MSC: 47H04, 47H10, 34A60

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ON FIXED POINTS OF MULTI-VALUED MAPS IN METRIC SPACES AND DIFFERENTIAL INCLUSIONS[1]

A generalization of the Nadler fixed point theorem for multi-valued maps acting in metric spaces is proposed. The obtained result allows to study the existence of fixed points for multi-valued maps that have as images any arbitrary sets of the corresponding metric space and are not necessarily contracting, or even continuous, with respect to the Hausdorff metric. The mentioned result can be used for investigating differential and functional-differential equations with discontinuities and inclusions generated by multi-valued maps with arbitrary images. In the second part of the paper, as an application, conditions of existence and continuation of solutions to the Cauchy problem for a differential inclusion with noncompact in $\mathbb{R}^n$ right-hand side are derived.

Keywords: multi-valued map, fixed point, differential inclusion.

Introduction

Let $\mathbb{R}^n$ be the n-dimensional real space with the norm $|\cdot|$. Using the standard notation, by $C^n[a,b]$ we mean the set of all continuous functions $x:[a,b]\to\mathbb{R}^n$, by $AC^n[a,b]$ the set of absolutely continuous, by $L^n[a,b]$ the set of Lebesgue integrable, and by $L^n_\infty[a,b]$ the set of essentially bounded $x:[a,b]\to\mathbb{R}^n$. We omit here the index n in the case of $n=1$. Given a metric space (X,ϱ_X), we use the following notation: $\overline{M}=X\setminus M$ is the complement of a set $M\subset X$; $\operatorname{int}M$ is the interior of M; $S_X(x_0,r)\doteq\{x\in X:\varrho_X(x,x_0)=r\}$, $B^o_X(x_0,r)\doteq\{x\in X:\varrho_X(x,x_0)<r\}$, $B_X(x_0,r)\doteq\{x\in X:\varrho_X(x,x_0)\leqslant r\}$ are, respectively, a sphere, an open, and a closed ball of radius $r>0$ centered at x_0 in the space X; $\varrho_X(x,M)\doteq\inf\limits_{y\in M}\varrho_X(x,y)$ is the distance from a point x to a set M in X; $d_X(M_1,M_2)\doteq\sup\limits_{x\in M_1}\varrho_X(x,M_2)$ is the Hausdorff semideviation from a set M_1 to M_2; $\operatorname{dist}_X(M_1,M_2)=\max\big\{d_X(M_1,M_2);d_X(M_2,M_1)\big\}$ is the Hausdorff distance between sets M_1 and M_2.

Let $\operatorname{clos}(X)$, $\operatorname{clbd}(X)$, and $\operatorname{comp}(X)$ stand for the spaces of *nonempty closed*, *nonempty closed bounded,* and *nonempty compact* subsets of X, respectively. These spaces we endow with the metric defined by the Hausdorff distance dist_X. In the case of closed unbounded sets this metric is usually called the *extended Hausdorff metric* since it may have infinite values.

While studying the problems of qualitative theory of differential inclusions and controlled systems [1–3] there has appeared the necessity of considering in the space $\operatorname{clos}(\mathbb{R}^n)$ a metric that would have finite values only, guarantee the completeness of $\operatorname{clos}(\mathbb{R}^n)$, and be such that convergence of a sequence $\{H_i\}_{i=1}^\infty\subset\operatorname{clos}(\mathbb{R}^n)$ with respect to this metric would mean convergence (in the Hausdorff metric), for every $r>0$, of the sequence of sets $H^i_r\in\operatorname{clbd}(\mathbb{R}^n)$ such that, for every i, $H^i_r\subset H^i$, $\bigcup\limits_{r>0}H^i_r=H^i$, and $H^i_{r_1}\subset H^i_{r_2}$ when $r_1<r_2$. It was managed to build such a metric for the space of nonempty closed convex subsets of $\mathbb{R}^n$ (see [3]). In the work [4], there was presented a metric in the whole of $\operatorname{clos}(\mathbb{R}^n)$; with respect to this metric, the convergence of a sequence $\{H_i\}_{i=1}^\infty$ is equivalent to convergence (in the Hausdorff metric), for every $r>0$, of the sequence of sets $H^i_r\in\operatorname{clbd}(\mathbb{R}^n)$ such that $H^i_r\cap B_{\mathbb{R}^n}(0,r)=H^i\cap B_{\mathbb{R}^n}(0,r)$. In [5], a similar metric was obtained in $\operatorname{clos}(X)$ with X an arbitrary metric space. In construction of the metric the following obvious properties of the Hausdorff

[1]The work is supported by the Russian Foundation for Basic Research (projects no. 11–01–00645 and 11–01–00626).

distance were used: for any $F, G \in \operatorname{clos}(X)$, $\theta \in X$, and $r \geqslant 0$ one has

$$\operatorname{dist}_X \left(F \cup \overline{B^o_X(\theta, r)}, G \cup \overline{B^o_X(\theta, r)}\right) < \infty, \tag{1}$$

$$\operatorname{dist}_X \left(F \cup \overline{B^o_X(\theta, r)}, G \cup \overline{B^o_X(\theta, r)}\right) \leqslant \operatorname{dist}_X(F, G). \tag{2}$$

Relations (1), (2) can be also utilized for «improving» topological properties of multi-valued maps. Let us have a map $X \ni x \mapsto \Phi(x) \in \operatorname{clos}(X)$ which is not continuous (upper semicontinuous or lower semicontinuous) in the Hausdorff metric, moreover, for any $x_1, x_2 \in X$, $x_1 \neq x_2$, there can be held $\operatorname{dist}_X \big(\Phi(x_1), \Phi(x_2)\big) = \infty$. Build a new «improved» map $\widetilde{\Phi} : X \to \operatorname{clos}(X)$ defined by

$$\widetilde{\Phi}(x) \doteq \Phi(x) \cup \overline{B^o_X(\theta, r)}, \tag{3}$$

where $\theta \in X$, $r > 0$. This map, according to (1) and (2), can posses already some continuity properties or even be Lipschitz. In such a case, there appears a possibility to investigate the corresponding equations and inclusions using the known methods. For example, if there can be justified the existence of a fixed point $\bar{x}$ for the map $\widetilde{\Phi}$, and if one is able to prove that $\bar{x} \notin \overline{B^o_X(\theta, r)}$, then $\bar{x}$ will also be a fixed point for the initial «bad» map Φ.

This idea was implemented in [5] in order to get a result on existence of fixed points for a multi-valued map not satisfying the assumptions of the Nadler fixed point theorem or other known fixed point principles. In this work we present more general statement based on a rather universal way of constructing the map $\widetilde{\Phi}$. The method turns out to be efficient when the images of the original map Φ are arbitrary subsets of the space X, and defining Φ by equality (3) represents one of its particular cases. The mentioned result we apply then for studying the solvability of ordinary differential inclusions generated by multi-valued maps with noncompact in $\mathbb{R}^n$ right-hand sides. Among numerous situations in which the conditions of the classical existence theorems for differential inclusions do not hold, one usually chooses to investigate the case when the corresponding multi-valued map has unbounded closed images. In many works the Nadler principle or its generalizations are used in order to prove the existence of solutions for such inclusions, and one has to assume that the corresponding multi-valued map with closed images is Lipschitz with respect to the extended Hausdorff metric (see. e.g., [6,7]). Using the fixed point theorem derived in this paper allows to relax considerably these requirements and to enlarge the class of inclusions under discussion.

§ 2. On fixed points of multi-valued maps

Recall some facts about multi-valued maps acting in metric spaces.

Given metric spaces (Ω, ϱ_Ω), (X, ϱ_X), we use the notation $\Phi : \Omega \multimap X$ to indicate a multi-valued map $\Omega \ni \omega \mapsto \Phi(\omega) \subset X$. We write $\Phi : \Omega \to \operatorname{clos}(X)$ if it is known that for every $\omega \in \Omega$ the set $\Phi(\omega)$ is nonempty and closed, $\Phi : \Omega \to \operatorname{clbd}(X)$ if $\Phi(\omega)$ is nonempty bounded and closed, and $\Phi : \Omega \to \operatorname{comp}(X)$ in the case of compact images. Let $q \geqslant 0$ and $\Omega_0 \subset \Omega$. A map $\Phi : \Omega \multimap X$ is called *q-Lipschitz* (or *Lipschitz with constant q*) *on the set* Ω_0 if the inequality

$$\operatorname{dist}_X \big(\Phi(\omega_1), \Phi(\omega_2)\big) \leqslant q\varrho_\Omega(\omega_1, \omega_2) \tag{4}$$

holds for every $\omega_1, \omega_2 \in \Omega_0$. In the case of $\Omega = X$ and $q < 1$, a map Φ satisfying (4) is called *contracting* or *q-contracting on the set* $\Omega_0 \subset X$. An element $x \in X$ such that $x \in \Phi(x)$ is called *a fixed point of* $\Phi : X \multimap X$.

In the following statement a technique analogous to (3) is used to «improve» the properties of multi-valued maps that are not contracting; it gives an opportunity to apply the fixed point principles and, in many cases, to verify the existence of fixed points.

Theorem 1. *Let X be a complete metric space and $\Phi : X \multimap X$. Assume there exist maps $\mathcal{A} : X \multimap X$, $\mathcal{B} : X \multimap X$ such that for the map*

$$X \ni x \mapsto \widetilde{\Phi}(x) \doteq \big(\Phi(x) \cap \mathcal{A}(x)\big) \cup \mathcal{B}(x) \subset X \tag{5}$$

there are $x_0 \in X$, $q \in (0,1)$, *and* $r_0 > 0$ *satisfying the following conditions:*

1) *the set* $\widetilde{\Phi}(x)$ *is nonempty and closed in* X *for every* $x \in B^o_X(x_0, r_0)$;

2) *the map* $\widetilde{\Phi}$ *is* q*-contracting on the ball* $B^o_X(x_0, r_0)$;

3) $(1-q)^{-1}\varrho_X\big(x_0, \widetilde{\Phi}(x_0)\big) < r_0$.

If for every $x \in B^o_X(x_0, r_0)$, *the set* $\mathcal{B}(x)$ *is either empty or such that*

$$\varrho_X\big(x_0, \mathcal{B}(x)\big) > r_0, \tag{6}$$

then for every r *satisfying the inequality*

$$(1-q)^{-1}\varrho_X\big(x_0, \widetilde{\Phi}(x_0)\big) < r \leqslant r_0, \tag{7}$$

the map Φ *has a fixed point* $\bar{x} \in B^o_X(x_0, r)$.

P r o o f. According to the fixed point principle [8, p. 42], for any r satisfying inequality (7), there exists a point $\bar{x} \in X$ such that

$$\bar{x} \in \widetilde{\Phi}(\bar{x}), \quad \varrho_X(x_0, \bar{x}) < r.$$

If $\mathcal{B}(\bar{x}) = \varnothing$, then, obviously, $\bar{x} \in \Phi(\bar{x}) \cap \mathcal{A}(\bar{x})$. In the case when $\mathcal{B}(\bar{x}) \neq \varnothing$, for $x = \bar{x}$, inequality (6) takes place, from which it follows that $\varrho_X(x_0, \bar{x}) < \varrho_X\big(x_0, \mathcal{B}(\bar{x})\big)$, and hence $\bar{x} \notin \mathcal{B}(\bar{x})$. So, in this situation, the inclusion $\bar{x} \in \Phi(\bar{x}) \cap \mathcal{A}(\bar{x})$ is also true. Thus, $\bar{x}$ is a fixed point of the map Φ. □

Remark 1. The hypothesis of $\widetilde{\Phi}$ to be q-contracting can be relaxed; it suffices to note that in the proof of the fixed point principle [8, p. 42] used in Theorem 1, the property of q-contraction is needed only for the sequence of iterations, so one can require that the inequality

$$\mathrm{dist}_X\big(\widetilde{\Phi}(x_1), \widetilde{\Phi}(x_2)\big) \leqslant q\varrho_X(x_1, x_2)$$

takes place not for all elements $x_1, x_2 \in B^o_X(x_0, r_0)$, but for those connected by the relation $x_2 \in \widetilde{\Phi}(x_1)$. This observation has been used before, for example, in [9].

In order to use efficiently the theorem proved above one needs to know how to choose properly the maps $\mathcal{A}$, $\mathcal{B}$ playing a crucial role in construction (5) of the map $\widetilde{\Phi}$. It is obvious that the map $\widetilde{\Phi}$ deserves to be called «improving» for the map Φ if for any x_1, x_2

$$\mathrm{dist}_X\big(\widetilde{\Phi}(x_1), \widetilde{\Phi}(x_2)\big) \leqslant \mathrm{dist}_X\big(\Phi(x_1), \Phi(x_2)\big). \tag{8}$$

We consider some recipes of defining the maps $\mathcal{A}$, $\mathcal{B}$.

First of all, one may set $\mathcal{A}(x) \equiv X$. In this case inequality (8) holds if the map $\mathcal{B} : X \to \mathrm{clos}(X)$ is such that

$$\mathrm{dist}_X\big(\mathcal{B}(x_1), \mathcal{B}(x_2)\big) \leqslant \mathrm{dist}_X\big(\Phi(x_1), \Phi(x_2)\big)$$

for all x_1, x_2. The latter obviously takes place when $\mathcal{B}(x) \equiv \mathcal{B} = \mathrm{const}$. In [5], the complement of an open ball, i.e., the set $\overline{B^o_X(\theta, r)}$, $\theta \in X$, $r > 0$, was taken as $\mathcal{B}$. Using such a design of the map $\widetilde{\Phi}$ one can get, for example, the following analog of the fixed point theorem obtained in [9, p. 75].

Corollary 1. *Let* X *be a complete metric space,* $\Phi : X \multimap X$, *and let there exist* $x_0 \in X$, $q \in (0,1)$, *and* $r_0 > 0$ *such that:*

1) *the set* $\Phi(x) \cap B_X(x_0, r_0)$ *is nonempty and closed for every* $x \in B^o_X(x_0, r_0)$;

2) *for any* $x_1, x_2 \in B^o_X(x_0, r_0)$ *there holds the inequality*

$$d_X\big(\Phi(x_1) \cap B_X(x_0, r_0), \Phi(x_2)\big) \leqslant q\varrho_X(x_1, x_2); \tag{9}$$

3) $(1-q)^{-1}\varrho_X\big(x_0, F(x_0)\big) < r_0$.

Then for every r *satisfying the inequality* $(1-q)^{-1}\varrho_X\big(x_0, \Phi(x_0)\big) < r \leqslant r_0$, *the map* Φ *has a fixed point* $\bar{x} \in B^o_X(x_0, r)$.

P r o o f follows directly from Theorem 1 if in definition (5) of the map $\widetilde{\Phi} : X \multimap X$ we set $\mathcal{A}(x) \equiv X$, $\mathcal{B}(x) \equiv \overline{B^o_X(x_0, r_0)}$. Indeed, in this case, for any $x_1, x_2 \in B^o_X(x_0, r_0)$, there holds the estimate

$$d_X\big(\Phi(x_1) \cap B_X(x_0, r_0), \Phi(x_2)\big) \geqslant$$
$$\geqslant d_X\big((\Phi(x_1) \cap B_X(x_0, r_0)) \cup \overline{B^o_X(x_0, r_0)},\ \Phi(x_2) \cup \overline{B^o_X(x_0, r_0)}\big) = d_X\big(\widetilde{\Phi}(x_1), \widetilde{\Phi}(x_2)\big).$$

Then, from inequality (9), we get

$$\mathrm{dist}_X\big(\widetilde{\Phi}(x_1), \widetilde{\Phi}(x_2)\big) = \max\Big\{d_X\big(\widetilde{\Phi}(x_1), \widetilde{\Phi}(x_2)\big);\ d_X\big(\widetilde{\Phi}(x_2), \widetilde{\Phi}(x_1)\big)\Big\} \leqslant q\varrho_X(x_1, x_2),$$

so condition 2) of Theorem 1 is satisfied. All the other requirements of Theorem 1 are also fulfilled, it follows directly from the hypotheses of the corollary. □

We continue to discuss the ways of choosing the maps $\mathcal{A}$, $\mathcal{B}$ and consider the situation when X is a linear normed space. In this case, it is convenient to define $\mathcal{A}(x) \equiv B_X(\theta, r)$, $\mathcal{B}(x) \equiv S_X(\theta, r)$, where $\theta \in X$, $r > 0$; then the map

$$\widetilde{\Phi} : X \to \mathrm{clos}(X), \quad \widetilde{\Phi}(x) \doteq \big(\Phi(x) \cap B_X(\theta, r)\big) \cup S_X(\theta, r) \tag{10}$$

will satisfy inequality (8). This construction was used in [4] for the space $\mathbb{R}^n$. In the paper [5], there is given an example of the map $\Phi : \mathbb{R} \to \mathrm{clos}(\mathbb{R})$ such that $\mathrm{dist}_{\mathbb{R}}\big(\Phi(x_1), \Phi(x_2)\big) = \infty$ for any $x_1, x_2 \in \mathbb{R}$, $x_1 \neq x_2$, but the corresponding map (10) is contracting; based on this property, there is proved the existence of a fixed point for the initial map Φ.

Here we give an example illustrating how Theorem 1 can be applied to studying functional equations and inclusions; we prove the existence of a fixed point for a map which is neither continuous, no bounded, and even not defined on the whole of the metric space considered.

Example 1. Let us have measurable functions $h : [0,1] \to [0,1]$, $g : [0,1] \to \mathbb{R}_+$. Consider the equation

$$|x(t)| = -\ln|\cos x(h(t))| + g(t), \quad t \in [0,1], \tag{11}$$

with respect to the unknown function $x \in L_\infty[0,1]$. Assume that for some measurable set $T \subset [0,1]$ there holds

$$\mu(T) = 0 \quad \Rightarrow \quad \mu(h^{-1}(T)) = 0 \tag{12}$$

(here μ stands for the Lebesgue measure). According to [10, p. 706], this condition is necessary and sufficient for the composition $x(h(\cdot))$ to be measurable for any measurable function $x(\cdot)$. We prove that, *if*

$$\operatorname*{ess\,sup}_{t\in[0,1]} \big|\cos g(h(t))\big|^{-1} < \sqrt[4]{5/4}, \tag{13}$$

then equation (11) *has a solution.*

We will use the usual metric in the space $L_\infty[0,1]$, i.e., $\varrho_{L_\infty}(x_1, x_2) \doteq \operatorname*{ess\,sup}_{t\in[0,1]} |x_1(t) - x_2(t)|$.

Put $\chi \doteq \{\frac{\pi}{2} + \pi k,\ k \in \mathbb{Z}\}$, $B(\chi, \varepsilon) \doteq \bigcup_{k\in\mathbb{Z}}[\frac{\pi}{2} + \pi k - \varepsilon, \frac{\pi}{2} + \pi k + \varepsilon]$, $\varepsilon > 0$. For an arbitrary function $x \in L_\infty[0,1]$, define the set

$$\Xi(x, \varepsilon) \doteq \{t \in [0,1] :\ x(h(t)) \in B(\chi, \varepsilon)\}.$$

According to (12), this set is measurable. Next, split the space $L_\infty[0,1]$ into two classes: $\mathcal{L}^0_\infty[0,1]$ and $\mathcal{L}^+_\infty[0,1]$. The elements of $\mathcal{L}^+_\infty[0,1]$ are the functions x such that $\mu(\Xi(x, \varepsilon)) > 0$ for all $\varepsilon > 0$;

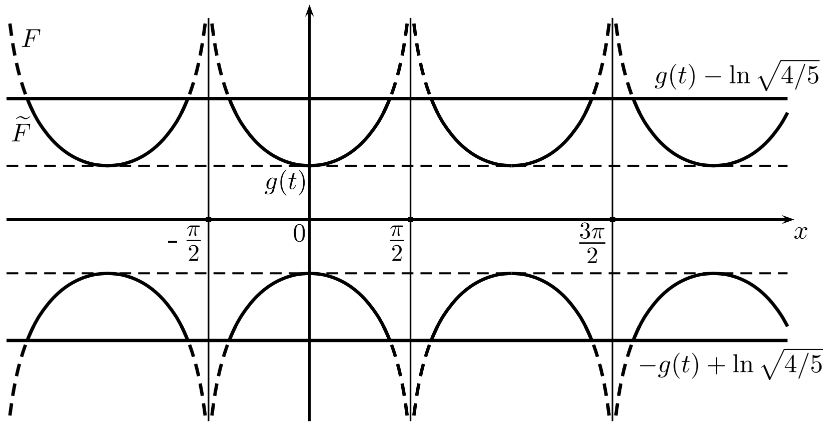

Fig 1. The graphs of the functions $x \mapsto F(t,x)$ and $x \to \widetilde{F}(t,x)$ in Example 1

a function x belongs to $\mathcal{L}^0_\infty[0,1]$ if and only if there exists an $\varepsilon > 0$ such that $\mu(\Xi(x,\varepsilon)) = 0$. Consider the maps

$$F : [0,1]\times\mathbb{R} \multimap \mathbb{R},\ \ F(t,x) \doteq \begin{cases} \{\pm\big(-\ln|\cos x| + g(t)\big)\} & \text{for } x \notin \chi, \\ \varnothing & \text{for } x \in \chi; \end{cases}$$

$$\Phi : L_\infty[0,1] \multimap L_\infty[0,1],\ \ \Phi(x) \doteq \begin{cases} \{y:\ y(t) \in F\big(t, x(h(t))\big)\} & \text{for } x \in \mathcal{L}^0_\infty[0,1], \\ \varnothing & \text{for } x \in \mathcal{L}^+_\infty[0,1]. \end{cases}$$

Then equation (11) is equivalent to the inclusion $x \in \Phi\, x$; we investigate the latter with the help of Theorem 1.

Put $\vartheta \doteq \{\, x \in \mathbb{R} :\ |\cos x| \leqslant \sqrt{4/5}\,\}$ and let the map $\mathfrak{a} : [0,1]\times\mathbb{R} \multimap \mathbb{R}$ be defined by

$$\mathfrak{a}(t,x) \doteq \begin{cases} \mathbb{R} & \text{for } x \notin \vartheta, \\ \varnothing & \text{for } x \in \vartheta. \end{cases}$$

Next, define the maps $\mathfrak{b}, \widetilde{F} : [0,1]\times\mathbb{R} \to \mathrm{clos}(\mathbb{R})$, $\widetilde{\Phi} : L_\infty[0,1] \to \mathrm{clos}(L_\infty[0,1])$ as follows:

$$\mathfrak{b}(t,x) \doteq \Big\{\pm\Big(g(t) - \ln\sqrt{4/5}\,\Big)\Big\};\quad \widetilde{F}(t,x) \doteq \big(F(t,x) \cap \mathfrak{a}(t,x)\big) \cup \mathfrak{b}(t,x);$$

$$\widetilde{\Phi}(x) \doteq \Big\{y:\ y(t) \in \widetilde{F}\big(t, x(h(t))\big)\Big\}$$

(the graphs of the multi-valued maps $F(t,\cdot)$ and $\widetilde{F}(t,\cdot)$ are shown in Figure 1).

In definition (5), for the map $\widetilde{\Phi}$ considered, the operator $\mathcal{A} : L_\infty[0,1] \multimap L_\infty[0,1]$ has values $\mathcal{A}(x) = L_\infty[0,1]$ for any x satisfying $\mu\{t \in [a,b] :\ |\cos x(h(t))| \leqslant \sqrt{4/5}\,\} = 0$, and $\mathcal{A}(x) = \varnothing$ otherwise. Next, for a function $y \in \widetilde{\Phi}(x)$, one has $y \in \mathcal{B}(x)$ if and only if the equality $|y(t)| = g(t) - \ln\sqrt{4/5}$ takes place on some set of positive measure.

We show now that the multi-valued map $\widetilde{\Phi}$ is contracting with coefficient 1/2. First of all, for any $x \notin \vartheta$ there holds the estimate

$$\left|\frac{d}{dt}\ln|\cos x|\right| = |\mathrm{tg}\, x| = \sqrt{\cos^{-2}x - 1} < \frac{1}{2}$$

according to which the map $\widetilde{F}(t,\cdot) : \mathbb{R} \to \mathrm{clos}(\mathbb{R})$ is 1/2-contracting for a.e. $t \in [0,1]$. Pick arbitrary $x_1, x_2 \in L_\infty[0,1]$, denote $\tilde{r} \doteq \varrho_{L_\infty}(x_1,x_2)$, and choose any function $y_1 \in \widetilde{\Phi}(x_1)$; so, $y_1(t) \in \widetilde{F}\big(t, x_1(h(t))\big)$ a.e. on $[0,1]$. Since the map $\widetilde{F}(t,\cdot)$ is 1/2-contracting, for any $\delta > 0$ and a.e. $t \in [0,1]$, the set

$$\mathcal{C}(t) \doteq B_{\mathbb{R}}\left(y_1(t), 2^{-1}(\tilde{r} + \delta)\right) \cap \widetilde{F}\big(t, x_2(h(t))\big)$$

is not empty, and the multi-valued map $t \mapsto \mathcal{C}(t)$ is measurable. Thus, there exists a measurable selection, say a function $y_2(\cdot)$, such that $\varrho_{L_\infty}(y_1, y_2) \leqslant 2^{-1}(\tilde{r} + \delta)$ and $y_2(t) \in \widetilde{F}\big(t, x_2(h(t))\big)$ for a.e. $t \in [0,1]$. From the arbitrariness of $\delta > 0$ it follows that $\operatorname{dist}_{L_\infty}\big(\widetilde{\Phi}(x_1), \widetilde{\Phi}(x_2)\big) \leqslant 2^{-1}\tilde{r}$. Hence, the map $\widetilde{\Phi}$ is 1/2-contracting.

Let $x_0 = g$. Then taking into account relation (13), we obtain

$$(1-q)^{-1}\varrho_{L_\infty}\big(x_0, \widetilde{\Phi}(x_0)\big) = 2\varrho_{L_\infty}\big(x_0, \Phi(x_0)\big) = \operatorname*{ess\,sup}_{t\in[0,1]}\big(-\ln\big|\cos g(h(t))\big|\big) =$$
$$= \ln\Big(\operatorname*{ess\,sup}_{t\in[0,1]}\big|\cos g(h(t))\big|^{-1}\Big) < \ln\sqrt{5/4},$$

and since

$$\varrho_{L_\infty}\big(x_0, \mathcal{B}(x)\big) = \ln\sqrt{5/4}$$

for every x, there exists an r_0 satisfying all the hypotheses of Theorem 1. Thus, the solvability of inclusion $x \in \Phi\, x$ is proved and so is the solvability of equation (11).

Concluding the discussion of Theorem 1, note that the presented method of «correcting» maps can be called universal in the sense that, by means of equality (5) and appropriate maps $\mathcal{A}$, $\mathcal{B}$, one can turn any given map $\Phi : X \multimap X$ into any required map $\widetilde{\Phi} : X \multimap X$. In order to do so, one should choose the maps $\mathcal{A}, \mathcal{B} : X \multimap X$ so that $\mathcal{A}(x) \cap \big(\Phi(x) \setminus \widetilde{\Phi}(x)\big) = \varnothing$ and $\widetilde{\Phi}(x) \setminus \mathcal{A}(x) \subset \mathcal{B}(x) \subset \widetilde{\Phi}(x)$ for all $x \in X$. Moreover, if $\widetilde{\Phi} : X \to \operatorname{clos}(X)$, then $\mathcal{A}, \mathcal{B}$ may also have closed images; it suffices to set $\mathcal{A}(x) = \overline{O\big(F(x) \setminus \mathcal{F}(x)\big)}$, where $O(M)$ is any open set containing $M \subset X$, and define a closed set $\mathcal{B}(x)$ as mentioned above.

§ 2. Cauchy problem for a differential inclusion with noncompact right-hand side

Let $F : \mathbb{R} \times \mathbb{R}^n \multimap \mathbb{R}^n$. Consider the ordinary differential inclusion

$$\dot{x} \in F(t, x). \tag{14}$$

We are concerned with conditions of existence and extendability of solutions to this inclusion, first of all, in the case when its right-hand side is unbounded and not necessarily closed-valued. Following the definition in [11, pp. 7, 53], as *a solution of inclusion* (14) *we understand a function defined on an interval I (compact or noncompact, finite or infinite) that is absolutely continuous on every compact interval $[a, b] \subset I$ and satisfies the inclusion a.e. on I.*

Given a number t_0 and a vector $\alpha_0 \in \mathbb{R}^n$, the system consisting of inclusion (14) and the initial condition

$$x(a) = \alpha_0 \tag{15}$$

is called the Cauchy problem for inclusion (14). The following theorem on solvability of problem (14), (15) is based on rewriting this problem as an operator inclusion in a space of measurable functions; we replace the multi-valued Nemytskii operator generated by the map F with the operator defined by formula (10), which represents the Nemytskii operator generated by the map $\widetilde{F}$ connected with the initial map F by means of the relation analogous to (10).

Theorem 2. *Suppose there exist $T > 0$, $\sigma > 0$, and integrable functions $R_0 : [t_0, t_0 + T] \to \mathbb{R}_+$, $\theta : [t_0, t_0 + T] \to \mathbb{R}^n$ such that the map $\widetilde{F} : [t_0, t_0 + T] \times B_{\mathbb{R}^n}(\alpha_0, \sigma) \multimap \mathbb{R}^n$ defined by the equality*

$$\widetilde{F}(t, x) \doteq \big(F(t, x) \cap B_{\mathbb{R}^n}(\theta(t), R_0(t))\big) \cup S_{\mathbb{R}^n}\big(\theta(t), R_0(t)\big) \tag{16}$$

satisfies the following conditions:

1) $\widetilde{F}(t, x) \in \operatorname{comp}(\mathbb{R}^n)$ *for every $x \in B_{\mathbb{R}^n}(\alpha_0, \sigma)$ and a.e. $t \in [t_0, t_0 + T]$;*

2) *the map* $\widetilde{F}(\cdot,x):[t_0,t_0+T]\to\operatorname{comp}(\mathbb{R}^n)$ *is measurable for every* $x\in B_{\mathbb{R}^n}(\alpha_0,\sigma)$;
3) *there is an integrable function* $k:[t_0,t_0+T]\to\mathbb{R}_+$ *such that the function*

$$\nu:[t_0,t_0+T]\to\mathbb{R}_+,\quad \nu(t)\doteq k(t)/R_0(t)$$

is essentially bounded and

$$\operatorname{dist}_{\mathbb{R}^n}\big(\widetilde{F}(t,x_1),\widetilde{F}(t,x_2)\big)\leqslant k(t)|x_1-x_2| \tag{17}$$

for a.e. $t\in[t_0,t_0+T]$ *and any* $x_1,x_2\in B_{\mathbb{R}^n}(\alpha_0,\sigma)$.

If the function $x_*(t)\doteq\alpha_0+\int_{t_0}^{t}\theta(s)\,ds$ *satisfies the inequality*

$$\operatorname*{ess\,sup}_{t\in[t_0,\,t_0+T]}\frac{\varrho_{\mathbb{R}^n}\big(\theta(t),F(t,x_*(t))\big)}{R_0(t)}<1, \tag{18}$$

then there exists a $\tau>0$ *such that the Cauchy problem* (14), (15) *has a solution defined on* $[t_0,t_0+\tau]$.

P r o o f. First of all, for a.e. $t\in[t_0,t_0+T]$, there holds the equality

$$\varrho_{\mathbb{R}^n}\big(\theta(t),\widetilde{F}(t,x_*(t))\big)=\varrho_{\mathbb{R}^n}\big(\theta(t),F(t,x_*(t))\big). \tag{19}$$

Indeed, for a.e. $t\in[t_0,t_0+T]$, from assumption (18) it follows that

$$\varrho_{\mathbb{R}^n}\big(\theta(t),F(t,x_*(t))\big)<R_0(t),$$

and hence,

$$\varrho_{\mathbb{R}^n}\big(\theta(t),\ F(t,x_*(t))\big)=\varrho_{\mathbb{R}^n}\big(\theta(t),\ F(t,x_*(t))\cap B_{\mathbb{R}^n}(\theta(t),R_0(t))\big),$$
$$\varrho_{\mathbb{R}^n}\big(\theta(t),\ F(t,x_*(t))\big)<\varrho_{\mathbb{R}^n}\big(\theta(t),\ S_{\mathbb{R}^n}(\theta(t),R_0(t))\big).$$

Taking into account definition (16) of the map $\widetilde{F}$, from these relations we get (19).

Now, find a $q\in(0,1)$ such that

$$\operatorname*{ess\,sup}_{t\in[t_0,t_0+T]}\frac{\varrho_{\mathbb{R}^n}\big(\theta(t),\ \widetilde{F}(t,x_*(t))\big)}{R_0(t)}<1-q \tag{20}$$

(according to estimate (18) and equality (19), such a q does exist). From essential boundedness of the function ν and absolute continuity of the Lebesgue integral it follows that there exists a $\tau\in(0,T)$ such that

$$\operatorname*{ess\,sup}_{t\in[t_0,\,t_0+\tau]}\Big(\nu(t)\int_{t_0}^{t}R_0(s)\,ds\Big)\leqslant q, \tag{21}$$

$$\int_{t_0}^{\tau}\big(R_0(s)+|\theta(s)|\big)\,ds\leqslant\sigma. \tag{22}$$

Define the functional space

$$\mathcal{L}\doteq\left\{y:[t_0,t_0+\tau]\to\mathbb{R}^n\ :\ \operatorname*{ess\,sup}_{t\in[t_0,\,t_0+\tau]}\frac{|y(t)-\theta(t)|}{R_0(t)}<\infty\right\}.$$

Note that the space $\mathcal{L}$ is not empty (for example, the function θ belongs to this space); one has $y\in\mathcal{L}$ if and only if there exists a $\lambda\geqslant0$ such that the inequality $|y(t)-\theta(t)|\leqslant\lambda R_0(t)$ takes place for a.e. $t\in[t_0,t_0+\tau]$. This means that $\mathcal{L}\subset L^n[t_0,t_0+\tau]$. We define a metric in $\mathcal{L}$ by the equality

$$\varrho_{\mathcal{L}}(y,z)\doteq\operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]}\frac{|y(t)-z(t)|}{R_0(t)},\quad y,z\in\mathcal{L}\,.$$

Obviously, the space $\mathcal{L}$ with respect to this metric is complete. It should be noticed that, if a function $y \in \mathcal{L}$ satisfies the inequality $\varrho_{\mathcal{L}}(y,\theta) < 1$, then the inclusion $y(t) \in B^{o}_{\mathbb{R}^n}(\theta(t), R_0(t))$ holds for a.e. $t \in [t_0, t_0+\tau]$.

Denote by $F_\tau : [t_0, t_0+\tau] \times \mathbb{R}^n \multimap \mathbb{R}^n$ the restriction of the map F, define the corresponding Nemytskii operator

$$N : C^n[t_0, t_0+\tau] \multimap \mathcal{L}, \quad Nx \doteq \big\{y \in \mathcal{L} : y(t) \in F_\tau(t, x(t)) \text{ for a.e. } t \in [t_0, t_0+\tau]\big\}$$

and the integral operator

$$I : B_{\mathcal{L}}(\theta, 1) \to C^n[t_0, t_0+\tau], \quad (Iz)(t) \doteq \alpha_0 + \int_{t_0}^{t} z(s)ds, \quad t \in [t_0, t_0+\tau].$$

By using Theorem 1, we show that the map $NI : B_{\mathcal{L}}(\theta,1) \multimap \mathcal{L}$ has a fixed point $\bar{y}$, so there will be justified the existence of the solution $\bar{x} = I\bar{y} \in AC^n[t_0, t_0+\tau]$ to the Cauchy problem (14), (15). All the successive reasoning is verifying the conditions of Theorem 1 for the operator $\Phi = NI$.

Define the function $\bar{\alpha}_0(t) \equiv \alpha_0$, $t \in [t_0, t_0+\tau]$. From inequality (22) there follows the inclusion $I(B_{\mathcal{L}}(\theta,1)) \subset B_{C^n[t_0,t_0+\tau]}(\bar{\alpha}_0, \sigma)$.

Denote by $\widetilde{F}_\tau : [t_0, t_0+\tau] \times B_{\mathbb{R}^n}(\alpha_0, \sigma) \to \operatorname{comp}(\mathbb{R}^n)$ the restriction of the map $\widetilde{F}$ given by (16). According to condition (17), the map $\widetilde{F}_\tau(t,\cdot) : B_{\mathbb{R}^n}(\alpha_0,\sigma) \to \operatorname{comp}(\mathbb{R}^n)$ is continuous for a.e. $t \in [t_0, t_0+\tau]$; from assumption 1) it follows that $\widetilde{F}_\tau(\cdot, x) : [t_0, t_0+\tau] \to \operatorname{comp}(\mathbb{R}^n)$ is measurable for every $x \in B_{\mathbb{R}^n}(\alpha_0,\sigma)$. Thus, the map $\widetilde{F}_\tau$ satisfies the Caratheodory conditions, and hence, for every $x \in B_{C^n[t_0,t_0+\tau]}(\bar{\alpha}_0,\sigma)$, there exists a measurable selection y of the map $\widetilde{F}_\tau(\cdot, x(\cdot))$ (see, e.g., [12]), so $y(t) \in \widetilde{F}_\tau(t, x(t))$ for a.e. $t \in [t_0, t_0+\tau]$. According to (16), any such selection is an element of the space $\mathcal{L}$, moreover, $\varrho_{\mathcal{L}}(y,\theta) \leqslant 1$. This means that we can define the Nemytskii operator generated by the map $\widetilde{F}_\tau$:

$$\widetilde{N} : B_{C^n[t_0,t_0+\tau]}(\bar{\alpha}_0, \sigma) \multimap \mathcal{L}, \quad \widetilde{N}x \doteq \big\{y \in \mathcal{L} : y(t) \in \widetilde{F}_\tau(t, x(t)) \text{ for a.e. } t \in [t_0, t_0+\tau]\big\}. \tag{23}$$

Let us verify that $\widetilde{N}x \in \operatorname{clbd}(\mathcal{L})$ for any $x \in B_{C^n[t_0,t_0+\tau]}(\bar{\alpha}_0,\sigma)$. From what is said above it follows that the set $\widetilde{N}x$ is bounded and $\widetilde{N}x \subset B_{\mathcal{L}}(\theta,1)$; show that $\widetilde{N}x$ is closed. Consider a sequence $\{y_i\}_{i=1}^{\infty} \subset \widetilde{N}x$ such that $\varrho_{\mathcal{L}}(y_i, y) \to 0$, $i \to \infty$. Then $|y_i(t) - y(t)| \to 0$ for a.e. $t \in [t_0, t_0+\tau]$, and since the set $\widetilde{F}_\tau(t,x(t))$ is closed in $\mathbb{R}^n$, there holds the inclusion $y(t) \in \widetilde{F}_\tau(t,x(t))$. Therefore, $y \in \widetilde{N}x$, and hence, the set $\widetilde{N}x$ is closed.

Now we prove that the superposition $\widetilde{N}I : B_{\mathcal{L}}(\theta,1) \to \operatorname{clbd}\big(\mathcal{L}\big)$ is q-contracting. Let $x_1, x_2 \in B_{C^n[t_0,t_0+\tau]}(\bar{\alpha}_0,\sigma)$. Take an arbitrary $y_1 \in \widetilde{N}x_1$ and consider the ball $B_{\mathbb{R}^n}(y_1(t), r_\varepsilon(t))$ of radius $r_\varepsilon(t) = k(t)|x_1(t) - x_2(t)| + \varepsilon$, where $\varepsilon > 0$. From estimate (17) it follows that for a.e. $t \in [t_0, t_0+\tau]$ the set $B_{\mathbb{R}^n}(y_1(t), r_\varepsilon(t)) \cap \widetilde{F}(t, x_2(t))$ is not empty. Next, the map $t \mapsto B_{\mathbb{R}^n}(y_1(t), r_\varepsilon(t)) \cap \widetilde{F}(t,x_2(t))$ is measurable, hence it has a measurable selection, say y_2^ε. So we have $y_2^\varepsilon \in \widetilde{N}x_2$ and $|y_1(t) - y_2^\varepsilon(t)| \leqslant r_\varepsilon(t)$ for a.e. $t \in [t_0, t_0+\tau]$. Then

$$\varrho_{\mathcal{L}}(y_1, y_2^\varepsilon) = \operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \frac{|y_1(t) - y_2^\varepsilon(t)|}{R_0(t)} \leqslant \operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \frac{r_\varepsilon(t)}{R_0(t)}.$$

Similarly, it can be proved that for any $y_2 \in \widetilde{N}x_2$ there exists a $y_1^\varepsilon \in \widetilde{N}x_1$ such that

$$\varrho_{\mathcal{L}}(y_1^\varepsilon, y_2) \leqslant \operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \frac{r_\varepsilon(t)}{R_0(t)}.$$

From the arbitrariness of $\varepsilon > 0$ and the inequalities obtained we get the estimate

$$\operatorname{dist}_{\mathcal{L}}\big(\widetilde{N}x_1, \widetilde{N}x_2\big) \leqslant \operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \frac{k(t)|x_1(t) - x_2(t)|}{R_0(t)}.$$

Then for any $z_1, z_2 \in B_{\mathcal{L}}(\theta, 1)$ there holds

$$\operatorname{dist}_{\mathcal{L}}\left(\widetilde{N}Iz_1,\, \widetilde{N}Iz_2\right) \leqslant \operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \frac{k(t)}{R_0(t)} \left| \int_{t_0}^{t} (z_1(s)-z_2(s))ds \right| \leqslant$$
$$\leqslant \operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \frac{k(t)}{R_0(t)} \int_{t_0}^{t} R_0(s)\frac{|z_1(s)-z_2(s)|}{R_0(s)} ds = \varrho_{\mathcal{L}}(z_1,z_2) \operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \frac{k(t)}{R_0(t)} \int_{t_0}^{t} R_0(s)ds.$$

Now, taking into account inequality (21), we get

$$\operatorname{dist}_{\mathcal{L}}\left(\widetilde{N}Iz_1, \widetilde{N}Iz_2\right) \leqslant q\varrho_{\mathcal{L}}(z_1,z_2),$$

so the map $\widetilde{N}I : B_{\mathcal{L}}(\theta,1) \to \operatorname{clbd}(\mathcal{L})$ is q-contracting.

According to the definitions (16), (23) of the map $\widetilde{F}$ and corresponding Nemytskii operator $\widetilde{N}$, for any $x \in B_{C^n[t_0,t_0+\tau]}(\bar{\alpha}_0,\sigma)$, there takes place the equality

$$\widetilde{N}(x) = \big(N(x)\cap\mathcal{A}(x)\big)\cup\mathcal{B}(x),$$

where $\mathcal{A}(x) \equiv B_{\mathcal{L}}(\theta,1)$ and $\mathcal{B}(x)$ is the set of functions having the following property: for $y \in \widetilde{N}(x)$ the inclusion $y \in \mathcal{B}(x)$ is true if and only if there exists a set $E \subset [t_0, t_0+\tau]$ of measure $\mu(E) > 0$ such that $y(t) \in S_{\mathbb{R}^n}(\theta(t), R_0(t))$ for a.e. $t \in E$. Thus, for any $x \in B_{C^n[t_0,t_0+\tau]}(\bar{\alpha}_0,\sigma)$ and arbitrary $y \in \mathcal{B}(x)$, we have $\varrho_{\mathcal{L}}(y,\theta) = 1$.

So the map $\widetilde{N}I : B_{\mathcal{L}}(\theta,1) \to \operatorname{clbd}(\mathcal{L})$ is q-contracting,

$$\widetilde{N}I(y) = \big(NI(y)\cap B_{\mathcal{L}}(\theta,1)\big)\cup\mathcal{B}I(y), \text{ and } \varrho_{\mathcal{L}}\big(\theta,\mathcal{B}I(y)\big) = 1 \text{ for any } y \in B_{\mathcal{L}}(\theta,1).$$

To fulfill the conditions of Theorem 1, it only remains to show that $(1-q)^{-1}\varrho_{\mathcal{L}}\big(\theta,\, \widetilde{N}I(\theta)\big) < 1$ (we can take any number between $(1-q)^{-1}\varrho_{\mathcal{L}}\big(\theta,\, \widetilde{N}x_*\big)$ and 1 as r_0). From measurability of the functions $\theta(\cdot)$ and $\widetilde{F}_\tau(\cdot, x_*(\cdot))$ there follows the existence of a measurable function $u : [t_0, t_0+\tau] \to \mathbb{R}^n$ such that $u(t) \in \widetilde{F}_\tau(t, x_*(t))$ and $|\theta(t)-u(t)| = \varrho_{\mathbb{R}^n}\big(\theta(t), \widetilde{F}_\tau(t,x_*(t))\big)$ for a.e. $t \in [t_0, t_0+\tau]$ (see, e.g., [7]). Then, taking into account inequality (20), we get $\varrho_{\mathcal{L}}(\theta,u) < 1-q$. Thus, $\varrho_{\mathcal{L}}(\theta,\widetilde{N}x_*) < 1-q$, and the required estimate is derived.

So, all the assumptions of Theorem 1 are complied; the existence of a fixed point for the map NI is verified. $\square$

Remark 2. If the functions R_0, θ in Theorem 2 are defined on $[t_0-T, t_0]$ and all the other assumptions hold true for the map $\widetilde{F} : [t_0-T, t_0] \times B_{\mathbb{R}^n}(\alpha_0,\sigma) \multimap \mathbb{R}^n$ given by (16), then for some $\tau > 0$, there exists a solution to problem (14), (15) defined on $[t_0-\tau, t_0]$. Obviously, if the assumptions of Theorem 2 take place for $t \in [t_0-T, t_0+T]$, then the Cauchy problem has a solution defined on $[t_0-\tau, t_0+\tau]$.

From Theorem 2 one can deduce the known results about solvability of inclusions with closed-valued and not necessarily bounded right-hand sides. For example, there takes place the following generalization of the statements obtained in the papers [6, 7].

Corollary 2. *Let $T > 0$ and $F : [t_0, t_0+T] \times \mathbb{R}^n \multimap \mathbb{R}^n$ be such that, for some $\sigma > 0$, the following conditions hold:*
1) *$F(t,x) \in \operatorname{clos}(\mathbb{R}^n)$ for any $x \in B_{\mathbb{R}^n}(\alpha_0,\sigma)$ and a.e. $t \in [t_0, t_0+T]$;*
2) *the map $F(\cdot,x) : [t_0,t_0+T] \to \operatorname{clos}(\mathbb{R}^n)$ is measurable for any $x \in B_{\mathbb{R}^n}(\alpha_0,\sigma)$;*
3) *there exists an integrable function $k : [t_0, t_0+T] \to \mathbb{R}_+$ such that*

$$\operatorname{dist}_{\mathbb{R}^n}(F(t,x_1), F(t,x_2)) \leqslant k(t)|x_1 - x_2|$$

for any $x_1, x_2 \in B_{\mathbb{R}^n}(\alpha_0,\sigma)$ and a.e. $t \in [t_0, t_0+T]$.

If there exists a function $x_* \in AC^n[t_0, t_0+T]$ satisfying the condition $x_*(a) = \alpha_0$ and such that the function

$$[t_0, t_0+T] \ni t \mapsto \varrho_{\mathbb{R}^n}\big(\dot{x}_*(t),\ F(t, x_*(t))\big) \in \mathbb{R}_+$$

is integrable, then there is a $\tau > 0$ such that the Cauchy problem (14), (15) *has a solution defined on* $[t_0, t_0+\tau]$.

P r o o f. In definition (16) of the map $\widetilde{F}$, set

$$\theta(t) = \dot{x}_*, \quad R_0(t) = k(t) + 2\varrho_{\mathbb{R}^n}\big(\dot{x}_*(t),\ F(t, x_*(t))\big),\ t \in [t_0, t_0+T].$$

Obviously, the map $\widetilde{F}$ satisfies conditions 1), 2) of Theorem 2, moreover, $\nu(t) = k(t)/R_0(t) \leqslant 1$ for a.e. $t \in [t_0, t_0+T]$. Further, according to [4], for any $x_1, x_2 \in \mathbb{R}^n$ we get

$$\mathrm{dist}_{\mathbb{R}^n}\big(\widetilde{F}(t, x_1), \widetilde{F}(t, x_2)\big) \leqslant \mathrm{dist}_{\mathbb{R}^n}\big(F(t, x_1), F(t, x_2)\big),\ t \in [t_0, t_0+T],$$

so the map $\widetilde{F}$ complies condition 3) of Theorem 2. Finally, the last assumption of the theorem, inequality (18), is also satisfied; indeed,

$$\frac{\varrho_{\mathbb{R}^n}\big(\theta(t), F(t, x_*(t))\big)}{R_0(t)} = \frac{\varrho_{\mathbb{R}^n}\big(\theta(t), F(t, x_*(t))\big)}{k(t) + 2\varrho_{\mathbb{R}^n}\big(\theta(t),\ F(t, x_*(t))\big)} \leqslant \frac{1}{2},\ t \in [t_0, t_0+T].$$

□

Remark 3. Let us note that estimate (18) in condition 3) of Theorem 2 cannot be relaxed. It cannot be replaced, for instance, by the assumption that there exists an $\varepsilon > 0$ such that the inequality

$$\varrho_{\mathbb{R}^n}\big(\theta(t), F(t, x_*(t))\big) \leqslant R_0(t) - \varepsilon \tag{24}$$

is true for a.e. $t \in [t_0, t_0+T]$. Below we give an example that confirms this observation.

Example 2. Consider the map

$$F : \mathbb{R}_+ \times \mathbb{R} \to \mathrm{clos}(\mathbb{R}), \quad F(t, x) = \begin{cases} \{\, t^{-\frac{1}{2}}(x+3)\,\}, & \text{for } t \neq 0,\ x < 0, \\ \{\, t^{-\frac{1}{2}}(x-1)\,\}, & \text{for } t \neq 0,\ x \geqslant 0, \\ \{0\}, & \text{for } t = 0,\ x \in \mathbb{R} \end{cases}$$

(so, for any pair (t, x), the image $F(t, x)$ consists of a single point, see Figure 2). It can easily be verified that the Cauchy problem for corresponding differential inclusion (14) with the initial condition $x(0) = 0$ is not solvable on $[0, \tau]$, $\tau > 0$.

Let $\theta(t) = t^{-\frac{1}{2}}$, then $x_*(t) = 2t^{\frac{1}{2}}$, $F(t, x_*(t)) = \{2 - t^{-\frac{1}{2}}\}$. Next, set $R_0(t) = 2t^{-\frac{1}{2}}$ and take any $\sigma \in (0, 1]$. So, we have $\widetilde{F}(t, x) = F(t, x) \cup \{-t^{-\frac{1}{2}};\ 3t^{-\frac{1}{2}}\}$ for any $|x| \leqslant \sigma$, $t > 0$. Clearly, this map satisfies conditions 1), 2) of Theorem 2. Besides, for all $|x_1| \leqslant \sigma$, $|x_2| \leqslant \sigma$, there holds the estimate

$$\mathrm{dist}_{\mathbb{R}}\big(\widetilde{F}(t, x_1), \widetilde{F}(t, x_2)\big) \leqslant t^{-\frac{1}{2}}\,|x_1 - x_2|,$$

i.e., condition 3) is complied with $k(t) = t^{-\frac{1}{2}}$ and $\nu(t) \equiv 1/2$. Further, for a.e. $t > 0$ we have

$$\varrho\big(\theta(t), F(t, x_*(t))\big) = t^{-\frac{1}{2}} - \big(2 - t^{-\frac{1}{2}}\big) = 2t^{-\frac{1}{2}} - 2 = R_0(t) - 2,$$

so inequality (24) is true.

At the same time, condition (18) does not hold since for any $T > 0$,

$$\operatorname*{ess\,sup}_{t \in [0,T]} \frac{\varrho\big(\theta(t), F(t, x_*(t))\big)}{R_0(t)} = \operatorname*{ess\,sup}_{t \in [0,T]} \frac{2t^{-\frac{1}{2}} - 2}{2t^{-\frac{1}{2}}} = 1.$$

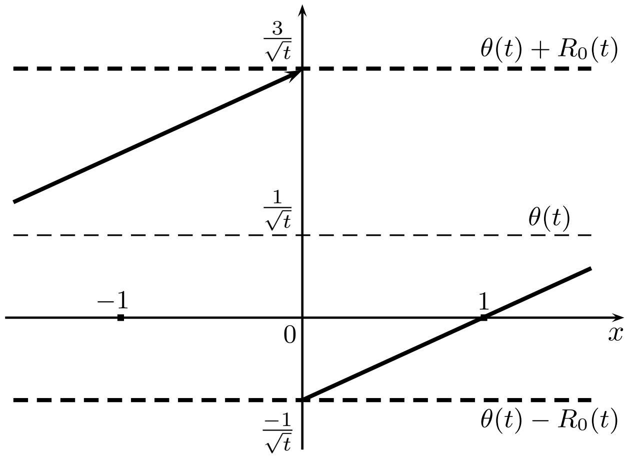

Fig 2. The graphs of the functions $x \mapsto F(t,x)$ and $x \to \widetilde{F}(t,x)$ for $t \neq 0$ in Example 2

Using Theorem 2 one can get conditions of continuation of solutions to differential inclusion (14). Let D be a closed bounded domain (i.e., the closure of a nonempty open bounded connected set) in $\mathbb{R} \times \mathbb{R}^n$.

Theorem 3. *Let $M : [a,b] \to \mathbb{R}_+$ be an integrable function, where $[a,b]$ is the projection of D onto $\mathbb{R}$, i.e., $[a,b] = \{t : \exists\, x\ (t,x) \in D\}$. Suppose that for every point $(t_0,\alpha_0) \in \operatorname{int} D$ there exist $T>0$, $\sigma>0$, and integrable functions $R_0 : [t_0-T,t_0+T] \to \mathbb{R}_+$, $\theta : [t_0-T,t_0+T] \to \mathbb{R}^n$ such that*

$$|\theta(t)| + R_0(t) \leqslant M(t), \quad t \in [t_0-T,t_0+T], \tag{25}$$

and the map $\widetilde{F} : [t_0-T,t_0+T] \times B_{\mathbb{R}^n}(\alpha_0,\sigma) \multimap \mathbb{R}^n$ given by (16) satisfies the conditions of Theorem 2 on the interval $[t_0-T,t_0+T]$. Then any solution $\bar{x}$ of inclusion (14) defined on some compact interval I and having the graph in $\operatorname{int} D$, meaning $\{(t,\bar{x}(t)), t \in I\} \subset \operatorname{int} D$, can be continued («on both sides») up to the boundary of D.

P r o o f. Denote by $\mathfrak{X}$ the set of solutions x to inclusion (14) that are defined on compact intervals, have graphes inside D, are continuations of the given on I solution $\bar{x}$, and satisfy the inequality

$$|\dot{x}(t)| \leqslant M(t) \tag{26}$$

for a.e. $t \notin I$. This set is not empty since $\bar{x} \in \mathfrak{X}$. We introduce on $\mathfrak{X}$ the order $\succcurlyeq$ in the following way: given two solutions $u,v \in \mathfrak{X}$ defined on compact intervals I_u and I_v respectively, we assume $u \succcurlyeq v$ if $I_u \supset I_v$ and $u(t)=v(t)$ for $t \in I_v$. According to the Hausdorff theorem (see, e.g., [13]), in $\mathfrak{X}$ there exists a maximal chain S containing $\bar{x}$. Each element of this chain is some solution x defined on $I_x=[c_x,d_x]$. Obviously, there exist $\widehat{c} \doteq \inf_{x\in S}\{c_x\}$ and $\widehat{d} \doteq \sup_{x\in S}\{d_x\}$. For any $t_0 \in (\widehat{c},\widehat{d}\,)$, find a solution $x \in S$ such that $t_0 \in I_x$ (such a solution does exist). Next, on the interval $(\widehat{c},\widehat{d}\,)$ define the function $\widehat{x}$ by the equality $\widehat{x}(t)=x(t)$, where $x \in S$ is a solution defined on the compact interval I_x containing t (this definition if well-posed, choosing any other element in S we get the same value $x(t)$). Since every $x \in S$ satisfies inequality (26), the function $\widehat{x}$ can be continued on the compact interval $[\widehat{c},\widehat{d}\,]$, and this continuation will be an element of $\mathfrak{X}$.

Now, we show that «the ends» of the solution found, the points $\big(\widehat{c},\, \widehat{x}(\widehat{c})\big)$, $\big(\widehat{d},\, \widehat{x}(\widehat{d})\big)$, belong to the boundary of D. Suppose this is not true, and, for example, $\big(\widehat{d},\, \widehat{x}(\widehat{d})\big) \in \operatorname{int} D$. Then, according to Theorem 2, there exists a solution of the Cauchy problem with the initial condition $x(\widehat{d}) = \widehat{x}(\widehat{d})$ defined on $[\widehat{d},\widehat{d}+\tau]$ and satisfying inequality (26). This means that one can find a continuation of the solution $\widehat{x}$ which is an element of $\mathfrak{X}$ and is greater than any element of the maximal chain S. This contradiction completes the proof. $\square$

We give an example which illustrates the importance of condition (25) in Theorem 3.

Example 3. Let $D = \{(t,x):\ t \in [0,1],\ x \in [0,2]\}$ and

$$f(t) \doteq (-1)^{i+1} t^{-2},\ t \in \big((i+1)^{-1}, i^{-1}\big],\ i = 1,2,\dots.$$

For $t \in (0,1]$, $x \in [0,2]$, define the map $F : \mathbb{R} \times \mathbb{R} \multimap \mathbb{R}$ by $F(t,x) \doteq \{f(t)\}$, and consider inclusion (14). The conditions of Theorem 2 are fulfilled for any initial point $(t_0, \alpha_0) \in \operatorname{int} D$. In order to prove this, one can choose $T, \sigma > 0$ so that $[t_0 - T, t_0 + T] \times B_{\mathbb{R}^n}(\alpha_0, \sigma) \subset \operatorname{int} D$ (obviously, such T and σ do exist) and set $\theta(t) = 0$, $k(t) = 0$, $R_0(t) = 2(t_0 - T)^{-2}$ for $t \in [t_0 - T, t_0 + T]$. At the same time, condition (25) does not hold. Indeed, from estimate (18) it follows that, for any choice of θ and R_0, one gets

$$|\theta(t)| + R_0(t) > |\theta(t)| + |f(t) - \theta(t)| \geqslant |f(t)| = t^{-2},\ t \in (0,1],$$

and the function $t \mapsto t^{-2}$ in not integrable on $[0,1]$.

The violation of condition (25) makes it impossible to continue a solution of inclusion (14) passing, for example, through the point $(1/2, 0)$, to the left up to the boundary of D. In fact, any solution x defined on $I_x \subset [0,1]$ and satisfying the condition $x(1/2) = 0$, is a restriction of the function

$$u(t) = \begin{cases} t^{-1} - 2i, & t \in \big((2i+1)^{-1}, (2i)^{-1}\big], \\ 2i - t^{-1}, & t \in \big((2i)^{-1}, (2i-1)^{-1}\big], \end{cases} \quad i = 1,2,\dots$$

(it is the only solution of the differential equation $\dot{x}(t) = f(t)$ with initial condition $x(1/2) = 0$), and this function has essential discontinuity at the point $t = 0$.

In conclusion, we consider an example of differential inclusion (14) with the map F satisfying all the assumptions of Theorem 3; so, for this inclusion the Cauchy problem with any initial condition has a solution, and every solution can be continued. It is interesting to note that the mentioned map F is obtained by changing «a little» the map studied in Example 2, for which the condition (18) does not hold and a solution of the corresponding Cauchy problem does not exist.

Example 4. Take any $\varkappa \in (0,1)$ and let the map $F : \mathbb{R} \times \mathbb{R} \to \operatorname{clos}(\mathbb{R})$ be given by

$$F(t,x) = \begin{cases} \{\, |t|^{-\frac{1}{2}}(x + 3 - \varkappa) \,\}, & \text{for } t \neq 0,\ x \in (-\infty, 0), \\ \{\, |t|^{-\frac{1}{2}}(x - 1);\ |t|^{-\frac{1}{2}}(x + 3 - \varkappa) \,\}, & \text{for } t \neq 0,\ x \in [0, \varkappa), \\ \{ |t|^{-\frac{1}{2}}(x - 1) \,\}, & \text{for } t \neq 0,\ x \in [\varkappa, \infty), \\ \{0\}, & \text{for } t = 0,\ x \in \mathbb{R} \end{cases}$$

(see Figure 3). Show that for any initial point (t_0, α_0), problem (14), (15) has a solution on the whole real line. We verify the conditions of Theorem 2; it is enough to consider the initial values $(t_0, 0)$ and $(t_0, \varkappa)$, $t_0 \in \mathbb{R}$ (for any other point the conditions are, obviously, satisfied).

Let $\alpha_0 = 0$. We restrict ourselves to the case of $t_0 = 0$, for $t_0 \neq 0$ the calculations will be very much the same. Put $T = 16^{-1}\varkappa^2$, $\sigma \in (0,1]$, $\theta(t) = |t|^{-\frac{1}{2}}$, $R_0(t) = 2|t|^{-\frac{1}{2}}$. Then

$$x_*(t) = \begin{cases} -2\,|t|^{\frac{1}{2}}, & t \in [-T, 0), \\ 2\,t^{\frac{1}{2}}, & t \in [0, T]; \end{cases} \qquad F(t, x_*(t)) = \begin{cases} \{-2 + |t|^{-\frac{1}{2}}(3 - \varkappa)\}, & t \in [-T, 0), \\ \{2 - t^{-\frac{1}{2}};\ 2 + t^{-\frac{1}{2}}(3 - \varkappa)\}, & t \in (0, T]. \end{cases}$$

Show that estimate (18) takes place. Find $\varrho(t) \doteq \varrho_{\mathbb{R}}\big(\theta(t), F(t, x_*(t))\big)$. For $t \in [-T, 0)$,

$$\varrho(t) = -2 + |t|^{-\frac{1}{2}}(3 - \varkappa) - |t|^{-\frac{1}{2}} = -2 + |t|^{-\frac{1}{2}}(2 - \varkappa).$$

For $t \in (0, T]$, one gets $\varrho(t) = \min\{\varrho_1(t), \varrho_2(t)\}$, where

$$\varrho_1(t) \doteq 2 + t^{-\frac{1}{2}}(3 - \varkappa) - t^{-\frac{1}{2}} = 2 + t^{-\frac{1}{2}}(2 - \varkappa);$$
$$\varrho_2(t) \doteq t^{-\frac{1}{2}} - \big(2 - t^{-\frac{1}{2}}\big) = -2 + t^{-\frac{1}{2}}.$$

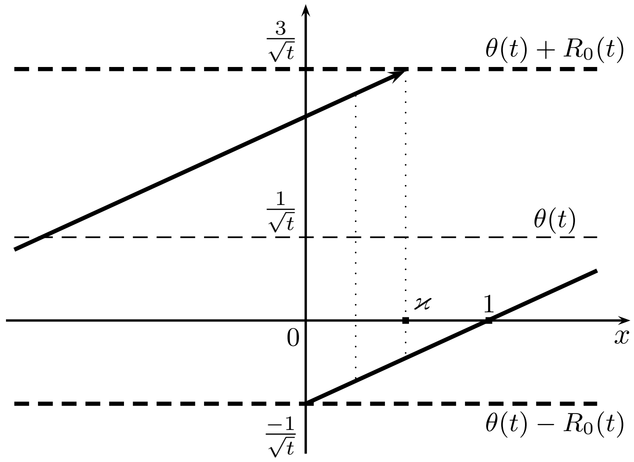

Fig 3. The graphs of the functions $x \mapsto F(t,x)$ and $x \to \widetilde{F}(t,x)$ for $t>0$ in Example 4

Since $T = 16^{-1}\varkappa^2$, it follows that $\varrho(t) = 2 + t^{-\frac{1}{2}}(2-\varkappa)$, $t \in (0,T]$. Thus,

$$\operatorname*{ess\,sup}_{t\in[-T,T]} \frac{\varrho(t)}{R_0(t)} = \sup\left\{ \operatorname*{ess\,sup}_{t\in[-T,0)} \frac{-2+|t|^{-\frac{1}{2}}(2-\varkappa)}{2|t|^{-\frac{1}{2}}},\ \operatorname*{ess\,sup}_{t\in[0,T]} \frac{2+t^{-\frac{1}{2}}(2-\varkappa)}{2t^{-\frac{1}{2}}} \right\} =$$

$$= \sup\left\{ \operatorname*{ess\,sup}_{t\in[-T,0)} \left(-|t|^{\frac{1}{2}} + 1 - \frac{\varkappa}{2}\right),\ \operatorname*{ess\,sup}_{t\in[0,T]} \left(t^{\frac{1}{2}} + 1 - \frac{\varkappa}{2}\right) \right\} = \sup\left\{1-\frac{\varkappa}{2}, 1-\frac{\varkappa}{4}\right\} = 1 - \frac{\varkappa}{4} < 1.$$

Next, for a.e. $t \in [-T,T]$, the map $x \mapsto \widetilde{F}(t,x)$ is, obviously, Lipschitz for $|x| \leqslant \sigma$; the other conditions of Theorem 2 are also fulfilled.

Now, let $\alpha_0 = \varkappa$. Here again we consider only the case of $t_0 = 0$ (for non-zero values of t_0 the reasoning will be analogous). As before, choose $T = 16^{-1}\varkappa^2$, $\sigma \in (0,1]$, $\theta(t) = |t|^{-\frac{1}{2}}$, $R_0(t) = 2|t|^{-\frac{1}{2}}$, and define on $[-T,T]$ the function

$$x_*(t) = \varkappa + \int_0^t \theta(s)\,ds = \begin{cases} \varkappa - 2\,|t|^{\frac{1}{2}}, & t \in [-T,0], \\ \varkappa + 2\,t^{\frac{1}{2}}, & t \in (0,T]. \end{cases}$$

Let us verify estimate (18); the other assumptions of Theorem 2 take place.

According to the definition of the map F, we get

$$F(t,x_*(t)) = \begin{cases} \left\{-2+|t|^{-\frac{1}{2}}(\varkappa-1);\ -2+3|t|^{-\frac{1}{2}}\right\}, & t \in [-T,0), \\ \left\{2+t^{-\frac{1}{2}}(\varkappa-1)\right\}, & t \in (0,T]. \end{cases}$$

Then

$$\varrho(t) \doteq \varrho_{\mathbb{R}}\big(\theta(t), F(t,x_*(t))\big) = \begin{cases} 2+|t|^{-\frac{1}{2}}(2-\varkappa), & t \in [-T,0), \\ -2+t^{\frac{1}{2}}(2-\varkappa), & t \in (0,T], \end{cases}$$

from which it follows that

$$\operatorname*{ess\,sup}_{t\in[-T,T]} \frac{\varrho(t)}{R_0(t)} = \sup\left\{ \operatorname*{ess\,sup}_{t\in[-T,0)} \left(|t|^{\frac{1}{2}} + 1 - \frac{\varkappa}{2}\right), \operatorname*{ess\,sup}_{t\in[0,T]} \left(-t^{\frac{1}{2}} + 1 - \frac{\varkappa}{2}\right) \right\} = 1 - \frac{\varkappa}{4} < 1\,.$$

Finally, to check condition (25) of Theorem 3 it suffices to set $M(t) = 3\,|t|^{-\frac{1}{2}}$ for all $t \neq 0$. So, according to Theorem 3, any solution $\bar{x}$ defined on a compact interval can be continued up to the boundary of any domain D. It is only left to note that, since every solution of the given inclusion should satisfy the estimate $|x(t)| \leqslant |x(t_0)| + 6\,|t|^{\frac{1}{2}}$, the solution $\bar{x}$ is the restriction of a solution defined on the whole of $\mathbb{R}$.

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Received 01.02.2013

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О неподвижных точках многозначных отображений метрических пространств и дифференциальных включениях

Ключевые слова: многозначное отображение, неподвижная точка, дифференциальное включение.

УДК 515.126.83+515.126.4+517.911.5

В работе предложено обобщение теоремы Надлера о неподвижных точках для многозначных отображений действующих в метрических пространствах. Полученный результат позволяет изучать существование неподвижных точек у многозначных отображений, которые не обязательно являются сжимающими, и даже непрерывными, относительно метрики Хаусдорфа, и образами которых могут быть произвольные множества соответствующего метрического пространства. Упомянутый результат можно использовать для исследования дифференциальных и функционально-дифференциальных уравнений с разрывами, а также включений, правые части которых порождены многозначными отображениями с произвольными образами. Во второй части работы, в качестве приложения, получены условия существования и продолжаемости решений задачи Коши для дифференциального включения с некомпактной правой частью в пространстве $\mathbb{R}^n$.

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Поступила в редакцию 01.02.2013

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