VESTNIK UDMURTSKOGO UNIVERSITETA. MATEMATIKA. MEKHANIKA. KOMP'UTERNYE NAUKI
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**SPECTRUM ASSIGNMENT AND STABILIZATION OF LINEAR DIFFERENTIAL EQUATIONS WITH DELAY BY STATIC OUTPUT FEEDBACK WITH DELAY**

A linear control system defined by a stationary differential equation with one lumped and one distributed delay is considered. In the system, the input is a linear combination of $m$ variables and their derivatives of order not more than $n-p$ and the output is a $k$-dimensional vector of linear combinations of the state and its derivatives of order not more than $p-1$. For this system, a spectrum assignment problem by linear static output feedback with delays is studied. Necessary and sufficient conditions are obtained for solvability of the arbitrary spectrum assignment problem by static output feedback controller of the same type as the system. Corollaries on stabilization of the system are obtained.

*Keywords*: linear differential equation, lumped delay, distributed delay, spectrum assignment, stabilization, static output feedback.

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## Introduction

A large number of works have been devoted to the problem of stability and stabilization of control systems with delays. A number of methods have been proposed to solving this problem. One of the common methods is known as the Lyapunov–Krasovsky functional approach [1, 2]. This approach is a development of the second Lyapunov method and allows one to obtain sufficient conditions for asymptotic and exponential stabilization of delayed systems. On the base of the Lyapunov-Krasovsky approach, various methods were obtained in [3–6]: in [3], a new technique is introduced based on the barycentric representation of a distributed delay system; in [4], a new stabilization method is proposed for linear systems with distributed input delay via reduction transformation and Riccati equation approach; in [5], conditions for stabilization are obtained by using the full-block $S$-procedure and a convex-hull relaxation in terms of a LMI; in [6] the problem of optimal stabilization is studied. Other approaches to solving the problem of stabilization of systems with distributed delay are presented in [7, 8]. In [7], it is shown how to obtain finite-time stabilization of linear systems with delays in the input by using an extension of Artstein's model reduction to nonlinear feedback. In [8], truncated predictor feedback approach is used for stabilization of time-varying linear systems with multiple and distributed input delays.

Another approach to problems of stability and stabilization of time-delay systems is an eigenvalue-based approach [9]. Here it is required to find conditions providing the desired placement of the spectrum of the system, that is, the sets of zeros of the characteristic function of the system. There are works on assignment of a given finite spectrum [10–16], spectral reducibility [17, 18], i.e., reduction of systems to a finite (but not given) spectrum, modal controllability [19–25]. In the present paper, necessary and sufficient conditions are obtained for arbitrary spectrum assignability by linear static output feedback for a control system defined by a linear differential equation of $n$-th order with one lumped and one distributed delay in the state variable.

## § 1. Main results

Let $\mathbb{K}=\mathbb{C}$ or $\mathbb{K}=\mathbb{R}$; $\mathbb{K}^n=\{x=\operatorname{col}(x_1,\dots,x_n)\colon x_i\in\mathbb{K}\}$ is the linear space of column vectors over $\mathbb{K}$; $M_{m,n}(\mathbb{K})$ is the space of $m\times n$-matrices over $\mathbb{K}$; $M_n(\mathbb{K}):=M_{n,n}(\mathbb{K})$; $I\in M_n(\mathbb{K})$ is the identity matrix; $\overline{a}$ is the complex conjugation of $a$; $T$ is the transposition of a vector or a matrix; $*$ is the Hermitian conjugation, i.e., $A^*=\overline{A}^T$; $\operatorname{Sp} H$ is the trace of a matrix $H\in M_n(\mathbb{K})$; for a matrix $H\in M_n(\mathbb{K})$, we use the denotation $H^0:=I$; $J:=\{\vartheta_{ij}\}\in M_n(\mathbb{R})$ where $\vartheta_{ij}=1$ for $j=i+1$ and $\vartheta_{ij}=0$ for $j\neq i+1$.

Consider a control system defined by a linear time-invariant differential equation of $n$-th order with a lumped and distributed delay in the state variable $x\in\mathbb{K}$; the input is a linear combination of $m$ variables and their derivatives of order $\leqslant n-p$; the output is a $k$-dimensional vector of linear combinations of the state $x$ and its derivatives of order $\leqslant p-1$:

$$\begin{gathered}
x^{(n)}(t)+a_{10}x^{(n-1)}(t)+a_{11}x^{(n-1)}(t-h)+\ldots+a_{n0}x(t)+a_{n1}x(t-h)+\\
+\int_{-h}^{0}g_1(\tau)x^{(n-1)}(t+\tau)\,d\tau+\ldots+\int_{-h}^{0}g_n(\tau)x(t+\tau)\,d\tau=\\
=b_{p1}u_1^{(n-p)}(t)+b_{p+1,1}u_1^{(n-p-1)}(t)+\ldots+b_{n1}u_1(t)+\ldots\\
\ldots+b_{pm}u_m^{(n-p)}(t)+\ldots+b_{nm}u_m(t),\quad t>0,
\end{gathered}\tag{1.1}$$

$$\begin{gathered}
y_1(t)=\overline{c}_{11}x(t)+\overline{c}_{21}x'(t)+\ldots+\overline{c}_{p1}x^{(p-1)}(t),\ \ldots,\\
y_k(t)=\overline{c}_{1k}x(t)+\overline{c}_{2k}x'(t)+\ldots+\overline{c}_{pk}x^{(p-1)}(t),
\end{gathered}\tag{1.2}$$

with initial conditions $x^{(n-i)}(\tau)=\phi_i(\tau)$, $\tau\in[-h,0]$; here $h>0$ is a constant delay, $\phi_i:[-h,0]\to\mathbb{K}$ are continuous functions; $a_{ij}$, $b_{l\alpha}$, $c_{\nu\beta}\in\mathbb{K}$, $i=\overline{1,n}$, $j=\overline{0,1}$, $l=\overline{p,n}$, $\alpha=\overline{1,m}$, $\nu=\overline{1,p}$, $\beta=\overline{1,k}$; $g_i:[-h,0]\to\mathbb{K}$ are integrable functions ($i=\overline{1,n}$); $u=\operatorname{col}(u_1,\dots,u_m)\in\mathbb{K}^m$ is a control vector and $y=\operatorname{col}(y_1,\dots,y_k)\in\mathbb{K}^k$ is an output vector; $p\in\{\overline{1,n}\}$; the complex conjugation to $c_{\nu\beta}$ is used for convenience of notation.

For system (1.1), (1.2) without delays ($a_{i1}=0$, $g_i(\tau)\equiv 0$, $i=\overline{1,n}$), the problem of assigning an arbitrary finite spectrum by static output feedback was studied in [26], and the problem of (robust) exponential stabilization by static output feedback was studied in [27] when the coefficients $a_{i0}=a_{i0}(t)$, $i=\overline{1,n}$, are uncertain bounded functions. For system (1.1), (1.2) only with lumped delays, without distributed delays ($g_i(\tau)\equiv 0$, $i=\overline{1,n}$), the problem of assigning an arbitrary spectrum by static output feedback was studied in [28].

Suppose that the controller in system (1.1), (1.2) has the form of linear static output feedback with a lumped and distributed delay:

$$u(t)=Q_0y(t)+Q_1y(t-h)+\int_{-h}^{0}R(\tau)y(t+\tau)\,d\tau,\tag{1.3}$$

$y(\xi)=0$, $\xi<-h$. Here $Q_j=\{q^j_{\alpha\beta}\}\in M_{m,k}(\mathbb{K})$ are constant matrices ($j=\overline{0,1}$), $R(\tau)=\{r_{\alpha\beta}(\tau)\}\in M_{m,k}(\mathbb{K})$, $r_{\alpha\beta}:[-h,0]\to\mathbb{K}$ are integrable functions, $\alpha=\overline{1,m}$, $\beta=\overline{1,k}$. By (1.2), we have $y_\beta(t)=\sum\limits_{\nu=1}^{p}\overline{c}_{\nu\beta}x^{(\nu-1)}(t)$, $\beta=\overline{1,k}$. Hence,

$$\begin{gathered}
u_\alpha(t)=\sum_{\beta=1}^{k}\Big[q^0_{\alpha\beta}\Big(\sum_{\nu=1}^{p}\overline{c}_{\nu\beta}x^{(\nu-1)}(t)\Big)+q^1_{\alpha\beta}\Big(\sum_{\nu=1}^{p}\overline{c}_{\nu\beta}x^{(\nu-1)}(t-h)\Big)+\\
+\int_{-h}^{0}r_{\alpha\beta}(\tau)\Big(\sum_{\nu=1}^{p}\overline{c}_{\nu\beta}x^{(\nu-1)}(t+\tau)\Big)d\tau\Big],\quad \alpha=\overline{1,m}.
\end{gathered}$$

The closed-loop system (1.1), (1.2), (1.3) take the form

$$\begin{gathered} x^{(n)}(t)+\sum_{i=1}^{n}a_{i0}x^{(n-i)}(t)+\sum_{i=1}^{n}a_{i1}x^{(n-i)}(t-h)+\sum_{i=1}^{n}\int_{-h}^{0}g_i(\tau)x^{(n-i)}(t+\tau)\,d\tau \\ -\sum_{\alpha=1}^{m}\sum_{l=p}^{n}b_{l\alpha}\Big(\sum_{\beta=1}^{k}\Big[q^0_{\alpha\beta}\big(\sum_{\nu=1}^{p}\overline{c}_{\nu\beta}x^{(\nu-1)}(t)\big)+q^1_{\alpha\beta}\big(\sum_{\nu=1}^{p}\overline{c}_{\nu\beta}x^{(\nu-1)}(t-h)\big)+ \\ +\int_{-h}^{0}r_{\alpha\beta}(\tau)\big(\sum_{\nu=1}^{p}\overline{c}_{\nu\beta}x^{(\nu-1)}(t+\tau)\big)d\tau\Big]\Big)^{(n-l)}=0. \end{gathered} \tag{1.4}$$

Denote by $\psi(\lambda)$ the characteristic function of the closed-loop system (1.4). Then

$$\begin{gathered} \psi(\lambda)=\lambda^n+\sum_{i=1}^{n}\Big(a_{i0}\lambda^{n-i}+a_{i1}\lambda^{n-i}e^{-\lambda h}+\int_{-h}^{0}g_i(\tau)\lambda^{n-i}e^{\lambda\tau}\,d\tau\Big) \\ -\sum_{\alpha=1}^{m}\sum_{l=p}^{n}b_{l\alpha}\Big(\sum_{\nu=1}^{p}\Big[\sum_{\beta=1}^{k}\big(q^0_{\alpha\beta}\overline{c}_{\nu\beta}+q^1_{\alpha\beta}\overline{c}_{\nu\beta}e^{-\lambda h}\big)+\int_{-h}^{0}r_{\alpha\beta}(\tau)\overline{c}_{\nu\beta}e^{\lambda\tau}\,d\tau\Big]\lambda^{n-l+\nu-1}\Big). \end{gathered} \tag{1.5}$$

The set $\Lambda=\{\lambda\in\mathbb{C}:\psi(\lambda)=0\}$ is called *the spectrum* of system (1.4). If the spectrum of system (1.4) lies in the left half-plane, then system (1.4) is exponentially stable. The spectrum of system (1.4) is uniquely determined by coefficients of system (1.4). Therefore, the spectrum assignment problem for system (1.4) can be considered as the problem of control over coefficients of system (1.4). We study the problem of assigning an arbitrary spectrum that system (1.4) can only have.

**Definition 1.** System (1.1), (1.2) is said to be *arbitrary spectrum assignable by the static output feedback controller* (1.3) if for any numbers $\gamma_{i0},\gamma_{i1}\in\mathbb{K}$ and for any integrable functions $\delta_i:[-h,0]\to\mathbb{K}$, $i=\overline{1,n}$, there exist constant matrices $Q_0,Q_1\in M_{m,k}(\mathbb{K})$ and an integrable matrix function $R:[-h,0]\to M_{m,k}(\mathbb{K})$ such that the characteristic function $\psi(\lambda)$ of the closed-loop system (1.4) satisfies the equality

$$\psi(\lambda)=\lambda^n+\sum_{i=1}^{n}\lambda^{n-i}\Big(\gamma_{i0}+\gamma_{i1}e^{-\lambda h}+\int_{-h}^{0}\delta_i(\tau)e^{\lambda\tau}\,d\tau\Big).$$

On the basis of system (1.1), (1.2), let us construct the matrices $B=\{b_{l\alpha}\}$, $l=\overline{1,n}$, $\alpha=\overline{1,m}$, and $C=\{c_{\nu\beta}\}$, $\nu=\overline{1,n}$, $\beta=\overline{1,k}$, where $b_{l\alpha}:=0$ for $l<p$ and $c_{\nu\beta}:=0$ for $\nu>p$.

Let us give an auxiliary assertion.

**Lemma 1.** *Suppose that $F=\{f_{l\alpha}\}\in M_{n,m}(\mathbb{K})$, $G=\{g_{\beta\nu}\}\in M_{k,n}(\mathbb{K})$ are arbitrary matrices ($l=\overline{1,n}$, $\alpha=\overline{1,m}$, $\beta=\overline{1,k}$, $\nu=\overline{1,n}$) and $D_j=GJ^jF$ ($j\in\{\overline{0,n-1}\}$), $D_j=\{d^j_{\beta\alpha}\}$, $\beta=\overline{1,k}$, $\alpha=\overline{1,m}$. Then $d^j_{\beta\alpha}=\sum\limits_{l=j+1}^{n}g_{\beta,l-j}f_{l\alpha}$.*

Lemma 1 is proved in [28, Lemma 1].

**Theorem 1.** *System (1.1), (1.2) is arbitrary spectrum assignable by the static output feedback controller (1.3) if and only if the matrices*

$$C^*J^0B,\quad C^*J^1B,\quad \ldots,\quad C^*J^{n-1}B \tag{1.6}$$

*are linearly independent.*

P r o o f. Consider the spectrum assignment problem for system (1.1), (1.2) by the output feedback controller (1.3). Let a function

$$\varphi(\lambda)=\lambda^n+\sum_{i=1}^{n}\lambda^{n-i}\Big(\gamma_{i0}+\gamma_{i1}e^{-\lambda h}+\int_{-h}^{0}\delta_i(\tau)e^{\lambda\tau}\,d\tau\Big) \tag{1.7}$$

be given, where $\gamma_{i0},\gamma_{i1}\in\mathbb{K}$ are some numbers and $\delta_i:[-h,0]\to\mathbb{K}$ are some integrable functions. One needs to construct matrices $Q_0,Q_1\in M_{m,k}(\mathbb{K})$ and integrable matrix function $R:[-h,0]\to M_{m,k}(\mathbb{K})$ such that the characteristic function $\psi(\lambda)$ of the closed-loop system (1.4) satisfies the equality

$$\psi(\lambda)=\varphi(\lambda). \tag{1.8}$$

Let us write the characteristic function (1.5) of the closed-loop system (1.4) in the form

$$\psi(\lambda)=\lambda^n+\sum_{i=1}^{n}\Big(a_{i0}\lambda^{n-i}+a_{i1}\lambda^{n-i}e^{-\lambda h}+\int_{-h}^{0}g_i(\tau)\lambda^{n-i}e^{\lambda\tau}\,d\tau\Big)-\Delta, \tag{1.9}$$

where

$$\Delta=\sum_{\alpha=1}^{m}\sum_{\beta=1}^{k}\sum_{l=p}^{n}\sum_{\nu=1}^{p}b_{l\alpha}\overline{c}_{\nu\beta}\lambda^{n-l+\nu-1}\Big(q^0_{\alpha\beta}+q^1_{\alpha\beta}e^{-\lambda h}+\int_{-h}^{0}r_{\alpha\beta}(\tau)e^{\lambda\tau}d\tau\Big). \tag{1.10}$$

Let us replace the last summation index $\nu$ by $i=l-\nu+1$ in (1.10). Since $\nu$ ranges from 1 to $p$, hence, $i$ ranges from $l-p+1$ to $l$. So,

$$\Delta=\sum_{\alpha=1}^{m}\sum_{\beta=1}^{k}\sum_{l=p}^{n}\sum_{i=l-p+1}^{l}b_{l\alpha}\overline{c}_{l+1-i,\beta}\lambda^{n-i}\Big(q^0_{\alpha\beta}+q^1_{\alpha\beta}e^{-\lambda h}+\int_{-h}^{0}r_{\alpha\beta}(\tau)e^{\lambda\tau}d\tau\Big).$$

If $i\in\{\overline{1,l-p}\}$, then $l+1-i\geqslant p+1$, hence, $c_{l+1-i,\beta}=0$. Thus,

$$\Delta=\sum_{\alpha=1}^{m}\sum_{\beta=1}^{k}\sum_{l=p}^{n}\sum_{i=1}^{l}b_{l\alpha}\overline{c}_{l+1-i,\beta}\lambda^{n-i}\Big(q^0_{\alpha\beta}+q^1_{\alpha\beta}e^{-\lambda h}+\int_{-h}^{0}r_{\alpha\beta}(\tau)e^{\lambda\tau}d\tau\Big).$$

If $l\in\{\overline{1,p-1}\}$, then $b_{l\alpha}=0$, hence,

$$\Delta=\sum_{\alpha=1}^{m}\sum_{\beta=1}^{k}\sum_{l=1}^{n}\sum_{i=1}^{l}b_{l\alpha}\overline{c}_{l+1-i,\beta}\lambda^{n-i}\Big(q^0_{\alpha\beta}+q^1_{\alpha\beta}e^{-\lambda h}+\int_{-h}^{0}r_{\alpha\beta}(\tau)e^{\lambda\tau}d\tau\Big).$$

Let us change the summation order: we replace $\sum\limits_{l=1}^{n}\sum\limits_{i=1}^{l}$ by $\sum\limits_{i=1}^{n}\sum\limits_{l=i}^{n}$; then we obtain

$$\Delta=\sum_{\alpha=1}^{m}\sum_{\beta=1}^{k}\sum_{i=1}^{n}\sum_{l=i}^{n}b_{l\alpha}\overline{c}_{l+1-i,\beta}\lambda^{n-i}\Big(q^0_{\alpha\beta}+q^1_{\alpha\beta}e^{-\lambda h}+\int_{-h}^{0}r_{\alpha\beta}(\tau)e^{\lambda\tau}d\tau\Big). \tag{1.11}$$

Let $D_{i-1}=C^*J^{i-1}B$, $D_{i-1}=\{d^{i-1}_{\beta\alpha}\}$, $i\in\{\overline{1,n}\}$, $\beta=\overline{1,k}$, $\alpha=\overline{1,m}$. Let us apply Lemma 1 to $G=C^*$, $F=B$: we have $g_{\beta\nu}=\overline{c}_{\nu\beta}$, $f_{l\alpha}=b_{l\alpha}$. So, by Lemma 1 for $j=i-1$, we have

$$d^{i-1}_{\beta\alpha}=\sum_{l=i}^{n}\overline{c}_{l+1-i,\beta}b_{l\alpha}. \tag{1.12}$$

For every $i \in \{\overline{1,n}\}$, consider the matrices $C^*J^{i-1}BQ_0$, $C^*J^{i-1}BQ_1$, $C^*J^{i-1}BR(\tau)$. Let us find its traces. Using (1.12), we obtain

$$\mathrm{Sp}\,(C^*J^{i-1}BQ_0) = \mathrm{Sp}\,(D_{i-1}Q_0) = \sum_{\alpha=1}^{m}\sum_{\beta=1}^{k} d_{\beta\alpha}^{i-1} q_{\alpha\beta}^{0} = \sum_{\alpha=1}^{m}\sum_{\beta=1}^{k}\sum_{l=i}^{n} \overline{c}_{l+1-i,\beta} b_{l\alpha} q_{\alpha\beta}^{0}. \tag{1.13}$$

Similarly,

$$\mathrm{Sp}\,(C^*J^{i-1}BQ_1) = \sum_{\alpha=1}^{m}\sum_{\beta=1}^{k}\sum_{l=i}^{n} \overline{c}_{l+1-i,\beta} b_{l\alpha} q_{\alpha\beta}^{1}, \tag{1.14}$$

$$\int_{-h}^{0} \mathrm{Sp}\,(C^*J^{i-1}BR(\tau))e^{\lambda\tau}d\tau = \int_{-h}^{0}\sum_{\alpha=1}^{m}\sum_{\beta=1}^{k}\sum_{l=i}^{n} \overline{c}_{l+1-i,\beta} b_{l\alpha} r_{\alpha\beta}(\tau) e^{\lambda\tau}d\tau. \tag{1.15}$$

From (1.11), (1.13), (1.14), and (1.15), it follows that

$$\begin{aligned}\Delta = \sum_{i=1}^{n}\Big(&\mathrm{Sp}\,(C^*J^{i-1}BQ_0)\lambda^{n-i} + \mathrm{Sp}\,(C^*J^{i-1}BQ_1)\lambda^{n-i}e^{-\lambda h} + \\ &+ \int_{-h}^{0}\mathrm{Sp}\,(C^*J^{i-1}BR(\tau))\lambda^{n-i}e^{\lambda\tau}d\tau\Big).\end{aligned} \tag{1.16}$$

Substituting (1.16) in (1.9), we obtain

$$\begin{aligned}\psi(\lambda) = \lambda^n + \sum_{i=1}^{n}\lambda^{n-i}\Big(&\Big(a_{i0} - \mathrm{Sp}\,(C^*J^{i-1}BQ_0)\Big) + \Big(a_{i1} - \mathrm{Sp}\,(C^*J^{i-1}BQ_1)\Big)e^{-\lambda h} + \\ &+ \int_{-h}^{0}\Big(g_i(\tau) - \mathrm{Sp}\,(C^*J^{i-1}BR(\tau))\Big)e^{\lambda\tau}d\tau\Big).\end{aligned} \tag{1.17}$$

Taking into account (1.17), (1.8), and (1.7), we obtain that system (1.1), (1.2) is arbitrary spectrum assignable by (1.3) iff there exist $Q_0, Q_1 \in M_{m,k}(\mathbb{K})$ and integrable matrix function $R : [-h, 0] \to M_{m,k}(\mathbb{K})$ such that for all $i = \overline{1,n}$ the following equalities hold:

$$\begin{aligned}\gamma_{i0} &= a_{i0} - \mathrm{Sp}\,(C^*J^{i-1}BQ_0),\\ \gamma_{i1} &= a_{i1} - \mathrm{Sp}\,(C^*J^{i-1}BQ_1),\\ \delta_i(\tau) &= g_i(\tau) - \mathrm{Sp}\,(C^*J^{i-1}BR(\tau)).\end{aligned} \tag{1.18}$$

Every system of (1.18) consists of $n$ equations with $mk$ unknown entries of matrices $Q_0$, $Q_1$, $R(\tau)$. Let us rewrite (1.18) in the vector form. Denote by $\mathrm{vec} : M_{p,q}(\mathbb{K}) \to \mathbb{K}^{pq}$ the mapping, which "unrolls" a matrix $Z = \{z_{ij}\}$, $i = \overline{1,p}$, $j = \overline{1,q}$, by rows into the column vector $\mathrm{vec}\,Z = \mathrm{col}\,(z_{11}, \ldots, z_{1q}, \ldots, z_{p1}, \ldots, z_{pq}) \in \mathbb{K}^{pq}$. Then $\mathrm{Sp}\,(XY) = (\mathrm{vec}\,X)^T \cdot (\mathrm{vec}\,Y^T)$ for any $X \in M_{p,q}(\mathbb{K})$, $Y \in M_{q,p}(\mathbb{K})$. Let us apply this equality to the matrix $X = C^*J^{i-1}B$ ($i = \overline{1,n}$) and to the matrices $Y = Q_0$, $Y = Q_1$, and $Y = R(\tau)$ one by one. Let us construct the $mk \times n$-matrix

$$P := [\mathrm{vec}\,(C^*B), \mathrm{vec}\,(C^*JB), \ldots, \mathrm{vec}\,(C^*J^{n-1}B)].$$

Denote $v_0 := \mathrm{vec}\,(Q_0^T) \in \mathbb{K}^{mk}$, $v_1 := \mathrm{vec}\,(Q_1^T) \in \mathbb{K}^{mk}$, $v_2(\tau) := \mathrm{vec}\,(R^T(\tau)) \in \mathbb{K}^{mk}$,

$$\begin{aligned}w_0 &:= \mathrm{col}\,(a_{10} - \gamma_{10}, \ldots, a_{n0} - \gamma_{n0}) \in \mathbb{K}^n,\\ w_1 &:= \mathrm{col}\,(a_{11} - \gamma_{11}, \ldots, a_{n1} - \gamma_{n1}) \in \mathbb{K}^n,\\ w_2(\tau) &:= \mathrm{col}\,(g_1(\tau) - \delta_1(\tau), \ldots, g_n(\tau) - \delta_n(\tau)) \in \mathbb{K}^n.\end{aligned}$$

Then systems (1.18) can be rewritten in the vector form

$$P^T v_0 = w_0, \qquad P^T v_1 = w_1, \qquad P^T v_2(\tau) = w_2(\tau). \tag{1.19}$$

System (1.1), (1.2) is arbitrary spectrum assignable by the feedback controller (1.3) iff systems (1.19) are solvable with respect to $v_0$, $v_1$, $v_2(\tau)$ for any $\gamma_{i0}, \gamma_{i1} \in \mathbb{K}$ and any integrable functions $\delta_i : [-h, 0] \to \mathbb{K}$. This is equivalent to linear independency of the matrices (1.6). In that case, systems (1.19) have the particular solutions

$$v_0 = P(P^T P)^{-1} w_0, \qquad v_1 = P(P^T P)^{-1} w_1, \qquad v_2(\tau) = P(P^T P)^{-1} w_2(\tau). \tag{1.20}$$

The required matrices $Q_0$, $Q_1$ and $R(\tau)$ can be found from the equalities $Q_0 = (\mathrm{vec}^{-1} v_0)^T$, $Q_1 = (\mathrm{vec}^{-1} v_1)^T$, $R(\tau) = (\mathrm{vec}^{-1} v_2(\tau))^T$. The theorem is proved. □

**Remark 1.** Note that a necessary condition for linear independence of matrices (1.6) is the condition $mk \geqslant n$.

## § 2. Corollaries

If the characteristic function of the closed-loop system (1.4) turns into a polynomial then spectrum $\Lambda$ of system (1.4) is finite. We say that system (1.1), (1.2) is *arbitrary finite spectrum assignable by the static output feedback controller* (1.3) if for any $\omega_i \in \mathbb{K}$, $i = \overline{1, n}$, there exist constant matrices $Q_0, Q_1 \in M_{m,k}(\mathbb{K})$ and an integrable matrix function $R : [-h, 0] \to M_{m,k}(\mathbb{K})$ such that the characteristic function $\psi(\lambda)$ of the closed-loop system (1.4) satisfies the equality

$$\psi(\lambda) = \lambda^n + \omega_1 \lambda^{n-1} + \ldots + \omega_n.$$

**Corollary 1.** *System* (1.1)*,* (1.2) *is arbitrary finite spectrum assignable by the static output feedback controller* (1.3) *iff matrices* (1.6) *are linearly independent.*

Corollary 1 follows from Theorem 1: the problem under consideration is equivalent to solvability of system (1.19), where $\gamma_{i0} = \omega_i$, $\gamma_{i1} = 0$, $\delta_i(\tau) \equiv 0$, $\tau \in [-h, 0]$, $i = \overline{1, n}$: if matrices (1.6) are linearly independent, then system (1.19) is resolvable for any $\omega_i$; if not, then system (1.19) is resolvable not for any $\omega_i$, $i = \overline{1, n}$.

**Corollary 2.** *If matrices* (1.6) *are linearly independent, then system* (1.1)*,* (1.2) *is exponentially stabilizable by the static output feedback controller* (1.3)*.*

Corollary 2 follows from Corollary 1, if one take, e.g., numbers $\omega_i$, $i = \overline{1, n}$, such that $\lambda^n + \sum\limits_{i=1}^{n} \omega_i \lambda^{n-1} = (\lambda + 1)^n$.

Next, consider system (1.1), (1.2) containing only a distributed delay:

$$x^{(n)} + \sum_{i=1}^{n} \Big( a_{i0} x^{(n-i)}(t) + \int_{-h}^{0} g_i(\tau) x^{(n-i)}(t + \tau)\, d\tau \Big) = \sum_{\alpha=1}^{m} \sum_{l=p}^{n} b_{l\alpha} u_\alpha^{(n-l)}(t). \tag{2.1}$$

Let the controller in system (2.1), (1.2) have the same form as the left-hand side of (2.1), i.e., contains only a distributed delay:

$$u(t) = Q_0 y(t) + \int_{-h}^{0} R(\tau) y(t + \tau)\, d\tau. \tag{2.2}$$

Consider the problem of assigning an arbitrary spectrum that the closed-loop system (2.1), (1.2), (2.2) can only have.

**Definition 2.** System (2.1), (1.2) is said to be *arbitrary spectrum assignable by the static output feedback controller* (2.2) if for any numbers $\gamma_{i0} \in \mathbb{K}$ and for any integrable functions $\delta_i : [-h, 0] \to \mathbb{K}$, $i = \overline{1,n}$, there exist a constant matrix $Q_0 \in M_{m,k}(\mathbb{K})$ and an integrable matrix function $R : [-h, 0] \to M_{m,k}(\mathbb{K})$ such that the characteristic function $\psi(\lambda)$ of the closed-loop system (2.1), (1.2), (2.2) satisfies the equality

$$\psi(\lambda) = \lambda^n + \sum_{i=1}^{n} \lambda^{n-i}\Big(\gamma_{i0} + \int_{-h}^{0} \delta_i(\tau)e^{\lambda\tau}\, d\tau\Big).$$

The following theorem take place.

**Theorem 2.** *System* (2.1)*,* (1.2) *is arbitrary spectrum assignable by the static output feedback controller* (2.2) *if and only if the matrices* (1.6) *are linearly independent.*

The proof of Theorem 2 repeats the proof of Theorem 1 with $a_{i1} = 0$, $\gamma_{i1} = 0$, $q^1_{\alpha\beta} = 0$, $i = \overline{1,n}$, $\alpha = \overline{1,m}$, $\beta = \overline{1,k}$.

**Corollary 3.** *System* (2.1)*,* (1.2) *is arbitrary finite spectrum assignable by the static output feedback controller* (2.2) *iff matrices* (1.6) *are linearly independent.*

**Corollary 4.** *If matrices* (1.6) *are linearly independent, then system* (2.1)*,* (1.2) *is exponentially stabilizable by the linear static output feedback controller* (2.2)*.*

Similar statements take place if system (1.1) and controller (1.3) contain only a lumped delay.

**Theorem 3.** *System*

$$x^{(n)} + \sum_{i=1}^{n}\Big(a_{i0}x^{(n-i)}(t) + a_{i1}x^{(n-i)}(t-h)\Big) = \sum_{\alpha=1}^{m}\sum_{l=p}^{n} b_{l\alpha}u_\alpha^{(n-l)}(t) \tag{2.3}$$

*with* (1.2) *is arbitrary spectrum assignable by the static output feedback controler*

$$u(t) = Q_0 y(t) + Q_1 y(t-h), \tag{2.4}$$

*i. e., for any numbers* $\gamma_{i0}, \gamma_{i1} \in \mathbb{K}$ *there exist constant matrices* $Q_0, Q_1 \in M_{m,k}(\mathbb{K})$ *such that the characteristic function* $\psi(\lambda)$ *of the closed-loop system* (2.3)*,* (1.2)*,* (2.4) *satisfies the equality*

$$\psi(\lambda) = \lambda^n + \sum_{i=1}^{n} \lambda^{n-i}\Big(\gamma_{i0} + \gamma_{i1}e^{-\lambda h}\Big),$$

*if and only if the matrices* (1.6) *are linearly independent.*

The proof of Theorem 3 repeats the proof of Theorem 1 with $g_i(\tau) \equiv 0$, $\delta_i(\tau) \equiv 0$, $r_{\alpha\beta}(\tau) \equiv 0$, $i = \overline{1,n}$, $\alpha = \overline{1,m}$, $\beta = \overline{1,k}$, $\tau \in [-h, 0]$.

**Corollary 5.** *System* (2.3)*,* (1.2) *is arbitrary finite spectrum assignable by the static output feedback controller* (2.4) *iff matrices* (1.6) *are linearly independent.*

**Corollary 6.** *If matrices* (1.6) *are linearly independent, then system* (2.3)*,* (1.2) *is exponentially stabilizable by the linear static output feedback controller* (2.4)*.*

Next, we can obtain the following statements if two equalities from (1.19) are nullified.

**Theorem 4.** *System*

$$x^{(n)} + \sum_{i=1}^{n} \int_{-h}^{0} g_i(\tau) x^{(n-i)}(t+\tau)\, d\tau = \sum_{\alpha=1}^{m} \sum_{l=p}^{n} b_{l\alpha} u_\alpha^{(n-l)}(t) \tag{2.5}$$

*with* (1.2) *is arbitrary spectrum assignable by the static output feedback controller*

$$u(t) = \int_{-h}^{0} R(\tau) y(t+\tau)\, d\tau. \tag{2.6}$$

*i. e., for any integrable functions* $\delta_i : [-h, 0] \to \mathbb{K}$, $i = \overline{1, n}$, *there exists an integrable matrix function* $R : [-h, 0] \to M_{m,k}(\mathbb{K})$ *such that the characteristic function* $\psi(\lambda)$ *of the closed-loop system* (2.5), (1.2), (2.6) *satisfies the equality*

$$\psi(\lambda) = \lambda^n + \sum_{i=1}^{n} \int_{-h}^{0} \delta_i(\tau) e^{\lambda\tau}\, d\tau,$$

*if and only if the matrices* (1.6) *are linearly independent.*

The proof of Theorem 4 repeats the proof of Theorem 1 with $a_{i0} = 0$, $a_{i1} = 0$, $\gamma_{i0} = 0$, $\gamma_{i1} = 0$, $q^0_{\alpha\beta} = 0$, $q^1_{\alpha\beta} = 0$, $i = \overline{1, n}$, $\alpha = \overline{1, m}$, $\beta = \overline{1, k}$.

**Remark 2.** Note that Theorem 4 is not contained in Theorem 2 and does not follow directly from it because the form of controller (2.6) is less general with respect to (2.2). In fact, any form of the controller that differs from others generates a separate problem.

**Theorem 5.** *System*

$$x^{(n)} + \sum_{i=1}^{n} a_{i1} x^{(n-i)}(t-h) = \sum_{\alpha=1}^{m} \sum_{l=p}^{n} b_{l\alpha} u_\alpha^{(n-l)}(t) \tag{2.7}$$

*with* (1.2) *is arbitrary spectrum assignable by the static output feedback controller*

$$u(t) = Q_1 y(t-h), \tag{2.8}$$

*i. e., for any numbers* $\gamma_{i1} \in \mathbb{K}$ *there exists a constant matrix* $Q_1 \in M_{m,k}(\mathbb{K})$ *such that the characteristic function* $\psi(\lambda)$ *of the closed-loop system* (2.7), (1.2), (2.8) *satisfies the equality*

$$\psi(\lambda) = \lambda^n + \sum_{i=1}^{n} \gamma_{i1} \lambda^{n-i} e^{-\lambda h},$$

*if and only if the matrices* (1.6) *are linearly independent.*

The proof of Theorem 5 repeats the proof of Theorem 1 with $a_{i0} = 0$, $g_i(\tau) \equiv 0$, $\gamma_{i0} = 0$, $\delta_i(\tau) \equiv 0$, $q^0_{\alpha\beta} = 0$, $r_{\alpha\beta}(\tau) \equiv 0$, $i = \overline{1, n}$, $\alpha = \overline{1, m}$, $\beta = \overline{1, k}$, $\tau \in [-h, 0]$.

**Theorem 6.** *System*

$$x^{(n)} + \sum_{i=1}^{n} a_{i1} x^{(n-i)}(t) = \sum_{\alpha=1}^{m} \sum_{l=p}^{n} b_{l\alpha} u_\alpha^{(n-l)}(t) \tag{2.9}$$

*with* (1.2) *is arbitrary* (*finite*) *spectrum assignable by the static output feedback controller*

$$u(t) = Q_0 y(t), \tag{2.10}$$

*i. e., for any numbers* $\gamma_{i0} \in \mathbb{K}$ *there exists a constant matrix* $Q_0 \in M_{m,k}(\mathbb{K})$ *such that the characteristic function* $\psi(\lambda)$ *of the closed-loop system* (2.9), (1.2), (2.10) *satisfies the equality*

$$\psi(\lambda) = \lambda^n + \sum_{i=1}^{n} \gamma_{i0} \lambda^{n-i},$$

*if and only if the matrices* (1.6) *are linearly independent.*

The proof of Theorem 6 repeats the proof of Theorem 1 with $a_{i1} = 0$, $g_i(\tau) \equiv 0$, $\gamma_{i1} = 0$, $\delta_i(\tau) \equiv 0$, $q^1_{\alpha\beta} = 0$, $r_{\alpha\beta}(\tau) \equiv 0$, $i = \overline{1,n}$, $\alpha = \overline{1,m}$, $\beta = \overline{1,k}$, $\tau \in [-h, 0]$.

**Remark 3.** Theorem 6 was proved in [26] for the case $\mathbb{K} = \mathbb{R}$.

**Remark 4.** Note that Theorems 5 and 6 are not contained in Theorem 3 and do not follow directly from it because the form of controllers (2.8) and (2.10) is less general with respect to (2.4).

**Corollary 7.** *If matrices* (1.6) *are linearly independent, then system* (2.9), (1.2) *is exponentially stabilizable by the linear static output feedback controller* (2.10).

**Remark 5.** Corollaries from Theorems 4 and 5 on assignment arbitrary finite spectrum do not take place. Corollaries from Theorems 4 and 5 on stabilization is questionable.

## § 3. Example

Consider the system with a lumped and distributed delay $h > 0$:

$$\begin{gathered} x'''(t) + 3x''(t) + 2x''(t-h) + 5x'(t) + 4x'(t-h) - x(t) + x(t-h) + \\ + \int_{-h}^{0} x''(t+\tau)\sin\tau d\tau + \int_{-h}^{0} 2x'(t+\tau)\cos\tau d\tau + \int_{-h}^{0} x(t+\tau)\sin 2\tau\, d\tau = -u_1(t) + u_2'(t), \\ y_1(t) = x(t), \quad y_2(t) = -x'(t), \end{gathered} \tag{3.1}$$

where $x \in \mathbb{R}$, $u = \mathrm{col}\,(u_1, u_2) \in \mathbb{R}^2$. We have

$$\begin{gathered} n = 3, \quad m = 2, \quad k = 2, \quad p = 2; \\ a_{10} = 3, \quad a_{20} = 5, \quad a_{30} = -1, \quad a_{11} = 2, \quad a_{21} = 4, \quad a_{31} = 1; \\ b_{21} = 0, \quad b_{31} = -1, \quad b_{22} = 1, \quad b_{32} = 0; \quad \bar{c}_{11} = 1, \quad \bar{c}_{21} = 0, \quad \bar{c}_{12} = 0, \quad \bar{c}_{22} = -1; \\ g_1(\tau) = \sin\tau, \quad g_2(\tau) = 2\cos\tau, \quad g_3(\tau) = \sin 2\tau. \end{gathered}$$

On the basis of (3.1), we construct the matrices $B$ and $C$: we obtain $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$. Hence,

$$C^*B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad C^*JB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C^*J^2B = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.2}$$

Obviously, the matrices (3.2) are linearly independent. Thus, by Theorem 1, system (1.1), (1.2) is arbitrary spectrum assignable by feedback (1.3). Let us construct such feedback controller. Suppose, for example, that

$$\psi(\lambda) = (\lambda+1)^2(\lambda+e^{-\lambda h}) = \lambda^3+2\lambda^2+\lambda+\lambda^2 e^{-\lambda h}+2\lambda e^{-\lambda h}+e^{-\lambda h}. \tag{3.3}$$

Then $\gamma_{10}=2$, $\gamma_{20}=1$, $\gamma_{30}=0$, $\gamma_{11}=1$, $\gamma_{21}=2$, $\gamma_{31}=1$, $\delta_1(\tau)=0$, $\delta_2(\tau)=0$, $\delta_3(\tau)=0$.

We have

$$\begin{aligned}
w_0 &= \operatorname{col}(a_{10}-\gamma_{10}, a_{20}-\gamma_{20}, a_{30}-\gamma_{30}) = (1,4,-1),\\
w_1 &= \operatorname{col}(a_{11}-\gamma_{11}, a_{21}-\gamma_{21}, a_{31}-\gamma_{31}) = (1,2,0),\\
w_2 &= \operatorname{col}(g_1(\tau)-\delta_1(\tau), g_2(\tau)-\delta_2(\tau), g_3(\tau)-\delta_3(\tau)) = (\sin\tau, 2\cos\tau, \sin 2\tau).
\end{aligned}$$

Calculating $v_0$, $v_1$, and $v_2(\tau)$ by formulas (1.20), we obtain

$$v_0=\operatorname{col}(1,2,2,-1),\quad v_1=\operatorname{col}(0,1,1,-1),\quad v_2=\operatorname{col}(-\sin 2\tau,\cos\tau,\cos\tau,-\sin\tau).$$

Therefore

$$Q_0=\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix},\quad Q_1=\begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix},\quad R(\tau)=\begin{bmatrix}-\sin 2\tau & \cos\tau\\ \cos\tau & -\sin\tau\end{bmatrix}.$$

The controller (1.3)

$$\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix} = Q_0\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix} + Q_1\begin{bmatrix}y_1(t-h)\\ y_2(t-h)\end{bmatrix} + \int_{-h}^{0} R(\tau)y(t+\tau)\,d\tau \tag{3.4}$$

has the components

$$u_1(t) = x(t)-2x'(t)-x'(t-h)-\int_{-h}^{0} x(t+\tau)\sin 2\tau\,d\tau-\int_{-h}^{0} x'(t+\tau)\cos\tau\,d\tau,$$

$$u_2(t) = 2x(t)+x'(t)+x(t-h)+x'(t-h)+\int_{-h}^{0} x(t+\tau)\cos\tau\,d\tau+\int_{-h}^{0} x'(t+\tau)\sin\tau\,d\tau.$$

System (3.1) closed-loop by feedback (3.4) take the form

$$x'''(t)+2x''(t)+x''(t-h)+x'(t)+2x'(t-h)+x(t-h)=0. \tag{3.5}$$

The characteristic function of system (3.5) is equal to (3.3).

## § 4. Conclusions and future works

Necessary and sufficient conditions are obtained for the problem of arbitrary spectrum assignment by static output feedback with a lumped and distributed delay for a linear differential equation with a lumped and distributed delay. Corollaries on stabilization are stated. An illustrative example is given.

In the future we expect to extend these results to control systems with several lumped and distributed delays. Moreover, this approach could be applied for the corresponding problems of eigenvalue assignment and stabilization by output feedback control for systems of differential equations (not just for one).

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Zaitsev Vasilii Aleksandrovich, Doctor of Physics and Mathematics, Head of the Laboratory of Mathematical Control Theory, Udmurt State University, ul. Universitetskaya, 1, Izhevsk, 426034, Russia.
E-mail: verba@udm.ru

Kim Inna Geraldovna, Researcher, Laboratory of Mathematical Control Theory, Udmurt State University, ul. Universitetskaya, 1, Izhevsk, 426034, Russia.
E-mail: kimingeral@gmail.com

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***В. А. Зайцев, И. Г. Ким***

**Управление спектром и стабилизация линейных дифференциальных уравнений с запаздыванием статической обратной связью по выходу с запаздыванием**

*Ключевые слова:* линейное дифференциальное уравнение, сосредоточенное запаздывание, распределенное запаздывание, управление спектром, стабилизация, статическая обратная связь по выходу.

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Рассматривается линейная система управления, заданная стационарным дифференциальным уравнением с одним сосредоточенным и одним распределенным запаздыванием. В системе на вход подается линейная комбинация из $m$ сигналов и их производных до порядка $n-p$ включительно, а выход представляет собой $k$-мерный вектор линейных комбинаций состояния и его производных до порядка не более $p-1$. Для этой системы исследуется задача управления спектром с помощью линейной статической обратной связи по выходу с сосредоточенным и распределенным запаздываниями. Получены необходимые и достаточные условия разрешимости задачи произвольного размещения спектра посредством статической обратной связи по выходу, имеющей тот же вид, что и система. Получены следствия о стабилизации системы.

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Зайцев Василий Александрович, д. ф.-м. н., заведующий лабораторией математической теории управления, Удмуртский государственный университет, 426034, Россия, г. Ижевск, ул. Университетская, 1.
E-mail: verba@udm.ru

Ким Инна Геральдовна, научный сотрудник, лаборатория математической теории управления, Удмуртский государственный университет, 426034, Россия, г. Ижевск, ул. Университетская, 1.
E-mail: kimingeral@gmail.com

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