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DIFFERENCE DERIVATIVE FOR AN INTEGRO-DIFFERENTIAL NONLINEAR VOLTERRA EQUATION

In this article, we propose a new numerical approximation method to deal with the unique solution of the nonlinear integro-differential Volterra equation. We are interested in a very particular form of this equation, in which the derivative of the sought solution appears under the integral sign in a nonlinear manner. Our vision is based on two different approaches: We use the Nyström method to transform the integral into a finite sum using a numerical integration formula, then we use the numerical backward difference derivative method to approach the derivative of our solution. This collocation between two different methods, the first outcome of the numerical processing of integral equations and the second outcome of the numerical processing of differential equations, gives a new nonlinear system for approaching the solution of our equation. We show that the system has a unique solution and that this numerical solution converges perfectly to our solution. A section is dedicated to numerical tests, in which we show the effectiveness of our new vision compared to two methods based only on numerical integration.

Keywords: Volterra integro-differential equation, nonlinear equation, fixed point, numerical derivative, Nyström method.

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Introduction

The Volterra nonlinear integral equations represent a very great interest in physics and mathematical modeling as history-dependent problems, systems theory, heat conduction and diffusion [1–5]

The analytical and numerical study of its non-differential type has been much studied [1, 6, 7] compared to the differential type [8, 9]. In this paper, we study a particular case of the integro-differential equations type: To find for a given function $f \in C^1(a,b)$, a unique solution $u \in C^1(a,b)$ such that

$$\forall t \in [a,b], \quad u(t) = \int_a^t K(t,s,u(s),u'(s))\,ds + f(t). \tag{0.1}$$

The peculiarity and interest of this equation lies in the fact that the derivative of the solution appears under the integral sign in a nonlinear way. Equations of this type are similar to those studied in [10, 11], and we can consider this equation as the regular version of the one studied in [12].

The analytical aspect of this equation has been studied in detail by Guebbai et *al* [10]. Using the hypotheses

$$(H)\left\|\begin{array}{ll} (H1) & \dfrac{\partial K}{\partial t} \in C([a,b]^2 \times \mathbb{R}^2), \\ (H2) & \exists M \in \mathbb{R}_+^*,\ \forall t,s \in [a,b],\ \forall x,y \in \mathbb{R}, \\ & \max\left(|K(t,s,x,y)|, \left|\dfrac{\partial K}{\partial t}(t,s,x,y)\right|\right) \le M, \\ (H3) & \exists A, B, \overline{A}, \overline{B} \in \mathbb{R}_+,\ \forall x,y,\overline{x},\overline{y} \in \mathbb{R},\ \forall t,s \in [0,T], \\ & |K(t,s,x,y) - K(t,s,\overline{x},\overline{y})| \le A|x-\overline{x}| + B|y-\overline{y}|, \\ & \left|\dfrac{\partial K}{\partial t}(t,s,x,y) - \dfrac{\partial K}{\partial t}(t,s,\overline{x},\overline{y})\right| \le \overline{A}|x-\overline{x}| + \overline{B}|y-\overline{y}|, \\ (H4) & B < 1, \end{array}\right.$$

and the fact that

$$\forall t \in [a,b], \quad u'(t) = K(t,t,u(t),u'(t)) + \int_a^t \frac{\partial K}{\partial t}(t,s,u(s),u'(s))\,ds + f'(t), \tag{0.2}$$

the authors showed that the equation has a unique solution $u \in C^1(a,b)$.

In the same paper [10], they use Nyström method [1] to build a numerical approximation method represented in the following nonlinear system:
$\forall N \in \mathbb{N}^*,\ h = \dfrac{b-a}{N},\ t_j = a + jh,\ 0 \le j \le N,$

$$U_0 = f(a), \tag{0.3}$$
$$\overline{U}_0 = f'(a) + K(a,a,U_0,\overline{U}_0), \tag{0.4}$$
$$U_i = f(t_i) + h\sum_{j=0}^{i} w_j K(t_i,t_j,U_j,\overline{U}_j), \quad 1 \le i \le N, \tag{0.5}$$
$$\overline{U}_i = f'(t_i) + K(t_i,t_i,U_i,\bar{U}_i) + h\sum_{j=0}^{i} w_j \frac{\partial K}{\partial t}(t_i,t_j,U_j,\overline{U}_j),\ 1 \le i \le N, \tag{0.6}$$

where, U_i approaches $u(t_i)$, $\bar{U}_i$ approaches $u'(t_i)$ and $\{w_j\}_{j=0}^N$ are the weights of an appropriate numerical integration method, supposed to verify:

$$\exists W > 0,\ \forall N \ge 1,\ \max_{0 \le j \le N} |w_j| \le W.$$

But, to obtain the convergence of this numerical scheme, they needed that $A < 1$ which is a very restrictive condition.

Recently, Segni et *al* [11] take up the numerical study of the equation and propose a new numerical scheme that converges under hypotheses (H) only. They start by changing the variable u by the variable $v = u'$, which transforms (0.2) into

$$\begin{aligned}\forall t \in [a,b], \quad v(t) &= K\left(t,t,f(a) + \int_a^t v(s)\,ds, v(t)\right)\\ &+ \int_a^t \frac{\partial K}{\partial t}\left(t,s,f(a) + \int_a^s v(\tau)\,d\tau, v(s)\right) ds + f'(t).\end{aligned}$$

Once the Nyström method is applied, they get the following numerical scheme:

$$V_0 = f'(a) + K(a,a,f(a),V_0) \tag{0.7}$$
$$\begin{aligned}V_n &= f'(t_n) + K\left(t_n,t_n,f(a) + h\sum_{i=0}^{n} w_i V_i, V_n\right)\\ &+ h\sum_{i=0}^{n} w_i \frac{\partial K}{\partial t}\left(t_n,t_i,f(a) + h\sum_{j=0}^{i} w_i V_i, V_i\right),\ 1 \le n \le N,\end{aligned} \tag{0.8}$$

where, V_n approaches $v(t_n) = u'(t_n)$ and they use $f(a) + h\sum_{i=0}^{n} w_i V_i$ to approach

$$f(a) + \int_a^{t_n} v(s)\,ds = u(t_n).$$

This last method converges perfectly without needing the additional condition $A < 1$.

In this work, we propose a new numerical scheme completely different from those proposed in [10] and [11], where we apply the numerical derivative in conjunction with Nyström method. By the way, it is known that the numerical derivative is more usual for partial derivative equations (PDE), but here we give a new vision of numerical derivative to approach the integro-differential equations. On the other hand, as all researchers' work is oriented to construct the weakest conditions that ensure the convergence of their methods, our proposed method is also very efficient, since we need only the hypotheses (H) to confirm its convergence without adding the restrictive condition $A < 1$.

§ 1. Numerical approximation method

In this section, we build our numerical technique to approach the solution of equation (0.1) based on two usual numerical principles, the first is the Nyström method based on numerical integration [1,7] and the second is the finite difference method based on numerical derivative [7]. For $N \in \mathbb{N}^*$, we define the following subdivision:

$$h = \frac{b-a}{N}, \quad t_j = a + jh, \quad 0 \le j \le N.$$

The formula of the numerical integration is given by:

$$\forall \xi \in C\,(a,b)\,, \ \int_a^b \xi(t)\,dt \simeq \tilde{T}\,(\xi, h) = \frac{h}{2}\xi(t_0) + h\sum_{j=1}^{N} \xi(t_j),$$

which can be called modified trapezoidal method.

And we use the following numerical derivative, called backward difference, to approach the derivative of our solution

$$\xi'(t_j) \approx \frac{\xi(t_j) - \xi(t_j - h)}{h} = \frac{\xi(t_j) - \xi(t_{j-1})}{h}, \quad 1 \le j \le N.$$

Applying these two numerical principles to equations (0.1) and (0.2), we obtain the new system

$$U_0 = f(a), \tag{1.1}$$

$$V_0 = f'(a) + K(a, a, U_0, V_0), \tag{1.2}$$

$$U_i = f(t_i) + \frac{h}{2}K(t_i, t_0, U_0, V_0) + h\sum_{j=1}^{i} K\left(t_i, t_j, U_j, \frac{U_j - U_{j-1}}{h}\right), \quad 1 \le i \le N, \tag{1.3}$$

$$V_i = f'(t_i) + K(t_i, t_i, U_i, V_i) + \frac{h}{2}\frac{\partial K}{\partial t}(t_i, t_0, U_0, V_0) + h\sum_{j=1}^{i} \frac{\partial K}{\partial t}(t_i, t_j, U_j, V_j), \ 1 \le i \le N, \tag{1.4}$$

where, U_i approaches $u(t_i)$ and V_i approaches $u'(t_i)$ for $1 \le i \le N$.

Our interest is to study the system (1.1)–(1.4): We show that it is well defined and converges to the solution of the equation (0.1). Numerical examples are developed to show its effectiveness compared to systems (0.3)–(0.6) and (0.7)–(0.8).

1.1. System study

In the next theorem, we proof the existence and uniqueness of the solution of our new discrete system by using hypotheses (H) only.

Theorem 1. *For h sufficiently small and under the hypotheses (H), the system (1.1)–(1.4) has a unique solution.*

P r o o f. First, we have $U_0 = f(a)$, and by using Banach's fixed point, it is clear that the equation (1.2) has a unique solution V_0.

We define $\psi_i : \mathbb{R} \to \mathbb{R}$, $\phi_i : \mathbb{R} \to \mathbb{R}$ for $1 \le i \le N$ and $X \in \mathbb{R}$, by:

$$\begin{aligned}\psi_i(X) &= f(t_i) + hK\left(t_i, t_i, X, \frac{X - U_{i-1}}{h}\right) + h\sum_{j=1}^{i-1} K\left(t_i, t_j, U_j, \frac{U_j - U_{j-1}}{h}\right)\\ &+ \frac{h}{2}K(t_i, t_0, U_0, V_0),\\ \phi_i(X) &= f'(t_i) + K(t_i, t_i, U_i, X) + h\frac{\partial K}{\partial t}(t_i, t_i, U_i, X) + h\sum_{j=1}^{i-1}\frac{\partial K}{\partial t}(t_i, t_j, U_j, V_j)\\ &+ \frac{h}{2}\frac{\partial K}{\partial t}(t_i, t_0, U_0, V_0).\end{aligned}$$

We have for all $X, Y \in \mathbb{R}$,

$$\begin{aligned}|\psi_i(X) - \psi_i(Y)| &\le \left|hK\left(t_i, t_i, X, \frac{X - U_{i-1}}{h}\right) - hK\left(t_i, t_i, Y, \frac{Y - U_{i-1}}{h}\right)\right|\\ &\le \left(h\left(A + \frac{B}{h}\right)\right)|X - Y| = (hA + B)\,|X - Y|\,.\end{aligned}$$

Then, for h sufficiently small, $(hA + B) < 1$, so ψ_i is a contraction and using Banach's fixed point we obtain that (1.3) has a unique solution U_i.

In the same way, we get for all $X, Y \in \mathbb{R}$,

$$\begin{aligned}|\phi_i(X) - \phi_i(Y)| \le |K(t_i, t_i, U_i, X) - K(t_i, t_i, U_i, Y)| +\\ + h\left|\frac{\partial K}{\partial t}(t_i, t_i, U_i, X) - \frac{\partial K}{\partial t}(t_i, t_i, U_i, Y)\right| \le \left(B + h\overline{B}\right)|X - Y|\,.\end{aligned}$$

Also, for h sufficiently small, $\left(B + h\overline{B}\right) < 1$, so ϕ_i is a contraction and using Banach's fixed point we obtain that (1.4) has a unique solution V_i. □

1.2. Error analysis

Now, we show that the solution obtained from our new numerical system converges to the exact solution of the equation (0.1). As we have approached $u'(t_i)$ twice, by V_i and using its numerical backward difference derivative formula, for this, we define for $i \ge 1$,

$$\varepsilon_i^1 = u(t_i) - U_i, \quad \varepsilon_i^2 = u'(t_i) - V_i, \quad \varepsilon_i^3 = u'(t_i) - \frac{U_i - U_{i-1}}{h}, \quad \varepsilon_i = \left|\varepsilon_i^1\right| + \left|\varepsilon_i^2\right| + \left|\varepsilon_i^3\right|.$$

Now, it is said that the method is convergent if

$$\lim_{h\to 0}\max_{1\le i\le N}\left(\left|\varepsilon_i^1\right| + \left|\varepsilon_i^2\right|\right) = 0,$$

this has been demonstrated in [10, 11]. In this paper, we show that

$$\lim_{h\to 0}\left(\max_{1\le i\le N}\varepsilon_i\right) = 0.$$

Which makes our new method more precise and more efficient.

For technical reasons, we define local consistency errors, for $i \geq 1$ and $\xi \in C^1(a,b)$, by:

$$\begin{aligned}
\delta_K(h,t_i,\xi) &= \int_a^{t_i} K(t_i,s,\xi(s),\xi'(s))\,ds \\
&- \frac{h}{2}K(t_i,t_0,\xi(t_0),\xi'(t_0)) - h\sum_{j=1}^{i} K(t_i,t_j,\xi(t_j),\xi'(t_j)), \\
\delta_{K_t}(h,t_i,\xi) &= \int_a^{t_i} \frac{\partial K}{\partial t}(t_i,s,\xi(s),\xi'(s))\,ds \\
&- \frac{h}{2}\frac{\partial K}{\partial t}(t_i,t_0,\xi(t_0),\xi'(t_0)) - h\sum_{j=1}^{i} \frac{\partial K}{\partial t}(t_i,t_j,\xi(t_j),\xi'(t_j)),
\end{aligned}$$

It is clear that our numerical method is consistent with (0.1), i. e.

$$\forall \xi \in C^1(a,b), \quad \lim_{h\to 0}\big(\max_{1\leq i\leq N}\{|\delta_K(h,t_i,\xi)|, |\delta_{K_t}(h,t_i,\xi)|\}\big) = 0.$$

Indeed, it suffices to show that our modified trapezoidal method is convergent. We recall that the classical trapezoidal formula is given for $N \geq 1$ by

$$\forall \xi \in C(a,b), \quad T(\xi,h) = \frac{h}{2}\xi(t_0) + h\sum_{j=1}^{N-1}\xi(t_j) + \frac{h}{2}\xi(t_N),$$

and knowing that

$$\forall \xi \in C(a,b), \quad \lim_{h\to 0}\left|\int_a^b \xi(s)\,ds - T(\xi,h)\right| = 0,$$

we can conclude that for all $\xi \in C(a,b)$,

$$\lim_{h\to 0}\left|\int_a^b \xi(s)\,ds - \tilde{T}(\xi,h)\right| \leq \lim_{h\to 0}\left|\int_a^b \xi(s)\,ds - T(\xi,h)\right| + \lim_{h\to 0}\frac{h}{2}|\xi(b)| = 0.$$

Theorem 2.

$$\lim_{h\to 0}\big(\max_{1\leq i\leq N}\varepsilon_i\big) = 0.$$

P r o o f. For $i \geq 1$, we have:

$$\begin{aligned}
\varepsilon_i^1 &= \int_a^{t_i} K(t_i,s,u(s),u'(s))\,ds - \frac{h}{2}K(t_i,t_0,U_0,V_0) - h\sum_{j=1}^{i} K\left(t_i,t_j,U_j,\frac{U_j-U_{j-1}}{h}\right), \\
&= \int_a^{t_i} K(t_i,s,u(s),u'(s))\,ds - \frac{h}{2}K(t_i,t_0,U_0,V_0) - h\sum_{j=1}^{i} K(t_i,t_j,u(t_j),u'(t_j)) \\
&+ h\sum_{j=1}^{i} K(t_i,t_j,u(t_j),u'(t_j)) - h\sum_{j=1}^{i} K\left(t_i,t_j,U_j,\frac{U_j-U_{j-1}}{h}\right) \\
&+ \frac{h}{2}K(t_i,t_0,u(t_0),u'(t_0)) - \frac{h}{2}K(t_i,t_0,U_0,V_0).
\end{aligned}$$

By using the previous definition of local consistency error, the hypotheses (H) and the equalities $u(t_0) = U_0, u'(t_0) = V_0$ we get for h small enough and $i \geq 1$,

$$\begin{aligned} |\varepsilon_i^1| &\leq |\delta_K(h, t_i, u)| + hA \sum_{j=1}^{i} |\varepsilon_j^1| + hB \sum_{j=1}^{i} \left|\varepsilon_j^3\right|, \\ |\varepsilon_i^1| &\leq \frac{|\delta_K(h, t_i, u)|}{1 - hA} + \frac{hA}{1 - hA} \sum_{j=1}^{i-1} |\varepsilon_j^1| + \frac{hB}{1 - hA} \sum_{j=1}^{i-1} \left|\varepsilon_j^3\right| + \frac{hB}{1 - hA} \left|\varepsilon_i^3\right|. \end{aligned} \tag{1.5}$$

In the same way, we get for h small enough and $i \geq 1$,

$$\begin{aligned} |\varepsilon_i^2| &\leq \frac{|\delta_{K_t}(h, t_i, u)|}{1 - (B + h\overline{B})} + \frac{A + h\overline{A}}{1 - (B + h\overline{B})} |\varepsilon_i^1| + \frac{h\overline{A}}{1 - (B + h\overline{B})} \sum_{j=1}^{i-1} |\varepsilon_j^1| \\ &+ \frac{h\overline{B}}{1 - (B + h\overline{B})} \sum_{j=1}^{i-1} |\varepsilon_j^2|. \end{aligned} \tag{1.6}$$

Now to estimate the quantity $\left| u'(t_i) - \dfrac{U_i - U_{i-1}}{h} \right|$, we begin by:

$$\frac{U_i - U_{i-1}}{h} = \frac{f(t_i) - f(t_{i-1})}{h} + \frac{h}{2} \frac{K(t_i, t_0, U_0, V_0) - K(t_{i-1}, t_0, U_0, V_0)}{h} +$$
$$+ K\left(t_i, t_i, U_i, \frac{U_i - U_{i-1}}{h}\right) + h \sum_{j=1}^{i-1} \frac{K\left(t_i, t_j, U_j, \frac{U_j - U_{j-1}}{h}\right) - K\left(t_{i-1}, t_j, U_j, \frac{U_j - U_{j-1}}{h}\right)}{h}.$$

As the function $K(t, s, x, y)$ is differentiable on the parameter t, then for h small enough, we can use the following approximation:

$$\begin{aligned} \frac{U_i - U_{i-1}}{h} &\approx f'(t_i) + \frac{h}{2} \frac{\partial K}{\partial t}(t_i, t_0, U_0, V_0) + K\left(t_i, t_i, U_i, \frac{U_i - U_{i-1}}{h}\right) \\ &+ h \sum_{j=1}^{i-1} \frac{\partial K}{\partial t}\left(t_i, t_j, U_j, \frac{U_j - U_{j-1}}{h}\right). \end{aligned}$$

On the other hand, we have:

$$\begin{aligned} \varepsilon_i^3 &= \int_a^{t_i} \frac{\partial K}{\partial t}(t_i, s, u(s), u'(s))\, ds - \frac{h}{2} \frac{\partial K}{\partial t}(t_i, t_0, u(t_0), u'(t_0)) - h \sum_{j=1}^{i} \frac{\partial K}{\partial t}(t_i, t_j, u(t_j), u'(t_j)) \\ &+ h \sum_{j=1}^{i-1} \frac{\partial K}{\partial t}(t_i, t_j, u(t_j), u'(t_j)) - h \sum_{j=1}^{i-1} \frac{\partial K}{\partial t}\left(t_i, t_j, U_j, \frac{U_j - U_{j-1}}{h}\right) \\ &+ \frac{h}{2} \frac{\partial K}{\partial t}(t_i, t_0, u(t_0), u'(t_0)) - \frac{h}{2} \frac{\partial K}{\partial t}(t_i, t_0, U_0, V_0) \\ &+ h \frac{\partial K}{\partial t}(t_i, t_i, u(t_i), u'(t_i)) + K(t_i, t_i, u(t_i), u'(t_i)) - K\left(t_i, t_i, U_i, \frac{U_i - U_{i-1}}{h}\right), \\ \left|\varepsilon_i^3\right| &\leq \frac{|\delta_{K_t}(h, t_i, u)|}{1 - B} + \frac{h\overline{A}}{1 - B} \sum_{j=1}^{i-1} \left|\varepsilon_j^1\right| + \frac{h\overline{B}}{1 - B} \sum_{j=1}^{i-1} \left|\varepsilon_j^3\right| + \frac{M}{1 - B} h + \frac{A}{1 - B} \left|\varepsilon_i^1\right|. \end{aligned} \tag{1.7}$$

Replacing (1.7) in (1.5), to get for h small enough and $i \geq 1$,

$$\begin{aligned} |\varepsilon_i^1| &\leq \alpha_1 \frac{|\delta_K(h,t_i,u)|}{1-hA} + \alpha_1 \frac{B|\delta_{K_t}(h,t_i,u)|}{(1-B)(1-hA)} + \alpha_1 \left(\frac{h^2 B\overline{A}}{(1-B)(1-hA)} + \frac{hA}{1-hA} \right) \sum_{j=1}^{i-1} |\varepsilon_j^1| \\ &+ \alpha_1 \left(\frac{h^2 B\overline{B}}{(1-B)(1-hA)} + \frac{hB}{1-hA} \right) \sum_{j=1}^{i-1} |\varepsilon_j^3| + \frac{\alpha_1 MB}{(1-B)(1-hA)} h^2, \qquad (1.8) \\ \alpha_1 &= \left(1 - \frac{hAB}{(1-B)(1-hA)} \right)^{-1}, \end{aligned}$$

and (1.8) in (1.6), to get for h small enough and $i \geq 1$,

$$\begin{aligned} |\varepsilon_i^2| &\leq \alpha_1\alpha_2 \frac{|\delta_K(h,t_i,u)|}{1-hA} + \left(\frac{1}{1-(B+h\overline{B})} + \frac{\alpha_1\alpha_2 B}{(1-B)(1-hA)} \right) |\delta_{K_t}(h,t_i,u)| \\ &+ \left(\frac{h\overline{A}}{1-(B+h\overline{B})} + \alpha_1\alpha_2 \left(\frac{h^2 B\overline{A}}{(1-B)(1-hA)} + \frac{hA}{1-hA} \right) \right) \sum_{j=1}^{i-1} |\varepsilon_j^1| \\ &+ \left(\frac{h\overline{B}}{1-(B+h\overline{B})} \right) \sum_{j=1}^{i-1} |\varepsilon_j^2| + \alpha_1\alpha_2 \left(\frac{h^2 B\overline{B}}{(1-B)(1-hA)} + \frac{hB}{1-hA} \right) \sum_{j=1}^{i-1} |\varepsilon_j^3| \\ &+ \frac{\alpha_1\alpha_2 MB}{(1-B)(1-hA)} h^2, \qquad (1.9) \\ \alpha_2 &= \frac{A+h\overline{A}}{1-(B+h\overline{B})}. \end{aligned}$$

Finally, we replace (1.8) in (1.7), to obtain for h small enough and $i \geq 1$,

$$\begin{aligned} \left|\varepsilon_i^3\right| &\leq \alpha_1\alpha_3 \frac{|\delta_K(h,t_i,u)|}{1-hA} + \left(\frac{1}{1-B} + \frac{\alpha_1\alpha_3 B}{(1-B)(1-hA)} \right) |\delta_{K_t}(h,t_i,u)| \\ &+ \left(\frac{h\overline{A}}{1-B} + \alpha_1\alpha_3 \left(\frac{h^2 B\overline{A}}{(1-B)(1-hA)} + \frac{hA}{1-hA} \right) \right) \sum_{j=1}^{i-1} \left|\varepsilon_j^1\right| \\ &+ \left(\frac{h\overline{B}}{1-B} + \alpha_1\alpha_3 \left(\frac{h^2 B\overline{B}}{(1-B)(1-hA)} + \frac{hB}{1-hA} \right) \right) \sum_{j=1}^{i-1} \left|\varepsilon_j^3\right| \\ &+ \frac{\alpha_1\alpha_3 MB}{(1-B)(1-hA)} h^2 + \frac{M}{1-B} h, \qquad (1.10) \\ \alpha_3 &= \frac{A}{1-B}. \end{aligned}$$

Now according to equations (1.8), (1.9) and (1.10) we obtain:

$$\varepsilon_i \leq \frac{\alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_3}{1-hA} |\delta_K(h,t_i,u)|$$

$$+ \left(\frac{1}{1-B} + \frac{1}{1-(B+h\overline{B})} + \frac{(\alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_3)B}{(1-B)(1-hA)} \right) |\delta_{K_t}(h,t_i,u)|$$

$$+ \left(\frac{h\overline{A}}{1-B} + \frac{h\overline{A}}{1-(B+h\overline{B})} + \frac{h^2 B\overline{A}(\alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_3)}{(1-B)(1-hA)} \right.$$

$$+ \frac{hA(\alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_3)}{1 - hA}\Bigg) \sum_{j=1}^{i-1} |\varepsilon_j^1| + \left(\frac{h\overline{B}}{1 - (B + h\overline{B})}\right) \sum_{j=1}^{i-1} |\varepsilon_j^2|$$

$$+ \left(\frac{h\overline{B}}{1 - B} + \frac{h^2 B\overline{B}(\alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_3)}{(1 - B)\,(1 - hA)} + \frac{hB(\alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_3)}{1 - hA}\right) \sum_{j=1}^{i-1} |\varepsilon_j^3|$$

$$+ \frac{MB(\alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_3)}{(1 - B)\,(1 - hA)} h^2 + \frac{M}{1 - B} h.$$

Then,

$$\begin{aligned}
\varepsilon_i \quad &\leq \quad h\gamma \sum_{j=1}^{i-1} \varepsilon_j + \left(\frac{1}{1 - B} + \frac{1}{1 - (B + h\overline{B})} + \frac{(\alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_3)B}{(1 - B)\,(1 - hA)}\right) |\delta_{K_t}(h, t_i, u)| \\
&+ \quad \frac{\alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_3}{1 - hA} |\delta_K(h, t_i, u)| + \frac{MB(\alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_3)}{(1 - B)\,(1 - hA)} h^2 + \frac{M}{1 - B} h,
\end{aligned}$$

where

$$\begin{aligned}
\gamma := \max \Bigg(&\frac{\overline{A}}{1 - B} + \frac{\overline{A}}{1 - (B + h\overline{B})} + \frac{hB\overline{A}(\alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_3)}{(1 - B)\,(1 - hA)} + \frac{A(\alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_3)}{1 - hA}, \\
&\frac{\overline{B}}{1 - (B + h\overline{B})}, \frac{\overline{B}}{1 - B} + \frac{hB\overline{B}(\alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_3)}{(1 - B)\,(1 - hA)} + \frac{B(\alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_3)}{1 - hA} \Bigg).
\end{aligned}$$

Applying Theorem 7.1 from [1], we get

$$\varepsilon_i \leq (1 + h\gamma)^{i-1} \delta,$$

where

$$\begin{aligned}
\delta := \max_{1 \leq j \leq i} \Bigg(&\left(\frac{1}{1 - B} + \frac{1}{1 - (B + h\overline{B})} + \frac{(\alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_3)B}{(1 - B)\,(1 - hA)}\right) |\delta_{K_t}(h, t_j, u)|\,, \\
&\frac{\alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_3}{1 - hA} |\delta_K(h, t_j, u)|, \frac{MB(\alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_3)}{(1 - B)\,(1 - hA)} h^2, \frac{M}{1 - B} h \Bigg).
\end{aligned}$$

On the other hand, we have:

$$(1 + h\gamma)^{i-1} \leq \left(1 + \frac{(b - a)\gamma}{N}\right)^N,$$

and

$$\lim_{N \to +\infty} \left(1 + \frac{(b - a)\gamma}{N}\right)^N < +\infty.$$

Then, $\exists \theta > 0$ such that

$$\forall N \in \mathbb{N}, \ \max_{1 \leq i \leq N} (1 + h\gamma)^{i-1} \leq \theta$$

which completes the demonstration. □

Table 1: Example 1: Numerical results with $\alpha = e$

N	E_1	E_2	E_3
200	2.64E-4	3.01E-3	2.97E-3
300	1.76E-4	2.10E-3	1.98E-3
500	1.05E-4	1.05E-3	1.19E-3
1000	5.29E-5	6.05E-4	5.95E-4
1500	3.53E-5	4.21E-4	3.96E-4

§ 2. Numerical result

In this section, to show the effectiveness of our method, we build two numerical examples. As discretization, we use the trapezoidal method, and we mention that the terms U_i and V_i are not calculated exactly, but they are approached using the Banach's iteration method from our systems (1.3) and (1.4), with the following stopping condition:

$$\|X_{new} - X_{old}\| \leq 10^{-7}$$

with a number of iterations that don't exceed 1000. To compare between the methods, we denote the error using our method (1.1)–(1.4) by:

$$E_1 = \max_{0\leq i\leq N} \left\{ |u(t_i) - U_i| + |u'(t_i) - V_i| \right\},$$

and by E_2, E_3 the errors obtained when we use the methods described in [10, 11] respectively.

Example 1. In the first example, we consider the following equation:

$$u(t) = \int_0^t \frac{se^{-\alpha u^2(s)}}{2 + t + u'(s)^2} ds + f(t), \quad \alpha > 0, \quad t \in [0, 1],$$

where

$$f(t) = t - \frac{1}{3+t}\left(\frac{1}{2\alpha} - \frac{1}{2\alpha}e^{-\alpha t^2}\right).$$

The exact solution is given by $u(t) = t$. The kernel of this example satisfies (H) with the parameter:

$$A = \sqrt{\frac{\alpha}{2e}}.$$

We notice that when $\alpha = e$, which ensures $A < 1$, our method is better than those developed in [10] and [11]. But, when we set $\alpha = 8e$ which gives $A > 1$, the method (0.3)–(0.6) diverges, unlike our new method which is faster than (0.7)–(0.8).

Example 2. In the second example, we consider the following equation:

$$u(t) = \frac{1}{2\pi}\int_0^t \cos\left(|t-2s|\,(t-2s) + e^s - \alpha u(s) + u'(s)\right) ds + f(t), \quad \alpha > 0, \quad t \in [0, \frac{\pi}{2}].$$

Table 2: Example 1: Numerical results with $\alpha = 8e$

N	E_1	E_2	E_3
200	1.07E-4	6.22E+1	3.04E-3
300	7.15E-5	6.34E+1	2.03E-3
500	4.30E-5	6.33E+1	1.22E-3
1000	2.15E-5	6.42E+1	6.04E-4
1500	1.43E-5	6.36E+1	4.06E-4

If we take $\alpha = 1$, and

$$\begin{aligned} f(t) &= \frac{1}{8\sqrt{\pi}}\left(\sin\left(\frac{t^2}{2}\right)\left(\mathrm{FreS}\left(\frac{3}{\sqrt{\pi}}t\right) - \mathrm{FreS}\left(\frac{1}{\sqrt{\pi}}t\right)\right)\right. \\ &- \left.\cos\left(\frac{t^2}{2}\right)\left(\mathrm{FreC}\left(\frac{3}{\sqrt{\pi}}t\right) - \mathrm{FreC}\left(\frac{1}{\sqrt{\pi}}t\right)\right)\right) \\ &+ \frac{1}{4\pi t}\sin(t^2) + te^t - 4t^2 - 8t - 8, \end{aligned}$$

we get

$$u(t) = 4t^2 + 8t + 8 - te^t.$$

We recall that FreC and FreS denote the Fresnel cosine integral function and the Fresnel sine integral function respectively. The kernel of this example satisfies (H) with the parameter:

$$A = \frac{\alpha}{2\pi} = \frac{1}{2\pi}.$$

But if we take $\alpha = 10$, and

$$\begin{aligned} f(t) &= \frac{1}{8\sqrt{\pi}}\left(\sin\left(\frac{t^2}{2}\right)\left(\mathrm{FreS}\left(\frac{3}{\sqrt{\pi}}t\right) - \mathrm{FreS}\left(\frac{1}{\sqrt{\pi}}t\right)\right)\right. \\ &- \left.\cos\left(\frac{t^2}{2}\right)\left(\mathrm{FreC}\left(\frac{3}{\sqrt{\pi}}t\right) - \mathrm{FreC}\left(\frac{1}{\sqrt{\pi}}t\right)\right)\right) \\ &+ \frac{1}{4\pi t}\sin(t^2) - \frac{1}{9}e^t - \left(\frac{2}{5}t^2 + \frac{2}{25}t + \frac{1}{125}\right), \end{aligned}$$

we get

$$u(t) = \frac{1}{9}e^t + \left(\frac{2}{5}t^2 + \frac{2}{25}t + \frac{1}{125}\right).$$

In this case,

$$A = \frac{5}{\pi}.$$

We obtain the same behavior above of the three methods when we vary the parameter A.

Table 3: Example 2: Numerical results with $\alpha = 1$

N	E_1	E_2	E_3
200	3.46E-4	7.55E-4	7.95E-4
300	1.53E-4	3.21E-4	3.54E-4
500	1.23E-4	1.09E-4	1.25E-4
1000	4.32E-5	9.62E-5	8.25E-5
1500	2.23E-5	6.15E-5	5.42E-5

Table 4: Example 2: Numerical results with $\alpha = 10$

N	E_1	E_2	E_3
200	5.12E-4	7.63E+1	8.05E-4
300	3.09E-4	7.57E+1	3.33E-4
500	9.36E-5	7.48E+1	1.12E-4
1000	3.52E-5	7.34E+1	8.52E-5
1500	1.98E-5	7.28E+1	5.51E-5

§ 3. Conclusion

In this paper we have built a new numerical method to approach the solution of an integro-differential nonlinear Volterra equation, based on the numerical backward difference derivative.

In practice the numerical derivative is used just for partial deferential equations (PDE), but by applying this numerical technique to our integro-differential equation, we have built a simple and clear approximate system, which is more efficient than those studied in [10] and [11].

As perspectives, according to the simplicity of our method, we will study how to apply this numerical idea for other types of integro-differential equation, as Fredholm equations [13] or equations with weakly singular kernel [12], also the integro-differential equation with higher order [9].

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Разностная производная для интегро-дифференциального нелинейного уравнения Вольтерра

Ключевые слова: интегро-дифференциальное уравнение Вольтерра, нелинейное уравнение, неподвижная точка, численная производная, метод Нистрёма.

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В этой статье мы предлагаем новый метод численной аппроксимации для решения единственного решения нелинейного интегро-дифференциального уравнения Вольтерра. Нас интересует особая форма этого уравнения, в которой производная искомого решения появляется под знаком интеграла нелинейным образом. Наше видение основано на двух разных подходах: мы используем метод Нистрёма для преобразования интеграла в конечную сумму, используя формулу численного интегрирования, затем мы используем метод численной обратной разностной производной для приближения к производной нашего решения. Такое сопоставление двух разных методов, первого результата численной обработки интегральных уравнений и второго результата численной обработки дифференциальных уравнений, дает новую нелинейную систему для приближения к решению нашего уравнения. Мы показываем, что система имеет единственное решение и что это численное решение идеально сходится к нашему решению. Раздел посвящен численным тестам, в которых мы показываем эффективность нашего нового видения по сравнению с двумя методами, основанными только на численном интегрировании.

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