathbb{R}^{n-1})$ by Theorem 1. Therefore $\sum_{j=0}^{r-1} T_j\gamma_j$ is bounded from $\Lambda^{s(\cdot)}(\overline{\mathbb{R}}^n_+)$ to $\Lambda^{s(\cdot',0)-m}(\mathbb{R}^{n-1})$ and the operator norm of this operator is estimated through a finite sum of seminorms of symbols $\underline{t}_j$. Namely, there exist a constant $C > 0$ and non-negative integers p, q, such that

$$\sum_{j=0}^{r-1} \left\|\underline{t}_j(x', D')\gamma_j u\right\|_{\Lambda^{s(\cdot',0)-m}(\mathbb{R}^{n-1})} \leq C \sum_{j=0}^{r-1} |\underline{t}_j|^{m-j}_{p,q} \|u\|_{\Lambda^{s(\cdot)}(\overline{\mathbb{R}}^n_+)}.$$

It is easy to notice that from condition (1.1) it follows that there exists $\varepsilon_1 > 0$ such that $s(x', x_n) - r + 1 \geq \varepsilon_1 > 0$ holds for any $x_n \in (0; 2^{1-\frac{S_1}{\varepsilon-\varepsilon_1}})$ and $x' \in \mathbb{R}^{n-1}$. Without loss of generality, we can assume that $2^{1-\frac{S_1}{\varepsilon-\varepsilon_1}} = \tau$. Suppose $\varphi_1(x_n) = \mathrm{cap}(2\frac{|x_n|}{\tau})$ and $\varphi_2 = 1 - \varphi_1$. Then $\varphi_1 \in C_0^\infty(\mathbb{R})$ and $\varphi_1(x_n) = 0$ when $|x_n| > \tau$. We split the operator T_0 into two terms $T_{0,l}u = T_0\varphi_l u$ when $l = 1, 2$. Let us consider the case $l = 1$. According to the condition (i) of the theorem, $\varphi_1(x_n)u(x', x_n) \in \in \Lambda^{s(\cdot',0)}(\overline{\mathbb{R}}^n_+)$ and besides $s(x', 0) \geq \varepsilon > 0$. Then $\Lambda^{s(\cdot',0)}(\overline{\mathbb{R}}^n_+) \subset \Lambda^{s(\cdot',0)}(\mathbb{R}^{n-1}; C_b(\overline{\mathbb{R}}_+))$ and the embedding is continuous. Let a Banach space $B_2 = \mathbb{C}$ and a Banach space $B_1 = C_b(\overline{\mathbb{R}}_+)$ and $a_t(x', \xi') \in B = \mathcal{B}(B_1, B_2)$ is an operator acting by the formula

$$a_t(x', \xi')v = \int_0^\infty \tilde{t}_{0,1}(x', x_n, \xi')v(x_n)\, dx_n,$$

where $v \in C_b(\overline{\mathbb{R}}_+)$. Then the trace operator $T_{0,1}$ of the zero class can be defined as a Banach-valued pseudodifferential operator

$$(T_{0,1}u)(x') = (2\pi)^{-n+1} \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} a_t(x', \xi') \cdot \acute{u}(\xi')\, d\xi'$$

with symbol $a_t(x', \xi') \in S^m(\mathbb{R}^{2(n-1)}; B)$. Then $T_{0,1}$ is bounded from $\Lambda^{s(\cdot',0)}(\mathbb{R}^{n-1}; C_b(\overline{\mathbb{R}}_+))$ into $\Lambda^{s(\cdot',0)-m}(\mathbb{R}^{n-1}; \mathbb{C}) = \Lambda^{s(\cdot',0)-m}(\mathbb{R}^{n-1})$. Therefore, the trace operator $T_{0,1}$ is bounded from $\Lambda^{s(\cdot)}(\overline{\mathbb{R}}^n_+)$ into $\Lambda^{s(\cdot',0)-m}(\mathbb{R}^{n-1})$, moreover, its norm is estimated through the seminorm of a symbol (or a symbol-kernel, which is equivalent) of the trace operator T_0 (see [13, p. 184]). That is, there exist a constant $C > 0$ and $\omega = (p, q, N) \in \mathbb{Z}^3_+$ such that

$$\|T_{0,1}u\|_{\Lambda^{s(\cdot',0)-m}(\mathbb{R}^{n-1})} \leq C|t_0|^{m,0}_\omega \|u\|_{\Lambda^{s(\cdot)}(\overline{\mathbb{R}}^n_+)}.$$

Now we assume $l = 2$. It is clear that $\varphi_2 u(x', x_n) \in \Lambda^{s(\cdot)}(\mathbb{R}^n)$ with an arbitrary extension of the function s on $\mathbb{R}^n_-$ and the operator of multiplication by the function φ_2 is a bounded operator acting from $\Lambda^{s(\cdot)}(\overline{\mathbb{R}}^n_+)$ into $\Lambda^{s(\cdot)}(\mathbb{R}^n)$. Let us represent the operator $T_{0,2}$ in the form $\gamma_0 \circ p(x', D)\varphi_2$ (see [13, p. 214]) where the pseudodifferential operator $p(x', D)$ of the order m is bounded from $\Lambda^{s(\cdot)}(\mathbb{R}^n)$ into $\Lambda^{s(\cdot)-m}(\mathbb{R}^n)$. By the pseudolocality property of the pseudodifferential operator, the distribution $T_{0,2}u$ is infinitely differentiable in a neighborhood of $x_n = 0$. Therefore, the operator γ_0 is defined on the image of the operator $p(x', D)$, moreover,

$$\|\gamma_0 p(x', D)\varphi_2 u\|_{\Lambda^{s(\cdot',0)-m}(\mathbb{R}^{n-1})} \leq c\|u\|_{\Lambda^{s(\cdot)}(\overline{\mathbb{R}}^n_+)},$$

where the constant c is estimated through the seminorm of $p(x', \xi)$, which is estimated through the seminorm of the symbol of the Trace operator T_0 (see [13, p. 214]). Thus, there exists $\omega = (p, q, N) \in \mathbb{Z}^3_+$ such that

$$\|T_{02}u\|_{\Lambda^{s(\cdot',0)-m}(\mathbb{R}^{n-1})} \leq C|t_0|^{m,0}_\omega \|u\|_{\Lambda^{s(\cdot)}(\overline{\mathbb{R}}^n_+)}. \qquad \square$$

It should be noted that the proof of the corresponding theorem for Hölder–Zygmund spaces with constant smoothness can be found in [8] and [13].

§ 5. Singular Green operator

Definition 8 (see [12, 13]). Let $g(x', \xi', \xi_n, \eta_n)$ be C^∞ function on $\mathbb{R}^{2(n-1)} \times \mathbb{R} \times \mathbb{R}$ admitting a series expansion:

$$g(x', \xi', \xi_n, \eta_n) = \sum_{j=0}^{r-1} g_j(x', \xi', \xi_n)\eta_n^j + \sum_{j,l\in\mathbb{Z}_+} g_{jl}(x', \xi') \frac{(\langle\xi'\rangle - i\xi_n)^j}{(\langle\xi'\rangle + i\xi_n)^{j+1}} \frac{(\langle\xi'\rangle + i\eta_n)^l}{(\langle\xi'\rangle - i\eta_n)^{l+1}},$$

where $g_j \in \mathcal{K}^{m-j}(\mathbb{R}^{n-1}, \mathbb{R}^n)$ is the Poisson symbol of order $m - j$, and g_{jl} is a rapidly decreasing double sequence in $S^{m+2}(\mathbb{R}^{2(n-1)})$, that is, for any seminorm $|\cdot|^{m+2}_{p,q}$ on $S^{m+2}(\mathbb{R}^{2(n-1)})$ and all $N \in \mathbb{Z}_+$ there exists a constant $C = C(p, q, N)$ such that

$$|g_{jl}|^{m+2}_{p,q} \leq C(1 + j + l)^{-N}.$$

The function $g(x', \xi', \xi_n, \eta_n)$ is called a *singular Green* symbol of order m and class r.

Definition 9 (see [12, 13]). The *singular Green* operator G of order m and class r with the symbol $g(x', \xi', \xi_n, \eta_n) \in \mathcal{G}^{m,r}(\mathbb{R}^{n-1}, \mathbb{R}^{n+1})$ is defined by the formula

$$Gu(x) = (2\pi)^{-n-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{ix\cdot\xi} g(x', \xi', \xi_n, \eta_n) \widehat{e^+u}(\xi', \eta_n)\, d\xi'\, d\xi_n\, d\eta_n.$$

The class of all such operators is denoted by $OP(\mathcal{G}^{m,r})$.

A singular Green operator arises, for example, as a composition of a Poisson operator and a trace operator. Let us define the Poisson symbols as

$$k_l(x', \xi', \xi_n) = \sum_{j=0}^{\infty} g_{jl}(x', \xi') \frac{(\langle\xi'\rangle - i\xi_n)^j}{(\langle\xi'\rangle + i\xi_n)^{j+1}},$$

then the symbol of a singular Green operator can be written as

$$g(x', \xi', \xi_n, \eta_n) = \sum_{j=0}^{r-1} g_j(x', \xi', \xi_n)\eta_n^j + \sum_{l=0}^{\infty} k_l(x', \xi', \xi_n) \frac{(\langle\xi'\rangle + i\eta_n)^l}{(\langle\xi'\rangle - i\eta_n)^{l+1}},$$

where $k_l(x', \xi', \xi_n)$ is a rapidly decreasing sequence in $\mathcal{K}^{m+1}(\mathbb{R}^{n-1}, \mathbb{R}^n)$, i.e., for any seminorm $|\cdot|_\omega^{m+1}$ on $\mathcal{K}^m(\mathbb{R}^{n-1}, \mathbb{R}^n)$ there is a constant $C = C(\omega)$, $\omega = (p, q, N)$ such that

$$|k_l|_\omega^{m+1} \le C(1 + l)^{-N}.$$

The space of all such symbols is denoted by $\mathcal{G}^{m,r}(\mathbb{R}^{n-1}, \mathbb{R}^{n+1})$, and the following set of seminorms is defined

$$|g|_\omega^{m,r} = \sum_{j=0}^{r-1} |g_j|_\omega^{m-j} + \sup_{j,l\in\mathbb{Z}_+} |g_{jl}|_{p,q}^{m+2} (1 + j + l)^N,$$

where $\omega = (p, q, N) \in \mathbb{Z}_+^3$, or an equivalent set of seminorms (see [13, p. 184])

$$|g|_\omega^{m,r} \approx \sum_{j=0}^{r-1} |g_j|_\omega^{m-j} + \sup_{l\in\mathbb{Z}_+} |k_l|_\omega^{m+1} (1 + l)^N.$$

The intersection over m of all classes $\mathcal{G}^{m,r}(\mathbb{R}^{n-1}, \mathbb{R}^{n+1})$ will be denoted by $\mathcal{G}^{-\infty,r}(\mathbb{R}^{n-1}, \mathbb{R}^{n+1})$.

Theorem 4 (see [12, 13]). *Any singular Green operator $G \in OP(\mathcal{G}^{m,r}(\mathbb{R}^{n-1}, \mathbb{R}^{n+1}))$ can be written in the form*

$$Gu = \sum_{j=0}^{r-1} \tilde{K}_j \gamma_j u + \sum_{l=1}^{\infty} K_l \circ T_l u, \quad u \in S(\overline{\mathbb{R}}_+^n),$$

where $\tilde{K}_j \in OP(\mathcal{K}^{m-j})$ and $K_l \in OP(\mathcal{K}^m)$, $T_l \in OP(\mathcal{L}^{0,0})$, and the corresponding series of symbols

$$\sum_{l=1}^{\infty} k_l(x', \xi', \xi_n) t_l(x', \xi', \eta_n)$$

together with the symbol of the first sum converges to the symbol of the operator G in $\mathcal{G}^{m,r}(\mathbb{R}^{n-1}, \mathbb{R}^{n+1})$.

Theorem 5. *The singular Green operator $G \in OP(\mathcal{G}^{m,r})$ is bounded*

$$G\colon \Lambda^{s(\cdot)}(\overline{\mathbb{R}}^n_+) \to \Lambda^{s(\cdot)-m-1}(\overline{\mathbb{R}}^n_+),$$

provided:
i) there is $\tau > 0$ such that $s(x', x_n) \geq s(x', 0)$ when $x_n \in (0;\tau)$ and $x' \in \mathbb{R}^{n-1}$;
ii) there is $\varepsilon > 0$ such that $s(x', 0) \geq \max\{r - 1 + \varepsilon,\ \varepsilon\}$ for any $x' \in \mathbb{R}^{n-1}$. Moreover, there exist a constant $C > 0$ and $\omega = (p, q, N) \in \mathbb{Z}^3_+$ such that for the operator norm the following estimate is fulfilled:

$$\|Gu\|_{\Lambda^{s(\cdot)-m-1}(\overline{\mathbb{R}}^n_+)} \leq C|g|^{m,r}_{\omega} \|u\|_{\Lambda^{s(\cdot)}(\overline{\mathbb{R}}^n_+)}.$$

P r o o f. By Theorem 4, the singular Green operator G of order m and class r can be represented as

$$G = \sum_{j=0}^{r-1} K_j \circ \gamma_j + G_0,$$

where K_j are the Poisson operators of order $m - j$, γ_j are traces, and

$$G_0 = \sum_{l=1}^{\infty} K_l \circ T_l,$$

where K_l are the Poisson operators of order m and T_l are the Trace operators of order 0 and class 0. Consequently Theorem 5 follows from Theorem 2 and Theorem 3. □

§ 6. Pseudodifferential operator with the transmission property

The following theorem is a transference of the theorems on the boundedness of a pseudodifferential operator with the transmission property ([13, p. 250], [8]) in Hölder–Zygmund spaces with variable smoothness.

Theorem 6. *Let the symbol $p(x,\xi)$ with the transmission property belong to $S^m(\mathbb{R}^{2n})$, $m \in \mathbb{Z}$. Then the operator $P_+ = r^+ p(x, D) e^+$ is bounded from $\Lambda^{s(\cdot)}(\overline{\mathbb{R}}^n_+)$ into $\Lambda^{s(\cdot)-m}(\overline{\mathbb{R}}^n_+)$ provided:*
i) there exists $\tau > 0$ such that $s(x', x_n) \geq s(x', 0)$ for $x_n \in (0;\tau)$,
ii) there exists $\varepsilon > 0$ such that $s(x', 0) \geq \varepsilon$ holds for any $x' \in \mathbb{R}^{n-1}$.

P r o o f. Let the distribution $u(x) \in \Lambda^{s(\cdot)}(\overline{\mathbb{R}}^n_+)$. Since

$$\|u\|_{\Lambda^{s(\cdot)}(\overline{\mathbb{R}}^n_+)} = \inf_v \{\|v\|_{\Lambda^{s(\cdot)}(\mathbb{R}^n)} \mid r^+ v = u\},$$

it follows that one can choose such expansion of distribution u (which will be denoted lu) that

$$\|lu\|_{\Lambda^{s(\cdot)}(\mathbb{R}^n)} \leq 2\|u\|_{\Lambda^{s(\cdot)}(\overline{\mathbb{R}}^n_+)}.$$

Then for the operator P_+ we have

$$r^+ p(x, D) e^+ u = r^+ p(x, D) lu + \left(r^+ p(x, D) e^+ - r^+ p(x, D) l\right) u = r^+ p(x, D) lu + Gu,$$

where G is a singular Green operator of order $m - 1$ and class 0 (see [13, p. 220]). Thereby

$$\|r^+ p(x, D) e^+ u\|_{\Lambda^{s(\cdot)-m}(\overline{\mathbb{R}}^n_+)} \leq \|r^+ p(x, D) lu\|_{\Lambda^{s(\cdot)-m}(\overline{\mathbb{R}}^n_+)} + \|Gu\|_{\Lambda^{s(\cdot)-m}(\overline{\mathbb{R}}^n_+)} \leq$$
$$\leq 2C|p|^m_{p,q}\|u\|_{\Lambda^{s(\cdot)}(\overline{\mathbb{R}}^n_+)} + C|g|^{m-1,0}_{\omega}\|u\|_{\Lambda^{s(\cdot)}(\overline{\mathbb{R}}^n_+)},$$

where $\omega = (p, q, N) \in \mathbb{Z}^3_+$. The symbol of the singular Green operator G calculated in [13] has $\mathcal{G}^{m-1,0}$ seminorms that are estimated by the corresponding seminorms of the symbol $p \in S^m(\mathbb{R}^{2n})$ with the transmission property:

$$|g|^{m-1,0}_{\omega} \leq C|p|^m_{\tilde{\omega}},$$

where $\tilde{\omega} = (\gamma', p', q', N') \in \mathbb{Z}^4_+$ and $\omega = (p, q, N) \in \mathbb{Z}^3_+$. Then

$$\|r^+ p(x, D) e^+ u\|_{\Lambda^{s(\cdot)-m}(\overline{\mathbb{R}}^n_+)} \leq C\left(|p|^m_{p,q} + |p|^m_{\tilde{\omega}}\right)\|u\|_{\Lambda^{s(\cdot)}(\overline{\mathbb{R}}^n_+)}.$$ □

§ 7. Green operators

Let us consider the matrix operator

$$\mathcal{A} = \begin{pmatrix} P_+ + G & K \\ T & Q \end{pmatrix} : \begin{matrix} S(\overline{\mathbb{R}}^n_+)\otimes\mathbb{C}^M \\ \oplus \\ S(\mathbb{R}^{n-1})\otimes\mathbb{C}^N \end{matrix} \to \begin{matrix} S(\overline{\mathbb{R}}^n_+)\otimes\mathbb{C}^{M'} \\ \oplus \\ S(\mathbb{R}^{n-1})\otimes\mathbb{C}^{N'} \end{matrix} \tag{7.1}$$

where

- $P_+ = r^+Pe^+$ is a $M' \times M$-matrix with pseudodifferential operators on a half-space, with symbols of the class $S^m(\mathbb{R}^{2n})$ with the transmission property as elements;
- G is a $M' \times M$-matrix with singular Green operators from $OP(\mathcal{G}^{m-1,r})$ as elements;
- K is a $M' \times M$-matrix with Poisson operators from $OP(\mathcal{K}^{\mu})$ as elements;
- T is a $M' \times M$-matrix with Trace operators from $OP(\mathcal{L}^{\gamma,r})$ as elements;
- Q is a $M' \times M$-matrix with pseudodifferential operators from $OPS^{1-m+\mu+\gamma}$ acting at the boundary as elements.

The operator $\mathcal{A}$ is called a *Green operator*, and the set of Green operators with the indicated orders is denoted by $OP(\mathfrak{G}^{m,\mu,\gamma,r})$.

Consider two spaces, defined by

$$\Lambda_1(\mathbb{R}^n_+) = \Lambda^{s(\cdot)}(\overline{\mathbb{R}}^n_+) \otimes \mathbb{C}^M \oplus \Lambda^{s(\cdot',0)-m+\mu+1}(\mathbb{R}^{n-1}) \otimes \mathbb{C}^N$$

and

$$\Lambda_2(\mathbb{R}^n_+) = \Lambda^{s(\cdot)-m}(\overline{\mathbb{R}}^n_+) \otimes \mathbb{C}^{M'} \oplus \Lambda^{s(\cdot',0)-\gamma}(\mathbb{R}^{n-1}) \otimes \mathbb{C}^{N'}.$$

Summarizing the results obtained in Theorems 2, 3, 5, 6 we arrive at the following

Theorem 7. *Suppose $\mathcal{A} \in OP(\mathfrak{G}^{m,\mu,\gamma,r})$; then it acts continuously*

$$\mathcal{A}\colon \Lambda_1(\mathbb{R}^n_+) \to \Lambda_2(\mathbb{R}^n_+)$$

provided:
i) there is $\tau > 0$ such that $s(x', x_n) \geq s(x', 0)$ holds for all $x_n \in (0;\tau)$ and arbitrary $x' \in \mathbb{R}^{n-1}$;
ii) there is $\varepsilon > 0$ such that $s(x', 0) \geq \max\{r - 1 + \varepsilon; \varepsilon\}$ for any $x' \in \mathbb{R}^{n-1}$.
The norm of the operator $\mathcal{A}$ is estimated by finite number of seminorms of symbols of operators P, G, K, T, Q.

In conclusion, note that condition i) in Theorem 7 is not necessary. For example, if the first column of matrix 7.1 contains only differential operators, then this condition can be removed. The authors do not know whether conditions i) and ii) can be weakened or removed in the general case.

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Kryakvin Vadim Donatovich, Candidate of Physics and Mathematics, I. I. Vorovich Institute of Mathematics, Mechanics, and Computer Science, Southern Federal University, ul. Mil'chakova, 8 a, Rostov-on-Don, 344090, Russia.
ORCID: https://orcid.org/0000-0001-6777-7971
E-mail: kryakvin@sfedu.ru

Omarova Gul'nara Peritkhanovna, Post-Graduate Student, I. I. Vorovich Institute of Mathematics, Mechanics, and Computer Science, Southern Federal University, ul. Mil'chakova, 8 a, Rostov-on-Don, 344090, Russia.
ORCID: https://orcid.org/0000-0001-6364-3159
E-mail: om.gulnara@yandex.ru

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В. Д. Кряквин, Г. П. Омарова

Операторы Буте де Монвеля в пространствах Гёльдера–Зигмунда переменной гладкости на $\mathbb{R}^n_+$

Ключевые слова: алгебра Буте де Монвеля, операторы Грина, пространства Гёльдера–Зигмунда, переменный показатель гладкости.

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Рассматриваются операторы Грина из алгебры Буте де Монвеля в пространствах Гёльдера–Зигмунда переменного порядка гладкости на $\overline{\mathbb{R}}^n_+$. Порядок гладкости зависит от точки пространства и может принимать отрицательные значения. Доказаны достаточные условия ограниченности оператора Буте де Монвеля в этих пространствах.

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Кряквин Вадим Донатович, к. ф.-м. н., доцент, Институт математики, механики и компьютерных наук им. И. И. Воровича, Южный федеральный университет, 344090, Россия, г. Ростов-на-Дону, ул. Мильчакова, 8 а.
ORCID: https://orcid.org/0000-0001-6777-7971
E-mail: kryakvin@sfedu.ru

Омарова Гульнара Перитхановна, аспирант, Институт математики, механики и компьютерных наук им. И. И. Воровича, Южный федеральный университет, 344090, Россия, г. Ростов-на-Дону, ул. Мильчакова, 8 а.
ORCID: https://orcid.org/0000-0001-6364-3159
E-mail: om.gulnara@yandex.ru

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