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STRONG COALITIONAL EQUILIBRIA IN GAMES UNDER UNCERTAINTY

The *Strong Coalitional Equilibrium* (SCE) is introduced for normal form games under uncertainty. This concept is based on the synthesis of the notions of individual rationality, collective rationality in normal form games without side payments, and a proposed coalitional rationality. For presentation simplicity, SCE is presented for 4-person games under uncertainty. Sufficient conditions for the existence of SCE in pure strategies are established via the saddle point of the Germeir's convolution function. Finally, following the approach of Borel, von Neumann and Nash, a theorem of existence of SCE in mixed strategies is proved under common minimal mathematical conditions for normal form games (compactness and convexity of players' strategy sets, compactness of uncertainty set and continuity of payoff functions).

Keywords: normal form game, uncertainty, guarantee, mixed strategies, Germeier convolution, saddle point, equilibrium.

Introduction

The theory of cooperative games has evolved in three directions. The first direction involves the introduction of equilibrium solutions for normal form games and their analysis. It is an extension of Nash theory [1,2]. The second direction is the characteristic function approach. In a characteristic function form game each coalition (subset of players) is associated with a value that it can afford. The third and most recent direction considers the coalition formation as a dynamic process. As the contribution of the present paper is related to the first direction of research, we briefly discuss the second and the third directions.

In characteristic function games, the most prominent solution concept is the core [3]. The core is based on the idea of blocking. A coalition can block an imputation if it can improve the outcomes of its members by deviating from the current imputation. An imputation is in the core if it cannot be blocked by any coalition. Many other concepts were introduced such as the nucleolus, the kernel and the Shapley value, just to mention few. The main drawback of the characteristic function form game and its concepts of solutions is that they do not incorporate the strategic interaction of players.

Due to the limitations of the characteristic form games and their solution concepts, the third direction of research appeared that considers the coalition formation in cooperative games as a dynamic process. The pioneering works in this direction are the contributions of [4] incorporating players' farsightedness in the game analysis, that is, considering that players care about long term outcomes of the game, [5] introducing coalition strategies to account for coalitional behavior of players during the game and [6] describing the coalition formation as a Markov process. For more details we refer the reader to [7].

Now, we turn to the first direction. Many coalition related concepts of equilibrium or solutions have been introduced for n-person normal form games. The main initial motivation for the inception of this direction of research is to overcome one of the drawbacks of Nash equilibrium (NE), namely, NE is not imune against coalition deviation. A coalition may improve the payoff of all its members by collectively deviating from NE. Aumann [8] introduced the strong equilibrium

(SE) that is imune against coalition deviation. It turned out that the set of SE is empty for most of the games. Later, the α-core and β-core were introduced by Aumann [9] to relax the conditions of SE. An action profile belongs to the core of a game if no group of players has an incentive to form a coalition and select a different action profile in which each of its members is made better-off, i. e. the action profile cannot be blocked by any coalition. The α-core and β-core differ on the definition of blocking. The α-core requires a blocking coalition to select a specific strategy independently of the complementary coalition's choice, the β-core allows a blocking coalition to vary its blocking strategy as a function of the complementary coalition's choice. Berge [10] introduced a very strong equilibrium, the Berge strong equilibrium (SBE), in the sense that if one of the players selects his strategy from SBE, the other players have no choice but to play their strategies from SBE. Berge equilibrium (BE), put forward by Zhukovskiy [11], is an equilibrium that reflects altruism and mutual support among players. A BE is a strategy profile where the payoff function of each player is maximized by all the other players. Recently, research on BE has gained some momentum Larbani and Zhukovskiy [12] as more impirical research showed that beside noncooperative behavior, cooperation, mutual support, reciprocity and caring about fairness can take place in interactions between individuals [13–16]. Bernheim noticed that in SE some deviations might not be self-enforcing [17], therefore, they may not be considered as credible threats. This led to the introduction of coalition proof Nash equilibrium (CPNE). In CPNE, only self-enforcing deviations are credible threats. A deviation by a coalition is self-enforcing if no subcoalition has an incentive to initiate a new deviation. Finally, some works combine different solution concepts such as the hybrid solution of [18], which assumes that a coalition structure is formed, and the game is noncooperative among coalitions but cooperative within coalitions so that Nash equilibrium is adopted in the former and the core in the latter as solutions.

The common drawback of the mentioned coalition equilibria and solutions is that their set is often empty; they do not exist under common conditions such as the compactness and convexity of strategy sets and continuity and quasiconcavity of payoff functions [17, 19–21], except for the α-core and the hybrid solution of Zhao. Using the notion of balancedness, Scarf [22] established the nonemptyness of the α-core under the compactness and convexity of strategy sets, and continuity and quasi-concavity of the payoff functions. However, no method of determination of an α-core element is derived from Scarf's theorem. Zhao's hybrid solution is established under similar conditions.

As most of the mentioned equilibrium concepts and solutions do not exist under common conditions in pure strategies in continuous games, a natural question arises: Do these concepts and solutions exist in mixed strategies? Works dealing with this question and related topics do not appear to exist in literature. Moreover, the existence of the mentioned concepts and solutions has not been considered in games involving uncertainty.

In this paper, we introduce a coalition rational equilibrium for a game under nonprobabilistic uncertainty and establish its existence in mixed strategies. Moreover, this concept generalizes many of the mentioned concepts.

A mathematical model of cooperation in game situation is represented in this paper by a 4-person normal form game under undetermined parameters (interval uncertainty). We have limited the presentation to 4-person game for simplicity of presentation. Regarding the undetermined parameters, it is assumed that the players know their range of variation only; no probability information is available (for known or unknown reasons). In the process of modelling game phenomena, considering uncertainty leads to more adequate results and decisions, which is supported by the numerous publications related to this domain (a google search on the topic «mathematical modelling under uncertainty», returns more than one million links to related works). The uncertainty appears because of incomplete information about the players' strategy sets, the strategies

being selected by each player and the related payoffs: «In these matters the only certainty is there is nothing certain» (Pliny the Elder[1]).

For example, an economic system, as a rule, undergoes unexpected and hard to predict perturbations that could be external or internal. External perturbations could be a change in quantity and nomenclature of supplies, jumps or drops in demand on products of the related industrial sector, changes in consumer behavior, variations in price of raw materials, like oil etc.; internal perturbations could be the introduction of new technologies, breakdowns or change of equipments, delays in replacement of equipment and supplies, etc. For instance, the mentioned perturbations can significantly affect the interactions of partners in a supply chain, which can be regarded as a game.

One more question arises: How can a player consider at the same time the game's strategic and cooperation aspects, and the presence of uncertainty when selecting his/her strategy? In this paper, the following approach to formalize the cooperation aspect of the game is adopted. It is assumed that the cooperation character of the game consists in the fact that any nonempty subset of players has the possibility to form a coalition through communication and coordination by agreeing to select a bundle of strategies to achieve the best possible payoff for all its members. This assumption means that the interests of all possible coalitions are considered. Further, it is also assumed that the game is without side payments or non-transferable utility (NTU). The concept of strong coalitional equilibrium (SCE) is introduced for the described game. A sufficient condition for its existence in pure strategies is provided and its existence in mixed strategies is established under minimal common conditions, the compactness and convexity of strategy sets, the compactness of uncertainty set and the continuity of payoff functions.

The paper is organized as follows. The game is described in Section 1. The SCE is introduced in Section 2. Sufficient condition for the existence of SCE in pure strategies is given in Section 3. A theorem of existence of SCE in mixed strategies is presented in Section 4. Section 5 concludes the paper.

§ 1. The game under uncertainty

In this section, we present the normal form game under uncertainty we consider. For simplicity of presentation, we consider a 4-person game only. All the results and definitions presented in this paper can be easily generalized to n-person games in a straightforward way.

Consider the 4-person normal form game under uncertainty

$$\Gamma = \langle N = \{1, 2, 3, 4\}, \{X_i\}_{i \in N}, Y, \{f_i(x, y)\}_{i \in N} \rangle,$$

where $N = \{1, 2, 3, 4\}$ is the set of players; player i is associated with the number $i \in N$; player $i \in N$ selects his/her strategy x_i from his/her strategy set $X_i \subset \mathrm{R}^{n_i}$, as a result, a strategy profile is obtained $x = (x_1, x_2, x_3, x_4) \in X = \prod_{i=1}^{4} X_i \subset \mathrm{R}^n$, $n = \sum_{i \in N} n_i$; the (*interval*) *uncertainty* $y \in Y \subset \mathrm{R}^m$ occurs independently of the players' actions; the payoff function of player $i \in N$ is the real-valued function $f_i(x, y)$, depending on the pair $(x, y) \in X \times Y$. The objective of player $i \in N$ in Γ is to select a strategy x_i that yields the largest possible payoff for him/her. This includes selecting strategies that maximize other players' payoffs if they benefit him/her. With this objective in view, the players should consider the possibility of formation of any coalition and the occurrence of any value of the uncertainty $y \in Y$. Considering the uncertainty $y \in Y$ leads to a multivalued payoff function of the form $x \to f_i(x, Y) = \bigcup_{y \in Y} f_i(x, y)$. Such mutivalued character of the payoff functions makes it difficult to study cooperative games of the form Γ.

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[1]Pliny the Elder (2379, BC), writer and scientist who lived in Rome, Italy

To consider the effect of uncertainty on their payoffs, the players need to adopt a principle of decision making under uncertainty [23] such as the maxmin principle [24], the minimax regret principle [25], etc. Moreover, a reasonable concept of solution of the game Γ must reflect the effect of the uncertainty on the players. As uncertainty is considered in equilibria of cooperative games for the first time, we assume that the players adopt a conservative (maxmin or risk averse) approach [24]. Considering other principles of decision under uncertainty in the game Γ can be the subject of other works. Thus, we propose to evaluate the payoff function of each player $i \in N$ not by its value $f_i(x, y)$, but by its guaranteed or secure level $f_i[x]$. A guarantee against the values $f_i(x, y)$, $y \in Y$ can be defined as follows

$$f_i[x] = \min_{y \in Y} f_i(x, y).$$

Indeed, we have $f_i[x] \leq f_i(x, y)$, $y \in Y$, therefore, the lower bound of the payoff function of the i-th player can be estimated by $f_i[x]$. We will see later that under common conditions the function $x \to f_i[x]$ is well-defined and continuous on X. In this section and in sections 2 and 3, we assume that the functions $x \to f_i[x]$, $i \in N$ are well defined and continuous on X. Thus, we obtain the guaranteed (conservative) game

$$\Gamma^g = \langle N = \{1, 2, 3, 4\}, \{X_i\}_{i \in N}, \{f_i[x]\}_{i \in N} \rangle.$$

In the next section we will introduce SCE of the game Γ via the game Γ^g.

§ 2. Coalitional rationality and strong coalitional equilibrium

In this section we first present the main properties of SCE then introduce it. To define the coalitional rationality, the following notations are convenient. For any nonempty subset K of N, denote by $-K$ the complement of K, that is, $N \setminus K$. Particularly, for each $i \in N$, denote by $-i$ the set $N \setminus \{i\}$ and for each $i, j \in N$, $i \neq j$, denote by $-(i, j)$ the set $N \setminus \{i, j\}$. The notion of partition of a set will be used as well. A partition of a set A is a family of disjoint subsets of A, the union of wich equals A. In game theory, a partition of the set of players is called a coalition structure. For a strategy profile $x \in X$, and $i \in N$ denote by $x = (x_i, x_{-i})$ and $X_{-i} = \prod_{j \in N \setminus \{i\}} X_j$.

In the game Γ, fifteen coalition structures can form, namely, $\{\{1\}, \{2\}, \{3\}, \{4\}\}$, $\{1, 2, 3, 4\}$, $K_{\{i\}} = \{\{i\}, \{-i\}\}$, $K_{\{i\}, \{j\}} = \{\{i\}, \{j\}, \{-(i, j)\}\}$, $K_{\{i, j\}} = \{\{i, j\}, \{-(i, j)\}\}$, for all $i, j \in N$, $i \neq j$. Let us recall some results of the theory of cooperative games without side payments [23]. For a strategy profile $x^* \in X$ in the game Γ^g, the following concepts are considered:

(a) x^* satisfies the *individual rationality condition* (IRC), if for all $i \in N$

$$f_i[x^*] \geq f_i^0 = \max_{x_i \in X_i} \min_{x_{-i} \in X_{-i}} f_i[x_i, x_{-i}] = \min_{x_{-i} \in X_{-i}} f_i[x_i^0, x_{-i}].$$

The value f_i^0 is the guaranteed payoff or security level of player $i \in N$. If player i selects his/her *maxmin* strategy x_i^0 then his/her payoff satisfies $f_i[x_i^0, x_{-i}] \geq f_i^0$, for all $x_{-i} \in X_{-i}$;

(b) x^* satisfies the *collective rationality condition* (CLRC) if x^* is maximal with respect to Pareto preference for the multiple criteria problem $\Gamma^P = \langle X, f_i[x]_{i \in N} \rangle$, that is, for all $x \in X$, the system of inequalities $f_i[x] \geq f_i[x^*]$, $i \in N$, with at least one strict inequality is impossible. Note that if for all $x \in X$, $\sum_{i \in N} f_i[x] \leq \sum_{i \in N} f_i[x^*]$, then x^* is Pareto maximal for the problem Γ^P;

(c) x^* satisfies the *coalitional rationality condition* (CRC) if

$$f_k[x^*] \geq f_k[x_i^*, x_{-i}], \quad \text{for all} \ \ x_{-i} \in X_{-i};$$

$$f_k[x^*] \geq f_k[x_i^*, x_j^*, x_{-(i,j)}], \quad \text{for all} \ \ x_{-(i,j)} \in X_{-(i,j)};$$

$$f_k[x^*] \geq f_k[x_i, x_{-i}^*], \quad \text{for all} \ \ x_i \in X_i,$$

for all $i, j, k \in N$, $i \neq j$, where $x = (x_i, x_j, x_{-(i,j)})$, $X_{-(i,j)} = \prod_{s \in N \setminus \{i,j\}} X_s$. This condition means that when a coalition K selects its strategy profile from x^*, then no player can improve his/her payoff if the counter coalition $-K$ deviates from its strategy profile in x^*.

Definition 1. A strategy profile $x^* \in X$ is called *strong coalitional equilibrium* (SCE) for the game Γ if it satisfies IRC, CLRC and CRC for the guaranteed game Γ^g.

Remark 1. IRC means that it makes sense for a player to form coalitions with other players if he/she gets a payoff not less than what he/she can guarantee by selecting his/her maxmin strategy. CLRC leads the players to a non-dominated strategy profile, with respect to Pareto preference. Finally, CRC means that the payoff of each player is immune against any deviations of individual players or coalitions from a strategy profile satisfying CRC. In other words, no player's payoff is increased when any coalition deviates from SCE. Thus, it is rational for all coalitions not to deviate from x^* as no player in a deviating coalition or outside it can benefit.

According to Definition 1, SCE must satisfy all the extremal constraints that define IRC, CLRC and CRC. However, all these constraints can be easily derived from the following seventeen of them

$$f_i[x_1^*, x_2, x_3, x_4] \leq f_i[x^*], \ \text{for all} \ x_k \in X_k, \ k = 2, 3, 4 \ \text{and} \ i = 1, 2, 3, 4;$$

$$f_i[x_1, x_2^*, x_3, x_4] \leq f_i[x^*], \ \text{for all} \ x_k \in X_k, \ k = 1, 3, 4 \ \text{and} \ i = 1, 2, 3, 4;$$

$$f_i[x_1, x_2, x_3^*, x_4] \leq f_i[x^*], \ \text{for all} \ x_k \in X_k, \ k = 1, 2, 4 \ \text{and} \ i = 1, 2, 3, 4; \qquad (2.1)$$

$$f_i[x_1, x_2, x_3, x_4^*] \leq f_i[x^*], \ \text{for all} \ x_k \in X_k, \ k = 1, 2, 3 \ \text{and} \ i = 1, 2, 3, 4;$$

$$\sum_{i \in N} f_i[x] \leq \sum_{i \in N} f_i[x^*], \ \text{for all} \ x \in X,$$

where $x^* = (x_1^*, x_2^*, x_3^*, x_4^*)$.

Henceforth, we will use the system of inequalities (2.1) to establish that a strategy profile is a SCE of the game Γ instead of the system of inequalities involved in the definitions of IRC, CLRC and CRC (see the items (a), (b) and (c) above). From (2.1) one can see that SCE has two interesting characteristics. First, once the players are in SCE, they do not have incentive to deviate from it individually, collectively or in coalitions. Second, if players are not in SCE, as soon as one player (or a coalition) declares that he/she (it) will select his/her (its) strategy (profile) from SCE, the other players have no choice but to select their strategies from SCE. In other words, any player or coalition can enforce a SCE.

Although SCE does not exist in pure strategies in most of continuous games, in finite games it is not the case. The following example, adapted from [5], shows that SCE exists in a class of games.

Example 1. Consider a three-player game where players 1, 2 and 3 choose rows, columns and boxes, respectively and are named accordingly. Let $\epsilon \in [0,1]$ and α, β, γ be nonnegative numbers such that at least one of the inequalities $\alpha + \beta + \gamma < 9$ holds. Each of the players has two strategies $\{T, B\}$ for player 1, $\{T, L\}$ for players 2 and 3. The minimum payoffs i.e. $f_i[x_1, x_2, x_3]$, where $x_1 = T, B$; $x_j = T, L$; $j = 2, 3$, and $i = 1, 2, 3$ are given as follows

	L	R	L	R
T	2, 2, 2	0, 0, ϵ	0, 0, 0	4, 4, 1
B	α, β, γ	0, 0, ϵ	0, 0, 0	3, 3, 1
		L		R

The strategy profile (T, R, R) is a SCE. Indeed, it is easy to see that this strategy profile satisfies the last inequality of the system (2.1), as the sum of payoffs at (T, R, R) is higher than the sum of payoffs at any other strategy profile, including (B, L, L) thanks to the inequality $\alpha + \beta + \gamma < 9$ and the fact that α, β, γ are nonnegative numbers, that is, (T, R, R) satisfies the last inequality of (2.1). Next, the possible deviations corresponding to the inequalities in (2.1) are (T, L, R), (T, R, L), (T, L, L) when player 1 chooses the SCE strategy T; (B, R, R), (T, R, L), (B, R, L) when player 2 chooses the SCE strategy R and (B, R, R), (T, L, R), (B, L, R) when player 3 chooses the SCE strategy R. At all the mentioned strategy profiles the payoffs of players 1, 2, and 3 are less or equal than their payoffs at (T, R, R), which are 4, 4, and 1, respectively.

2.1. Related concepts

In this section, we recall the most prominent cooperative solution for NTU games in normal form and compare them to SCE. We also compare SCE to solution concepts defined in dynamic context with respect to coalition deviation.

a) [8] A strategy profile $x^* \in X$ is a strong equilibrium (SNE) of the game Γ^g, if for all $S \subset N$, for all $y_{-S} \in X_{-S}$, the system of inequalities $f_i[x^*] < f_i[y_S, x^*_{-S}]$, for all $i \in S$, is impossible.

This definition means that no coalition can improve the payoff of all its members by deviating from SNE, when the other players stick to SNE.

b) [9] A strategy profile $x^* \in X$ is in the α-core of the game Γ^g, if for any coalition $S \subset N$, for each $y_S \in X_S$, there exists $z_{-S} \in X_{-S}$ such that the system of inequalities $f_i[x^*] < f_i[y_S, z_{-S}]$, for all $i \in S$, is impossible.

This definition means that if a coalition deviates from the α-core strategy profile x^*, the remaining players have a counter strategy profile to punish it in such a way that not all members of the coalition are better-off.

c) [9] A strategy profile $x^* \in X$ is in the β-core of the game Γ^g, if for each coalition $S \subset N$, there exists $z_{-S} \in Z_{-S}$ such that for all $y_S \in X_S$ the system of inequalities $f_i[x^*] < f_i[y_S, z_{-S}]$, for all $i \in S$, is impossible.

This definition means that for each coalition, the other players have a special strategy profile that they can use to punish it for any deviation from the β-core strategy profile x^* in such a way that not all members of the coalition are better-off.

d) [10] A strategy profile $x^* \in X$ is a strong Berge equilibrium (SBE) of the game Γ^g, if for all $i \in N$, $f_j[x^*_i, z_{-i}] \leq f_j[x^*]$ for all $z_{-i} \in Z_{-i}$ and $j \in -i$.

This definition means that no coalition of the form $-i$ can make any of its members better-off, if it deviates from SBE. When a player plays a strategy from SBE, the other players have no choice but to follow him/her by selecting their strategy from SBE.

e) [11] A strategy profile $x^* \in X$ is a Berge equilibrium (BE) of the game Γ^g, if for all $i \in N$, $f_i[x_i^*, z_{-i}] \leq f_i[x^*]$ for all $z_{-i} \in Z_{-i}$.

This definition means that at BE the players maximize each other payoff functions. This equilibrium reflects mutual support and altruism among players [12].

Using (2.1), it is easy to verify that SCE is also an SNE, SBE and BE. It is well-known that SNE is a CPNE, then CSE is also a CPNE. Next, SCE is an element of the α-core and the β-core. SCE has similarities with SBE. However, there are two important differences between SBE and SCE. The first is that in SBE the system of inequalities for each $i \in N$, $f_j[x_i^*, z_{-i}] \leq f_j[x^*]$ for all $z_{-i} \in Z_{-i}$ and $j \in -i$ does not include the inequality corresponding to player i, $f_i[x_i^*, z_{-i}] \leq f_i[x^*]$, which means that the other players do not care about player i's payoff when they select their strategy from x^* as his/her payoff is not maximized. In SCE the inequality $f_i[x_i^*, z_{-i}] \leq f_i[x^*]$ is included, which means that player i's payoff function is maximized by the other players. This shows that SCE involves mutual support, while SBE does not. The second is that SBE is not Pareto optimal in general. Pareto SBE is investigated in [12]. SCE has also some similarities with BE. However, there are important differences between the two equilibria. BE expresses mutual support and altruism and ignores individual player's interest; it is not a refinement of Nash equilibrium as BE may not satisfy IRC [12]. The difference between the SCE and the hybrid solution (HS) of [18] is that in HS it is assumed that a coalition structure is formed and there is no cooperation across the coalitions of the coalition structure, while in SCE such assumptions are not made.

Although the coalitional equilibrium of [5], the equilibrium binding agreement of [7], the equilibrium process of coalition formation of [6] and the consistent set of [4] are defined in a dynamic context, they can be compared to SCE based on when a coalition can deviate. In the mentioned concepts, a coalition deviates to another state or strategy profile only and only if all its members are better off, while in SCE a coalition can deviate if and only if all players of the game are better off (not only its members).

Moreover, in the concepts mentioned in this section, uncertainty as an exogenous factor, is not considered as in SCE.

§ 3. Sufficient conditions for the existence of SCE in pure strategies

In the previous section we have seen that SCE is also a SNE and SBE. Since these equilibria do not exist in pure strategies in most of the continuous games as pointed out in the introduction, SCE suffers also from this drawback. Nevertheless, we formulate sufficient conditions for its existence using the approach developed in [26]. The approach used in this section paves the way to the next section where the main result of this paper is presented. We first introduce the convolution [27] related to SCE

$$\varphi_1(x, z) = \max_{i \in N}\{f_i[z_1, x_2, x_3, x_4] - f_i[z]\},$$

$$\varphi_2(x, z) = \max_{i \in N}\{f_i[x_1, z_2, x_3, x_4] - f_i[z]\},$$

$$\varphi_3(x, z) = \max_{i \in N}\{f_i[x_1, x_2, z_3, x_4] - f_i[z]\}, \tag{3.1}$$

$$\varphi_4(x, z) = \max_{i \in N}\{f_i[x_1, x_2, x_3, z_4] - f_i[z]\},$$

$$\varphi_5(x, z) = \sum_{i \in N} f_i[x] - \sum_{i \in N} f_i[z],$$

$$\varphi(x, z) = \max_{r=1,\dots,5} \{\varphi_r(x, z)\},$$

where $x = (x_1, x_2, x_3, x_4)$ and $z = (z_1, z_2, z_3, z_4) \in X = \prod_{i \in N} X_i$.

A saddle point $(x^0, z^*) \in X \times X$ of the real-valued function $\varphi(x, z)$ in (3.1) is defined by the chain of inequalities

$$\varphi(x, z^*) \le \varphi(x^0, z^*) \le \varphi(x^0, z), \quad \text{for all } x, z \in X. \tag{3.2}$$

Theorem 1. *If $(x^0, z^*) \in X \times X$ is a saddle point of the function $\varphi(x, z)$, then the minimax strategy z^* is a SCE of the game Γ.*

P r o o f. Indeed, setting $z = x^0$ in (3.2), from (3.1) we get $\varphi(x^0, x^0) = 0$. Then by transitivity, from (3.2) we obtain

$$\varphi(x^0, z^*) \le \varphi(x^0, x^0) = 0 \Rightarrow \varphi(x, z^*) \le 0, \quad \text{for all } x \in X,$$

which implies (2.1). $\square$

Remark 2. According to Theorem 1, the determination of SCE reduces to the determination of a saddle point (x^0, z^*) of the Germeier convolution $\varphi(x, z)$ from (3.1). We obtain the following procedure for the determination of SCE of the game Γ.

Step 1. Construct the function $\varphi(x, z)$ by (3.1).

Step 2. Find a saddle point $(x^0, z^*) \in X \times X$ of the function $\varphi(x, z)$.

Step 3. Compute the four values $f_i[z^*]$, $i \in N$.

Then the pair $(z^*, f[z^*] = (f_1[z^*], f_2[z^*], f_3[z^*], f_4[z^*])) \in X \times \mathrm{R}^4$ consists of the SCE z^* and the corresponding payoffs of the four players. When the players select their strategies from the SCE z^*, they get the payoffs $f_i[z^*]$, $i \in N$, respectively.

Thus, when the bifunction $\varphi(x, z)$ has a saddle point, one can use the existing in literature numerical methods for computing saddle points.

§ 4. The existence of SCE in mixed strategies

Since SCE like SNE and SBE does not exist in pure strategies in most continuous games, one can naturally follow Borel, von Neumann and Nash [28, 29] and [1, 2] when they randomized strategies to establish the existence of Nash equilibrium in mixed strategies. Indeed, following these great scholars, we establish the existence of SCE in mixed strategies. For this purpose, we need some preliminary results that will help in proving the main existence theorem.

4.1. Preliminaries

First, we introduce some notations. Denote by $\mathrm{comp}\, \mathrm{R}^{n_i}$ and $\mathrm{co\, comp}\, \mathrm{R}^{n_i}$ the set of compact subsets of R^{n_i} and the set of convex and compact subsets of R^{n_i}, respectively, and by $C(X \times Y)$ the set of real-valued and continuous functions with domain $X \times Y$.

Assume that the elements of the game Γ satisfy the following condition.

Condition 1.

$$X_i \in \mathrm{co\, comp}\, \mathrm{R}^{n_i}, \quad Y \in \mathrm{co\, comp}\, \mathrm{R}^m, \ f_i(\cdot) \in C(X \times Y), \quad \text{for all } i \in N.$$

Then according to Berge' maximum theorem [30], the function $x \to f_i[x]$ is well-defined and continuous on X for all $i \in N$.

Next, we construct the mixed extension of the game Γ^g including mixed strategies sets and strategy profiles, and the expected value of the players' payoff functions.

First, to each strategy set $X_i \in \operatorname{co\,comp} \mathrm{R}^{n_i}$ associate the Borel σ-algebra $B(X_i)$, which consists of subsets $Q^{(i)}$ of X_i such that the intersection and union of a countable set of elements of $B(X_i)$ belong to $B(X_i)$, moreover, $B(X_i)$ is the minimal σ-algebra that contains all closed subsets of X_i. In game theory, a mixed strategy of the i-th player, $\nu_i(\cdot)$ can be identified as a *probability measure* on the compact set of pure strategies X_i. A probability measure is a non-negative function, $\nu_i(\cdot)$, defined on the Borel σ-algebra $B(X_i)$ and satisfies the following two conditions:

(1) $\nu_i \left(\bigcup_k Q_k^{(i)} \right) = \sum_k \nu_i \left(Q_k^{(i)} \right)$ for any sequence of disjoint elements $\{Q_k^{(i)}\}$ of $B(X_i)$ (countable additivity property);

(2) $\nu_i(X_i) = 1$ (normality property).

Note that (2) implies that $\nu_i \left(Q^{(i)} \right) \le 1$, for all $Q^{(i)} \in B(X_i)$.

Denote by $\{\nu_i\}$ the set of mixed strategies of the player $i \in N$. Then a mixed strategy profile of the game Γ^g can be formulated as a product-measure as follows

$$\nu(dx) = \nu_1(dx_1)\nu_2(dx_2)\nu_3(dx_3)\nu_4(dx_4),$$

the set of which we denote by $\{\nu\}$. The payoff of the i-th player corresponding to his/her payoff function in the game Γ^g is defined by $f_i[\nu] = \int_X f_i[x]\nu(dx)$. We obtain the mixed extension of the game Γ^g, as follows

$$\widetilde{\Gamma}^g = \langle N = \{1, 2, 3, 4\}, \{\nu_i\}_{i \in N}, \{f_i[\nu]\}_{i \in N} \rangle. \tag{4.1}$$

Here we have committed an abuse of notations denoting the expected value function of $f_i[x]$ by $f_i[\nu]$, the reader may distinguish between the two functions by the involved variable.

Now we can derive the following definition of equilibrium from Definition 1 and (2.1).

Definition 2. A mixed strategy profile $\nu^*(\cdot) \in \{\nu\}$ is called a *mixed strategy coalitional equilibrium* (MSCE) of the game Γ if it is a SCE of the mixed extension game (4.1), that is,

(i) $\nu^*(\cdot)$ satisfies individual rationality and coalitional rationality conditions IRC and CRC, which can be derived from the following inequalities

$$f_i[\nu_1^*, \nu_2, \nu_3, \nu_4] \le f_i[\nu^*], \text{ for all } \nu_k(\cdot) \in \{\nu_k\}, \ k = 2, 3, 4 \text{ and } i = 1, 2, 3, 4;$$

$$f_i[\nu_1, \nu_2^*, \nu_3, \nu_4] \le f_i[\nu^*], \text{ for all } \nu_k(\cdot) \in \{\nu_k\}, \ k = 1, 3, 4 \text{ and } i = 1, 2, 3, 4;$$

$$f_i[\nu_1, \nu_2, \nu_3^*, \nu_4] \le f_i[\nu^*], \text{ for all } \nu_k(\cdot) \in \{\nu_k\}, \ k = 1, 2, 4 \text{ and } i = 1, 2, 3, 4; \tag{4.2}$$

$$f_i[\nu_1, \nu_2, \nu_3, \nu_4^*] \le f_i[\nu^*], \text{ for all } \nu_k(\cdot) \in \{\nu_k\}, \ k = 1, 2, 3 \text{ and } i = 1, 2, 3, 4;$$

where $\nu^* = (\nu_1^*, \nu_2^*, \nu_3^*, \nu_4^*)$.

(ii) $\nu^*(\cdot)$ satisfies the collective rationality condition, CLRC, or it is Pareto maximal for the four-criteria optimization problem

$$\widetilde{\Gamma}_v^g = \langle \{\nu\}, \{f_i[\nu]\}_{i \in N} \rangle,$$

that is, for all $\nu(\cdot) \in \{\nu\}$ the following system of inequalities

$$f_i[\nu] \geq f_i[\nu^*], \quad i = 1, 2, 3, 4,$$

with at least one strict inequality, is impossible.

A sufficient condition for the Pareto optimality in condition (ii) is given in the following remark.

Remark 3. A mixed strategy profile $\nu^*(\cdot) \in \{\nu\}$ is Pareto optimal for the multiple criteria optimization problem $\widetilde{\Gamma}_v^g = \langle \{\nu\}, \{f_i[\nu]\}_{i \in N} \rangle$, if

$$\max_{\nu(\cdot) \in \{\nu\}} \sum_{i \in N} f_i[\nu] = \sum_{i \in N} f_i[\nu^*].$$

Consider the function $\varphi_i(x, z)$, $i = 1, 2, 3, 4$ and the function

$$\varphi(x, z) = \max_{r=1,\ldots,5} \{\varphi_r(x, z)\} \tag{4.3}$$

introduced in (3.1).

Proposition 1. *The following inequality holds*

$$\max_{r=1,\ldots,5} \int_{X \times X} \varphi_r(x, z) \mu(dx) \nu(dz) \leq \int_{X \times X} \max_{r=1,\ldots,5} \varphi_r(x, z) \mu(dx) \nu(dz) \tag{4.4}$$

for all $\nu(\cdot) \in \{\nu\}$ and $\mu(\cdot) \in \{\nu\}$.

P r o o f. In fact, from (4.3), for all $x, z \in X$, we get the following five inequalities

$$\varphi_r(x, z) \leq \varphi(x, z) = \max_{r=1,\ldots,5} \varphi_r(x, z), \quad r = 1, \ldots, 5.$$

Integrating both parts of these inequalities with an arbitrary product-measure $\mu(dx)\nu(dz)$ as an integration measure, we obtain

$$\int_{X \times X} \varphi_r(x, z) \mu(dx) \nu(dz) \leq \int_{X \times X} \max_{r=1,\ldots,5} \varphi_r(x, z) \mu(dx) \nu(dz),$$

for all $\mu(\cdot), \nu(\cdot) \in \{\nu\}$ and $r = 1, \ldots, 5$. Therefore,

$$\max_{r=1,\ldots,5} \int_{X \times X} \varphi_r(x, z) \mu(dx) \nu(dz) \leq \int_{X \times X} \max_{r=1,\ldots,5} \varphi_r(x, z) \mu(dx) \nu(dz),$$

for all $\mu(\cdot), \nu(\cdot) \in \{\nu\}$. Hence, (4.4) is satisfied. $\qquad \square$

Remark 4. Proposition 1 is, in fact, a generalization of the optimization operation: the maximum of the sum of some functions is less or equal than the sum of the maximums of these functions.

Proposition 2. *The function $\varphi(x, z)$ defined in (4.3) is continuous over $X \times Z$, where $Z = X$.*

The proof of an even more general result on the continuity of the maximum of a finite number of continuous functions over a compact set is available in many monographs, for instance in [31].

4.2. Existence Theorem

In this subsection, we prove the main result of this paper, the existence of MSCE for the game Γ.

Theorem 2. *Under the Condition 1, the game Γ has a MSCE.*

P r o o f. Consider the following two-person zero-sum game.

$$\Gamma^a = \langle \{1, 2\}, X, Z, \varphi(x, z) \rangle,$$

where $X = Z$. In the game Γ^a, the set X is the strategy set of both the maximizing and minimizing players; $\varphi(x, z)$ is the payoff function of maximizing player and $-\varphi(x, z)$ is the payoff function of the minimizing player. Any saddle point (x^0, z^*) of the function $\varphi(x, z)$ is a NE of the game Γ^a. Indeed, the saddle point definition

$$\varphi(x, z^*) \leq \varphi(x^0, z^*) \leq \varphi(x^0, z), \text{ for all } (x, z) \in X \times Z$$

means that the strategy profile (x^0, z^*) is a NE of the game Γ^a. Now with the game Γ^a, we associate its mixed strategy extension

$$\widetilde{\Gamma}^a = \langle \{1, 2\}, \{\mu\}, \{\nu\}, \varphi(\mu, \nu) \rangle,$$

where $\{\mu\}$ is the set of strategies of the maximizing player, $\{\nu\} = \{\mu\}$ is the set of strategies of the minimizing player and $\varphi(\mu, \nu)$ is the payoff (expected utility) of the maximizing player

$$\varphi(\mu, \nu) = \int\limits_{X \times X} \varphi(x, z) \mu(dx) \nu(dz). \tag{4.5}$$

Here also we have committed an abuse of notations by denoting the expected value function of the function $\varphi(x, z)$ by $\varphi(\mu, \nu)$. The reader can distinguish between the two by the involved variables. Similarly, any saddle point (μ^0, ν^*) of the function $\varphi(\mu, \nu)$ is a NE of the game $\widetilde{\Gamma}^a$. Indeed, the definition of a saddle point

$$\varphi(\mu, \nu^*) \leq \varphi(\mu^0, \nu^*) \leq \varphi(\mu^0, \nu), \text{ for all } (\mu, \nu) \in \{\nu\} \times \{\nu\} \tag{4.6}$$

means that the strategy profile (μ^0, ν^*) is a NE of the game $\widetilde{\Gamma}^a$.

In 1952 Gliksberg [32] established the existence of a NE in mixed strategies for N-person games with $N > 1$, from which we deduce (for the special case of two-person zero sum game Γ^a) the following statement. As the set of strategy profiles $X \subset \mathbb{R}^n$ is convex and compact and the function $\varphi(x, z)$ is continuous over $X \times X$ (see Proposition 2), the game Γ^a has a mixed strategy NE (μ^0, ν^*) satisfying (4.6).

Considering (4.3) and (4.5), the inequalities (4.6) take the form

$$\int_{X \times X} \max_{r=1,\ldots,5} \varphi_r(x,z) \mu(dx) \nu^*(dz) \leq$$

$$\leq \int_{X \times X} \max_{r=1,\ldots,5} \varphi_r(x,z) \mu^0(dx) \nu^*(dz) \leq$$

$$\leq \int_{X \times X} \max_{r=1,\ldots,5} \varphi_r(x,z) \mu^0(dx) \nu(dz),$$

for all $(\mu,\nu) \in \{\nu\} \times \{\nu\}$. Setting $\nu_i(dz_i) = \mu_i^0(dx_i)$, $i \in N$, which means $\nu(dz) = \mu^0(dx)$, in

$$\varphi(\mu^0, \nu) = \int\limits_{X \times X} \max_{r=1,\ldots,5} \varphi_r(x,z) \mu^0(dx) \nu(dz).$$

We get

$$\varphi(\mu^0, \mu^0) = \int\limits_{X \times X} \max_{r=1,\ldots,5} \varphi_r(x,x) \mu^0(dx) \mu^0(dx).$$

From (3.1), we get $\varphi_r(x,x)$, $r = 1,\ldots,5$ for all $x \in X$, then the previous integral gives $\varphi(\mu^0, \mu^0) = 0$. A similar reasoning leads to $\varphi(\nu^*, \nu^*) = 0$. Then from (4.6), we obtain

$$\varphi(\mu^0, \nu^*) = 0. \tag{4.7}$$

Using (4.7) and the inequalities in (4.6), by transitivity, we obtain

$$\varphi(\mu, \nu^*) = \int\limits_{X \times X} \max_{r=1,\ldots,5} \varphi_r(x,z) \mu(dx) \nu^*(dz) \leq 0, \quad \text{for all } \mu \in \{\nu\}.$$

According to Proposition 1,

$$\max_{r=1,\ldots,5} \int\limits_{X \times X} \varphi_r(x,z) \mu(dx) \nu^*(dz) \leq \int\limits_{X \times X} \max_{r=1,\ldots,5} \varphi_r(x,z) \mu(dx) \nu^*(dz) \leq 0,$$

for all $\mu \in \{\nu\}$. Therefore,

$$\int\limits_{X \times X} \varphi_r(x,z) \mu(dx) \nu^*(dz) \leq 0, \quad \text{for all } \mu \in \{\nu\} \text{ and for all } r = 1,\ldots,5. \tag{4.8}$$

We distinguish two cases.

   Case I. ($r = 1,\ldots,4$) Here according to (3.1), (4.8) and the fact that $\mu(\cdot)$ is normed i.e. $\int_X \mu(dx) = 1$, for instance, for $r = 1$, we get

$$f_i[\nu_1^*, \mu_2, \mu_3, \mu_4] - f_i[\nu^*] =$$

$$= \int\limits_{X \times X} f_i[z_1, x_2, x_3, x_4] \mu(dx) \nu^*(dz) - \int\limits_X f_i[z] \nu^*(dz) \int\limits_X \mu(dx) =$$

$$= \int\limits_{X \times X} f_i[z_1, x_2, x_3, x_4] \mu(dx) \nu^*(dz) - \int\limits_{X \times X} f_i[z] \mu(dx) \nu^*(dz) =$$

$$= \int\limits_{X \times X} (f_i[z_1, x_2, x_3, x_4] - f_i[z]) \mu(dx) \nu^*(dz) \leq$$

$$\leq \int\limits_{X \times X} \max_{i \in N}\{f_i[z_1, x_2, x_3, x_4] - f_i[z]\}\mu(dx)\nu^*(dz) =$$

$$= \int\limits_{X \times X} \varphi_1(x, z)\mu(dx)\nu^*(dz) \leq 0,$$

for all $i \in N$. Thus, $f_i[\nu_1^*, \mu_2, \mu_3, \mu_4] - f_i[\nu^*] \leq 0$, for all $i \in N$ and $\mu_k \in \{\nu_k(\cdot)\}$, $k = 2, 3, 4$.

In a similar way, the following three inequalities can be established for $r = 2, 3, 4$:

$$f_i[\mu_1, \nu_2^*, \mu_3, \mu_4] - f_i[\nu^*] \leq 0, \text{ for all } i \in N \text{ and } \mu_k \in \{\nu_k(\cdot)\}, k = 1, 3, 4,$$

$$f_i[\mu_1, \mu_2, \nu_3^*, \mu_4] - f_i[\nu^*] \leq 0, \text{ for all } i \in N \text{ and } \mu_k \in \{\nu_k(\cdot)\}, k = 1, 2, 4,$$

$$f_i[\mu_1, \mu_2, \mu_3, \nu_4^*] - f_i[\nu^*] \leq 0, \text{ for all } i \in N \text{ and } \mu_k \in \{\nu_k(\cdot)\}, k = 1, 2, 3.$$

According to Definition 2, the mixed strategy profile $\nu^*(\cdot)$ the four inequalities (4.2) of condition (i) of Definition 2. It remains to prove that $\nu^*(\cdot)$ satisfies condition (ii) of Definition 2, that is, it is Pareto optimal or satisfies CLRC. For this purpose, we will use Remark 3.

Case II. ($r = 5$) According to (3.1), (4.8) and the fact that $\mu(\cdot)$ and $\nu^*(\cdot)$ are normed i. e. $\int\limits_X \mu(dx) = 1$ and $\int\limits_X \nu^*(dx) = 1$, we get

$$\sum_{i \in N} f_i[\mu] - \sum_{i \in N} f_i[\nu^*] = \sum_{i \in N} \int\limits_X f_i[x]\mu(dx) - \sum_{i \in N} \int\limits_X f_i[z]\nu^*(dz) =$$

$$= \int\limits_X \sum_{i \in N} f_i[x]\mu(dx) \int_X \nu^*(dz) - \int\limits_X \sum_{i \in N} f_i[z]\nu^*(dz) \int_X \mu(dx) =$$

$$= \int\limits_{X \times X} \left[\sum_{i \in N} f_i[x] - \sum_{i \in N} f_i[z]\right] \mu(dx)\nu^*(dz) =$$

$$= \int\limits_{X \times X} \varphi_5(x, z)\mu(dx)\nu^*(dz) \leq 0.$$

Thus, $\sum_{i \in N} f_i[\mu] - \sum_{i \in N} f_i[\nu^*] \leq 0$, for all $\mu \in \{\nu\}$. Then according to Remark 3, the mixed strategy profile $\nu^*(\cdot)$ is Pareto optimal for the multiple criteria problem

$$\widetilde{\Gamma}_v^g = \langle \{\nu\}, \{f_i[\nu]\}_{i \in N} \rangle.$$

Thus, we have established that the mixed strategy profile $\nu^*(\cdot)$ is a SCE of the game $\widetilde{\Gamma}^g$. According to Definition 2, $\nu^*(\cdot)$ is a MSCE for the game Γ and $f[\nu^*]$ is the players' payoff vector. $\square$

Remark 5. The reader may be tempted to infer from Theorem 2 that any finite game has a mixed strategy SCE because its mixed strategy extension game satisfies all the conditions of Theorem 2. It is important to note that the conditions of Theorem 2 are imposed on the initial pure strategy game not on the mixed strategy extension of this game. Therefore, as strategy sets of finite games are not convex, Theorem 2 does not apply to them.

§ 5. Conclusion

The contribution of this paper to the theory of cooperative games in normal form consists of the following. First, the concept of the strong coalitional equilibrium (SCE) in normal form games under uncertainty is formalized. This concept considers the interests of all coalitions. Second, a constructive procedure for pure strategy SCE determination is provided that reduces to the determination of a saddle point of bifunction. Third, the existence of SCE in mixed strategies is proved under minimal common mathematical programming conditions (continuity of payoff functions and compactness and convexity of players' strategy sets and compactness of uncertainty set).

In our view, the following qualitative results that can be derived from this paper are important.

1. The results of this work can be extended to games with any finite number of players (more than four players).

2. A SCE $x^* \in X$ is stable against any deviation of any coalition of players and is attractive because when a coalition selects its strategies from x^*, all the other players will have incentive to select their strategies from it.

3. SCE could be applied even when the coalition structure changes.

4. SCE could be used for the formation of stable alliances.

5. So far, in game theory, the focus was on the individual rationality and collective rationality. Indeed, individual interests of players are represented by the prominent Nash equilibrium with its egocentric character (each player acts for himself only). Collective interests of players are represented by Berge equilibrium (see the definition and related discussion in Section 2.1) with its altruism (help others to the extent of forgetting about one's self interests) [33, 34]. Such an omission is not rooted in the human nature of the players. SCE partially addresses the incomplete representation of human behavior in the two mentioned concepts. Indeed, in the game Γ, in the context of SCE, when the first player chooses his/her SCE strategy, he/she does not forget his/her own interests as SCE is also a Nash equilibrium, and according to (2.1), he/she helps (maximizes the payoff of) all the other players (property of Berge equilibrium). The other players act in a similar way. Thus, SCE fills the gap between Nash equilibrium and Berge equilibrium by completing the former by the «caring about others» and the latter by «caring about oneself».

Finally, we point out two possible ways of extending the present work. The first is to investigate SCE in finite games. The second is to consider other principles of handling the uncertainty such as the maximin regret principle.

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Строгие коалиционные равновесия в играх при неопределенности

В статье для игр в нормальной формой при интервальной неопределенности вводится концепция сильного коалиционного равновесия. Эта концепция основана на синтезе трех понятий: индивидуальной рациональности, коллективной рациональности для игр в нормальной форме без побочных платежей и коалиционной рациональности. Для простоты изложения, сильное коалиционное равновесие рассматривается для игр 4 лиц при неопределенности. Достаточные условия существования сильного коалиционного равновесия в чистых стратегиях устанавливаются с помощью седловой точки специального вида свертки Гермейера. Наконец, следуя подходу Бореля, Неймана и Нэша, доказана теорема существования сильного коалиционного равновесия в смешанных стратегиях при стандартных для теории игр условиях (компактность и выпуклость множеств стратегий игроков, компактность множества неопределенностей и непрерывность функций выигрыша).

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