VESTNIK UDMURTSKOGO UNIVERSITETA. MATEMATIKA. MEKHANIKA. KOMP'YUTERNYE NAUKI
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INTEGRATION OF THE KORTEWEG–DE VRIES EQUATION WITH LOADED TERMS AND A SELF-CONSISTENT SOURCE IN THE CLASS OF RAPIDLY DECREASING FUNCTIONS

In this paper, we solve the Cauchy problem for the Korteweg–de Vries equation with loaded terms and a self-consistent source in the class of rapidly decreasing functions. To solve this problem, the method of the inverse scattering problem is used. The evolution of the scattering data of the self-adjoint Sturm–Liouville operator, whose coefficient is a solution of the Korteweg–de Vries equation with loaded terms and a self-consistent source, is obtained. Examples are given to illustrate the application of the obtained results.

Keywords: loaded Korteweg–de Vries equation, Jost solutions, inverse scattering problem, Gelfand–Levitan–Marchenko integral equation, evolution of the scattering data.

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Introduction

Inverse spectral problems play a significant role in the integration of some important evolutionary equations of mathematical physics and have their origins in the work of Gardner, Greene, Kruskal and Miura [1]. They managed to find a global solution of the Cauchy problem for the Korteweg–de Vries (KdV) equation by reducing it to the inverse scattering problem for a self-adjoint Sturm–Liouville operator. This inverse scattering problem was first solved in the work of L. D. Faddeev [2], then in the works of V. A. Marchenko [3], B. M. Levitan [4], etc. In the article [5], Lax showed the universality of the inverse scattering method and generalized the KdV equation, introducing the concept of the higher KdV equation. This theory is described in more detail in the monographs [3, 4, 6–8].

In modern scientific literature, integrable nonlinear evolution equations with sources attract much attention. They have important applications (see [9–12]) in plasma physics, hydrodynamics, solid state physics, etc. For example, the KdV equation, which contains a source, was considered in [10]. Such equations can describe the interaction of long and short capillary–gravity waves [11]. In addition, integrability of various nonlinear equations with sources was studied in [13–22].

In works [23, 24], the geophysical Korteweg–de Vries equation

$$u_t - \omega_0 u_x + \frac{3}{2} u u_x + \frac{1}{6} u_{xxx} = 0,$$

was studied. Here ω_0 is Coriolis effect parameter, u is function with respect to x, t. In particular, in [24], soliton solutions of the geophysical Korteweg–de Vries equation were found using the Hirota bilinear method.

Loaded differential equations in the literature are usually called equations containing in the coefficients or the right side any functionals of the solution, in particular, the values of the solution or its derivatives on manifolds of lower dimension. Among the works devoted to loaded equations, the works [25–31] and others should be noted.

In this paper, we study the loaded KdV equation with a source of the form:

$$\begin{gathered} u_t + P(u(x_0,t))(u_{xxx} - 6uu_x) + Q(u(x_1,t))u_x + 4\sum_{m=1}^{N} \frac{\partial}{\partial x}(|\varphi_m(x,t)|^2) = 0, \\ -\varphi_m'' + u(x,t)\varphi_m = \lambda_m(t)\varphi_m, \quad m = \overline{1,\,N}, \end{gathered} \tag{0.1}$$

where $P(u(x_0,t))$ and $Q(u(x_1,t))$ are some polynomials in $u(x_0,t)$ and $u(x_1,t)$ respectively, and x_0, $x_1 \in \mathbb{R}$. The equation (0.1) is considered under the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R},$$

where the initial function $u_0(x)$ has the following properties.

(1)
$$\int_{-\infty}^{\infty} (1 + |x|)|u_0(x)|dx < \infty; \tag{0.2}$$

(2) The operator $L(0) := -\frac{d^2}{dx^2} + u_0(x)$, $x \in \mathbb{R}$, is equal to N negative eigenvalues $\lambda_1(0)$, $\lambda_2(0), \ldots, \lambda_N(0)$.

It is assumed that
$$\int_{-\infty}^{\infty} |\varphi_m(x,t)|^2 dx = A_m(t), \quad m = \overline{1,\,N},$$
where $A_m(t) > 0$, $m = \overline{1,\,N}$, are given positive continuous functions.

Suppose that the function $u(x,t)$ has sufficient smoothness and tends to its limits rather quickly as $x \to \pm\infty$, i. e.,

$$\int_{-\infty}^{\infty} (1 + |x|)\Big(|u(x,t)| + \Big|\frac{\partial^j u(x,t)}{\partial x^j}\Big|\Big)\, dx < \infty, \quad j = 1, 2, 3. \tag{0.3}$$

In this paper, we propose an algorithm for constructing a solution $u(x,t)$, $\varphi_m(x,t)$, $x \in \mathbb{R}$, $t > 0$, $m = \overline{1,\,N}$, of problem (0.1)–(0.3) by the method of the inverse scattering problem for the Sturm–Liouville operator.

It should be noted that the Kortoweg–de Vries equations with variable coefficients also occur in applied mechanics. For example, in the works of Lugovtsov [32, 33], the system of equations describing the propagation of one-dimensional nonlinear waves in an inhomogeneous gas–liquid medium is reduced to a single equation of the form

$$u_\tau + \alpha(\tau)uu_\eta + \beta(\tau)u_{\eta\eta\eta} - \mu(\tau)u_{\eta\eta} + \left[\frac{k}{2\tau} + \sigma(\tau)\right]u = 0.$$

In particular, for $\mu = 0$, $k = 1$, $\sigma = 0$ it is shown that, under certain conditions, cylindrical waves can exist in the form of solitons.

§ 1. Preliminaries

Consider the Sturm–Liouville operator

$$L(0)y := -y'' + u_0(x)y = k^2 y, \quad x \in \mathbb{R}, \tag{1.1}$$

where the potential $u_0(x)$ satisfies the condition (0.2). In this section, information necessary for further presentation concerning the direct and inverse scattering problems for the equation (1.1) will be given. Denote by $f(x,k)$ and $g(x,k)$ the Jost solutions of the equation (1.1) with asymptotics

$$\begin{aligned} f(x,k) &= e^{ikx} + o(1), \quad x \to \infty \quad (\operatorname{Im} k = 0); \\ g(x,k) &= e^{-ikx} + o(1), \quad x \to -\infty \quad (\operatorname{Im} k = 0). \end{aligned} \tag{1.2}$$

Under the condition (0.2), such solutions exist and are uniquely determined by the asymptotics (1.2).

Solutions $f(x,k)$, $g(x,k)$ have the representations

$$\begin{aligned} f(x,k) &= e^{ikx} + \int_x^{\infty} K^+(x,z)e^{ikz}\,dz, \\ g(x,k) &= e^{-ikx} + \int_{-\infty}^{x} K^-(x,z)e^{-ikz}\,dz, \end{aligned} \tag{1.3}$$

where kernels $K^+(x,z)$, $K^-(x,z)$ are real functions related to the potential $u_0(x)$ as follows:

$$u_0(x) = \mp 2\frac{d}{dx}K^{\pm}(x,x).$$

For real k, pairs of functions $\{f(x,k),\, f(x,-k)\}$ and $\{g(x,k),\, g(x,-k)\}$ are pairs of linearly independent solutions to the equation (1.1), so

$$\begin{aligned} f(x,k) &= b(k)g(x,k) + a(k)g(x,-k), \\ g(x,k) &= -b(-k)f(x,k) + a(k)f(x,-k). \end{aligned}$$

Functions $r^+(k) = -\frac{b(-k)}{a(k)}$, $r^-(k) = \frac{b(k)}{a(k)}$ are called reflection coefficients (right and left, respectively). The coefficients $a(k)$, $b(k)$ and $r^+(k)$ have the following properties (see [4, p. 121]).

I. For real $k \neq 0$

$$a(-k) = \overline{a(k)}, \quad b(-k) = \overline{b(k)}, \quad |a(k)|^2 = 1 + |b(k)|^2, \tag{1.4}$$

$$\begin{aligned} a(k) &= \frac{1}{2ik}W\{g(x,k),\, f(x,k)\}, \quad b(k) = \frac{1}{2ik}W\{f(x,k),\, g(x,-k)\}, \\ a(k) &= 1 + O\left(\frac{1}{k}\right), \quad b(k) = O\left(\frac{1}{k}\right), \quad k \to \pm\infty, \end{aligned} \tag{1.5}$$

where

$$W\{f(x,k),\, g(x,k)\} \equiv f(x,k)g'(x,k) - f'(x,k)g(x,k).$$

II. The function $a(k)$ extends analytically to the half-plane $\operatorname{Im} k > 0$ and there it has a finite number of zeros $k_n = i\chi_n$, $\chi_n > 0$, $n = 1, 2, 3, \ldots, N$, these zeros are simple, and $\lambda_n = -\chi_n^2$ are the eigenvalues of the operator $L(0)$. In addition, following relation holds

$$g(x, i\chi_j) = B_j f(x, i\chi_j), \quad j = 1, 2, \ldots, N. \tag{1.6}$$

III. For real $k \neq 0$ the function $r^+(k)$ is continuous,

$$\begin{aligned} \overline{r^+(k)} = r^+(-k), \quad |r^+(k)| < 1, \quad r^+(k) = o\left(\frac{1}{k}\right), \quad |k| \to \infty, \\ k^2\big[1 - |r^+(k)|^2\big]^{-1} = O(1), \quad |k| \to 0. \end{aligned}$$

IV. The function $k(a(k)-1)$, where $a(k)$ is defined by the formula

$$a(k)=\prod_{j=1}^{N}\frac{k-i\chi_j}{k+i\chi_j}\exp\left\{-\frac{1}{2i\pi}\int_{-\infty}^{\infty}\frac{\ln(1-|r^+(\xi)|^2)}{\xi-k}\,d\xi\right\},\quad \operatorname{Im}k>0,$$

is continuous and bounded in $\operatorname{Im}k\geq 0$ and

$$(a(k))^{-1}=O(1),\quad |k|\to 0,\quad \operatorname{Im}k\geq 0,$$
$$\lim_{k\to 0}ka(k)(r^+(k)+1)=0,\quad \operatorname{Im}k=0.$$

V. The functions $R^{\pm}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}r^{\pm}(k)e^{\pm ikx}\,dk$ for each $a>-\infty$ satisfy the condition $(1+|x|)|R^{\pm}(\pm x)|\in L^1(a,\infty)$. The set $\{r^+(k),\chi_1,\chi_2,\dots,\chi_N,B_1,B_2,\dots,B_N\}$ is called the scattering data for the operator $L(0)$. The direct scattering problem is to determine the scattering data from the potential $u_0(x)$, and the inverse problem is to reconstruct the potential $u_0(x)$ from the scattering data of equation (1.1).

The kernel $K^+(x,y)$ in representation (1.3) is a solution to the Gelfand–Levitan–Marchenko integral equation

$$K^+(x,y)+F^+(x+y)+\int_x^{\infty}K^+(x,z)F^+(z+y)\,dz=0\quad (y>x),$$

where

$$F^+(x)=\sum_{j=1}^{N}\alpha_j^+e^{-\chi_jx}+\frac{1}{2\pi}\int_{-\infty}^{\infty}r^+(k)e^{ikx}\,dk,\quad \alpha_j^+=\frac{B_j}{i\frac{da(z)}{dz}\Big|_{z=i\chi_j}},\tag{1.7}$$

and $a(z)$ is the analytic continuation of the function $a(k)$ to the upper half-plane $\operatorname{Im}k>0$.

Lemma 1. *Let the functions $y(x,\lambda)$ and $z(x,\mu)$ be solutions of the following equations*

$$Ly(x,\lambda)=\lambda y(x,\lambda),\qquad Lz(x,\mu)=\mu z(x,\lambda)$$

respectively. In this case, the following equalities are true

$$\frac{d}{dx}W\{y(x,\lambda),z(x,\mu)\}=(\lambda-\mu)y(x,\lambda)z(x,\mu).$$

This lemma is proved by direct verification.

The following theorem is valid (see [4, p. 231]).

Theorem 1. *Specifying the scattering data uniquely determines the potential $u_0(x)$.*

§ 2. Evolution of scattering data

Consider the following equation

$$u_t+P(u(x_0,t))(u_{xxx}-6uu_x)=G(x,t),\tag{2.1}$$

where

$$G(x,t)=-Q(u(x_1,t))u_x-4\sum_{m=1}^{N}\frac{\partial}{\partial x}(|\varphi_m|^2).\tag{2.2}$$

For the equation (2.1), we will look for the Lax pair [5] in the form

$$
\begin{gathered}
-\phi_{xx} + (u(x,t) - k^2)\phi = 0, \\
\phi_t = P(u(x_0,t))(-u_x + 4ik^3)\phi + P(u(x_0,t))(2u + 4k^2)\phi_x + F(x,t).
\end{gathered} \tag{2.3}
$$

Using the identity $\phi_{xxt} = \phi_{txx}$, and taking into account the equalities (2.1)–(2.3), we get

$$
-F_{xx} + (u(x,t) - k^2)F = -G(x,t)\phi. \tag{2.4}
$$

Putting $\phi(x,t) = g(x,k,t)$, we look for the solution of the equation (2.4) in the form

$$
F(x,t) = B(x)g(x,k,t) + C(x)g(x,-k,t).
$$

Then to determine $B(x)$ and $C(x)$ we obtain the following system of equations

$$
\begin{gathered}
B'(x)g(x,k,t) + C'(x)g(x,-k,t) = 0, \\
B'(x)g'(x,k,t) + C'(x)g'(x,-k,t) = G(x,t)g(x,k,t),
\end{gathered}
$$

the solution of which has the form

$$
\begin{aligned}
B(x) &= -\frac{1}{2ik}\int_{-\infty}^{x} g(s,k,t)g(s,-k,t)G(s,t)\,ds, \\
C(x) &= \frac{1}{2ik}\int_{-\infty}^{x} g^2(s,k,t)G(s,t)\,ds.
\end{aligned}
$$

Therefore, using the expression (2.2), the equation (2.3) can be rewritten as follows

$$
\begin{aligned}
\frac{\partial g(x,k,t)}{\partial t} &= P(u(x_0,t))(-u_x + 4ik^3)g(x,k,t) + P(u(x_0,t))(2u + k^2)\frac{\partial g(x,k,t)}{\partial x} + \\
&+ \frac{Q(u(x_1,t))g(x,k,t)}{2ik}\int_{-\infty}^{x} g(s,k,t)\overline{g(s,k,t)}u_s(s,t)\,ds - \\
&- \frac{Q(u(x_1,t))\overline{g(x,k,t)}}{2ik}\int_{-\infty}^{x} g^2(s,k,t)u_s(s,t)\,ds + \\
&+ \frac{4g(x,k,t)}{2ik}\int_{-\infty}^{x} g(s,k,t)\overline{g(s,k,t)}\sum_{m=1}^{N}\frac{\partial}{\partial x}(|\varphi_m(s,t)|^2)ds - \\
&- \frac{4\overline{g(x,k,t)}}{2ik}\int_{-\infty}^{x} g^2(s,k,t)\sum_{m=1}^{N}\frac{\partial}{\partial x}(|\varphi_m(s,t)|^2)\,ds.
\end{aligned} \tag{2.5}
$$

Passing in the equality (2.5) to the limit $x \to \infty$, due to (1.4), (1.5) and the asymptotics of the Jost solution, we derive

$$
\begin{aligned}
\frac{da(k,t)}{dt} &= \frac{Q(u(x_1,t))a(k,t)}{2ik}\int_{-\infty}^{\infty} g(s,k,t)\overline{g(s,k,t)}u_s(s,t)\,ds + \\
&+ \frac{Q(u(x_1,t))b(k,t)}{2ik}\int_{-\infty}^{\infty} g^2(s,k,t)u_s(s,t)\,ds - \\
&- \frac{4a(k,t)}{2ik}\int_{-\infty}^{\infty} g(s,k,t)\overline{g(s,k,t)}\sum_{m=1}^{N}\frac{\partial}{\partial x}(|\varphi_m(s,t)|^2)\,ds - \\
&- \frac{4b(k,t)}{2ik}\int_{-\infty}^{\infty} g^2(s,k,t)\sum_{m=1}^{N}\frac{\partial}{\partial x}(|\varphi_m(s,t)|^2)\,ds,
\end{aligned} \tag{2.6}
$$

$$
\begin{aligned}
&\frac{db(-k,t)}{dt} = 8ik^3 P(u(x_0,t))b(-k,t) + \\
&+ \frac{Q(u(x_1,t))b(-k,t)}{2ik} \int_{-\infty}^{\infty} g(s,k,t)\overline{g(s,k,t)}u_s(s,t)\,ds + \\
&+ \frac{Q(u(x_1,t))a(-k,t)}{2ik} \int_{-\infty}^{\infty} g^2(s,k,t)u_s(s,t)\,ds - \\
&- \frac{4b(-k,t)}{2ik} \int_{-\infty}^{\infty} g(s,k,t)\overline{g(s,k,t)} \sum_{m=1}^{N} \frac{\partial}{\partial x}(|\varphi_m(s,t)|^2)\,ds - \\
&- \frac{4a(-k,t)}{2ik} \int_{-\infty}^{\infty} g^2(s,k,t) \sum_{m=1}^{N} \frac{\partial}{\partial x}(|\varphi_m(s,t)|^2)\,ds.
\end{aligned}
\tag{2.7}
$$

Now consider the following lemma:

Lemma 2. *The following equalities are true*

$$
\int_{-\infty}^{\infty} \frac{\partial}{\partial x}(|\varphi_m(x,t)|^2)g^2(x,k,t)\,dx = 0, \tag{2.8}
$$

$$
\int_{-\infty}^{\infty} \frac{\partial}{\partial x}(|\varphi_m(x,t)|^2)g(x,k,t)\overline{g(x,k,t)}\,dx = 0. \tag{2.9}
$$

P r o o f. Let's prove equation (2.8), for this firstly we divide equation (2.8) into two, that is, as follows

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \frac{\partial}{\partial x}(|\varphi_m(x,t)|^2)g^2(x,k,t)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial}{\partial x}(|\varphi_m(x,t)|^2)g^2(x,k,t)\,dx + \\
&+ \frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial}{\partial x}(|\varphi_m(x,t)|^2)g^2(x,k,t)\,dx = \frac{1}{2}g^2(x,k,t)|\varphi_m(x,t)|^2\Big|_{-\infty}^{\infty} - \\
&- \frac{1}{2}\int_{-\infty}^{\infty} |\varphi_m(x,t)|^2 2g(x,k,t)g'(x,k,t)\,dx + \frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial}{\partial x}|\varphi_m(x,t)|^2 g^2(x,k,t)\,dx = \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \left(\frac{\partial}{\partial x}(|\varphi_m(x,t)|^2)g^2(x,k,t) - |\varphi_m(x,t)|^2 2g(x,k,t)\overline{g(x,k,t)}\right)dx = \\
&= \frac{1}{2}\int_{-\infty}^{\infty} [(\varphi_m'\bar{\varphi}_m + \varphi_m\bar{\varphi}_m')g^2 - 2gg'\varphi_m\bar{\varphi}_m]\,dx = \\
&= \frac{1}{2}\int_{-\infty}^{\infty} (g\bar{\varphi}_m W\{g,\varphi_m\} + \varphi_m g W\{g,\bar{\varphi}_m\})\,dx.
\end{aligned}
$$

From this we generate the following expression using lemma 1,

$$
\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{\infty} (g\bar{\varphi}_m W\{g,\varphi_m\} + \varphi_m g W\{g,\bar{\varphi}_m\})\,dx = \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \left(\frac{1}{\lambda-\lambda_m}\left(\frac{d}{dx}W\{g,\bar{\varphi}_m\}\right)\cdot W\{g,\varphi_m\} + \frac{1}{\lambda-\lambda_m}\left(\frac{d}{dx}W\{g,\varphi_m\}\right)W\{g,\bar{\varphi}_m\}\right)dx = \\
&= \frac{1}{2(\lambda-\lambda_m)}\int_{-\infty}^{\infty} \frac{d}{dx}\{W\{g,\bar{\varphi}_m\}W\{g,\varphi_m\}\}\,dx = \frac{1}{2(\lambda-\lambda_m)}W\{g,\bar{\varphi}_m\}W\{g,\varphi_m\}\Big|_{-\infty}^{\infty} = 0.
\end{aligned}
$$

The equation (2.9) is proved in a similar way. □

Multiplying (2.7) by $a(k,t)$ and subtracting (2.6) multiplied by $b(-k,t)$, according to (1.7), we get

$$
\frac{dr^+(k,t)}{dt} = 8ik^3 P(u(x_0,t))r^+(k,t) - \frac{Q(u(x_1,t))}{2ika^2(k,t)}\int_{-\infty}^{\infty} g^2(s,k,t)u_s(s,t)\,ds.
$$

Now we calculate the integral on the right side of the previous equality. To do this, we use the formula (1.4) and have

$$\int_{-\infty}^{\infty} g^2(s,k,t)u_s(s,t)\,ds = g^2(s,k,t)u(s,t)\big|_{-\infty}^{\infty} -$$
$$-2\int_{-\infty}^{\infty} (g''(s,k,t)+k^2g(s,k,t))g'(s,k,t)\,ds =$$
$$= -2\int_{-\infty}^{\infty} g'(s,k,t)g''(s,k,t)ds - 2k^2\int_{-\infty}^{\infty} g(s,k,t)g'(s,k,t)\,ds =$$
$$-\int_{-\infty}^{\infty} (g'^2(s,k,t))'\,ds - k^2\int_{-\infty}^{\infty} (g^2(s,k,t))'\,ds =$$
$$= -g'^2(s,k,t)\big|_{-\infty}^{\infty} - k^2\, g^2(s,k,t)\big|_{-\infty}^{\infty} = k^2a^2(k,t)e^{-2ikx} + 2k^2a(k,t)b(-k,t) +$$
$$+ k^2b^2(-k,t)e^{2ikx} - k^2e^{-2ikx} - k^2a^2(k,t)e^{-2ikx} + 2k^2a(k,t)b(-k,t) -$$
$$- k^2b^2(-k,t)e^{2ikx} + k^2e^{-2ikx} = 4k^2a(k,t)b(-k,t).$$

According to this and the equality (2.6), we have

$$\frac{da(k,t)}{dt} = 0.$$

Therefore, we deduce that

$$\frac{d\lambda_j(t)}{dt} = 0, \tag{2.10}$$

$$\frac{dr^+(k,t)}{dt} = \big(8ik^3P(u(x_0,t)) - 2ikQ(u(x_1,t))\big)r^+(k,t). \tag{2.11}$$

Now we turn to finding the evolution of the normalization numbers B_n, $n = 1,2,\ldots,N$, corresponding to the eigenvalues λ_n, $n = 1,2,3,\ldots,N$. To do this, we rewrite the equality (2.5) in the following form

$$\frac{\partial g(x,k,t)}{\partial t} = P(u(x_0,t))(-u_x+4ik^3)g + P(u(x_0,t))(2u(x,t)+4k^2)\frac{\partial g(x,k,t)}{\partial x} +$$
$$+ \frac{Q(u(x_1,t))g(x,k,t)}{2ik}[g(x,k,t)\overline{g(x,k,t)}u(x,t) -$$
$$- \int_{-\infty}^{x} u(s,t)(g'(s,k,t)\overline{g(s,k,t)} + g(s,k,t)\overline{g}'(x,k,t))\,ds] -$$
$$- \frac{Q(u(x_1,t))\overline{g(x,k,t)}}{2ik}\Big[g^2(x,k,t)u(x,t) - \int_{-\infty}^{x} 2g'(s,k,t)g(s,k,t)u(s,t)\,ds\Big] +$$
$$+ \frac{4g(x,k,t)}{2ik}\int_{-\infty}^{x} g(s,k,t)\overline{g(s,k,t)}\sum_{m=1}^{N}\frac{\partial}{\partial x}(|\varphi_m(x,t)|^2)\,ds -$$
$$- \frac{4\overline{g(x,k,t)}}{2ik}\int_{-\infty}^{x} g^2(s,k,t)\sum_{m=1}^{N}\frac{\partial}{\partial x}(|\varphi_m(x,t)|^2)\,ds =$$
$$= P(u(x_0,t))(-u_x+4ik^3)g(x,k,t) + P(u(x_0,t))(2u+4k^2)\frac{\partial g(x,k,t)}{\partial x} -$$
$$- Q(u(x_1,t))g'(x,k,t) - ikQ(u(x_1,t))g(x,k,t) - 2g(x,k,t)\int_{-\infty}^{x} |\varphi_m(x,t)|^2\,ds.$$

Using the equality (1.6), setting $k = k_n$, taking into account the asymptotics of the Jost solution at $x \to +\infty$ and equating the coefficients at $e^{-\chi_n x}$, find an analogue of the Gardner–Greene–Kruskal–Miura equations

$$\frac{dB_n(t)}{dt} = \big(8\chi_n^3 P(u(x_0,t)) + 2\chi_n Q(u(x_1,t)) - 2A_n(t)\big)B_n(t), \quad n = \overline{1,N}. \tag{2.12}$$

Thus, the following theorem has been proved.

Theorem 2. *If the functions $u(x,t)$, $\varphi_m(x,t)$, $m = \overline{1,N}$, $x \in \mathbb{R}$, $t > 0$, are a solution to the problem (0.1)–(0.3), then the scattering data $\{r^+(k,t),\ \lambda_n(t),\ B_n(t), n = \overline{1,N}\}$ of the operator $L(t)$ with potential $u(x,t)$ satisfy the differential equations (2.10), (2.11) and (2.12).*

Remark 1. Consider the kernel of the Gelfand–Levitan–Marchenko integral equation

$$F^+(x,t) = \sum_{j=1}^{N} \alpha_j^+(t)e^{-\chi_j x} + \frac{1}{2\pi}\int_{-\infty}^{\infty} r^+(k,t)e^{ikx}\,dk$$

with scattering data from Theorem 2. Then the data

$$\{r^+(k,t),\ \chi_1(t),\ \ldots,\ \chi_N(t),\ B_1(t), \ldots,\ B_n(t)\}$$

satisfy conditions **I**–**V**. Therefore, according to Theorem 1, the potential $u(x,t)$ in the operator $L(t)$ is uniquely determined.

Remark 2. The relations obtained completely determine the evolution of the scattering data for the operator $L(t)$ and thus make it possible to apply the method of the inverse scattering problem to solve the problem (0.1)–(0.3).

Let a function $u_0(x)$ satisfying condition (0.2) be given. Then the solution of problem (0.1)–(0.3)is found according to the following algorithm.

1. We solve the direct scattering problem with the initial function $u_0(x)$ and obtain the scattering data $\{r^+(k),\ \chi_n,\ B_n,\ n = \overline{1,N}\}$ for the operator $L(0)$.
2. Using the results of Theorem 2, we find the scattering data for $t > 0$

$$\{r^+(k,t),\ \chi_n(t),\ B_n(t),\ n = \overline{1,N}\}.$$

3. Using the method based on the Gelfand–Levitan–Marchenko integral equation, we solve the inverse scattering problem, i. e. find $u(x,t)$ from the scattering data for $t > 0$, obtained in the previous step. After that, it is easy to find the solution $\varphi_m(x,t)$ of the equation $L(t)\varphi_m(x,t) := -\varphi_m''(x,t) + u(x,t)\varphi_m(x,t) = \lambda_m\varphi_m(x,t)$, $m = 1, 2, \ldots, N$.

Let's give a couple of examples illustrating the described algorithm.

Example 1. Consider the following problem

$$u_t + \beta(t)u(0,t)(u_{xxx} - 6uu_x) + \gamma(t)u(\ln 2,t)u_x + 4\frac{\partial}{\partial x}|\varphi_1|^2 = 0$$
$$-\varphi_1''(x,t) + u(x,t)\varphi_1(x,t) = \lambda(t)\varphi_1(x,t),$$
$$u(x,0) = -\frac{2}{\operatorname{ch}^2 x}, \quad x \in \mathbb{R},$$

where

$$\int_{-\infty}^{\infty} |\varphi_1(x,t)|^2\,dx = A_1(t) = \frac{1}{2(t+1)^2},$$

$$\beta(t) = -\left(\frac{\sqrt{t+1}}{8} + \frac{1}{8\sqrt{(t+1)^3}}\right)^2, \qquad \gamma(t) = \left(\frac{\sqrt{\frac{t}{2}+1}}{(t+1)^2} + \frac{\sqrt{t+2}}{4\sqrt{2}}\right)^2.$$

It is easy to find the scattering data for the operator $L(0)$:

$$\lambda(0) = -1, \quad r^+(k,0) = 0, \quad B_1(0) = 1.$$

By Theorem 2, we have

$$\lambda(t) = \lambda(0) = -1, \quad r^+(k,t) = 0, \quad B_1(t) = e^{\mu(t)},$$

where

$$\mu(t) = 8\int_0^t \beta(\tau)u(0,\tau)\,d\tau + 2\int_0^t \gamma(\tau)u(\ln 2,\tau)\,d\tau - 2\int_0^t A_1(t)\,dt.$$

Substituting these data into the formula (1.7), we find the kernel

$$F_+(x,t) = 2e^{-x+\mu(t)}$$

of the integral equation of Gelfand–Levitan–Marchenko. Further, solving the following integral equation

$$K_+(x,y,t) + 2e^{\mu(t)}\cdot e^{-(x+y)} + 2e^{\mu(t)}\cdot e^{-y}\int_x^{\infty} K_+(x,s,t)e^{-s}\,ds = 0$$

we get

$$K_+(x,y,t) = -\frac{2e^{\mu(t)}e^{-(x+y)}}{1+e^{\mu(t)}e^{-2x}}.$$

From this we find the solution of the Cauchy problem

$$u(x,t) = -\frac{2}{\operatorname{ch}^2(x-\ln(t+1))}, \quad \varphi_1(x,t) = \frac{1}{e^x + (t+1)^2 e^{-x}}.$$

Example 2. Consider the following problem

$$u_t + \beta(t)u(0,t)(u_{xxx} - 6uu_x) + \gamma(t)u(\ln 2,t)u_x + 4\frac{\partial}{\partial x}(|\varphi_1|^2 + |\varphi_2|^2) = 0,$$
$$-\varphi_j''(x,t) + u(x,t)\varphi_j(x,t) = \lambda(t)\varphi_j(x,t), \quad j = 1,2,$$
$$u(x,0) = -\frac{6}{\operatorname{ch}^2 x}, \quad x \in \mathbb{R},$$

where

$$\beta(t) = \frac{(\operatorname{ch} 18t + 3\operatorname{ch} 14t)^2}{-72 - 24\operatorname{ch} 32t - 96\operatorname{ch} 4t},$$
$$\gamma(t) = -\frac{(e^{-4t} + e^{-32t})(\operatorname{ch}(\ln 8 - 18t) + 3\operatorname{ch}(\ln 2 - 14t))^2}{72 + 24\operatorname{ch}(\ln(6-32t)) + 96\operatorname{ch}(\ln 4 - 4t)},$$
$$A_1(t) = \frac{1}{2e^{4t}}, \quad A_2(t) = \frac{1}{2e^{32t}}.$$

In this case, the scattering data of the operator $L(0)$ have the following form:

$$\lambda_1(0) = -1, \quad \lambda_2(0) = -4, \quad r^+(k,0) = 0, \quad \alpha_1^+(0) = 6, \quad \alpha_2^+(0) = 12.$$

By Theorem 2, we have

$$\lambda_1(t) = -1, \quad \lambda_2(t) = -4; \quad r^+(k,t) = 0, \quad \alpha_1^+(t) = 6e^{\mu_1(t)}, \quad \alpha_2^+(t) = 12e^{\mu_2(t)},$$

where

$$\mu_1(t) = 8\int_0^t \beta(\tau)u(0,\tau)\,d\tau + 2\int_0^t \gamma(\tau)u(\ln 2,\tau)\,d\tau - 2\int_0^t (A_1(\tau)+A_2(\tau))\,d\tau,$$
$$\mu_2(t) = 64\int_0^t \beta(\tau)u(0,\tau)\,d\tau + 4\int_0^t \gamma(\tau)u(\ln 2,\tau)\,d\tau - 4\int_0^t (A_1(\tau)+A_2(\tau))\,d\tau.$$

Now, we find a two-soliton solution to the problem (2.11)–(2.12):

$$u(x,t) = -12\frac{3+\operatorname{ch}(4x-32t)+4\operatorname{ch}(2x-4t)}{(\operatorname{ch}(3x-18t)+3\operatorname{ch}(x-14t))^2},$$
$$\varphi_1(x,t) = \frac{e^{-x}-e^{32t-5x}}{1+3e^{32t-4x}+3e^{4t-2x}+e^{36t-6x}},$$
$$\varphi_2(x,t) = \frac{e^{-2x}+e^{4t-4x}}{1+3e^{32t-4x}+3e^{4t-2x}+e^{36t-6x}}.$$

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Umid Azadovich Hoitmetov, Candidate of Physics and Mathematics, Associate Professor, Department of Applied Mathematics and Mathematical Physics, Urgench State University, ul. H. Alimdjan, 14, Urgench, 220100, Uzbekistan.
ORCID: https://orcid.org/0000-0002-5974-6603
E-mail: x_umid@mail.ru

Temur Gafurjonovich Khasanov, PhD student, Urgench State University, ul. H. Alimdjan, 14, Urgench, 220100, Uzbekistan.
ORCID: https://orcid.org/0000-0001-7321-5764
E-mail: temur.xasanov.2018@mail.ru

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ВЕСТНИК УДМУРТСКОГО УНИВЕРСИТЕТА. МАТЕМАТИКА. МЕХАНИКА. КОМПЬЮТЕРНЫЕ НАУКИ
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У. А. Хоитметов, Т. Г. Хасанов

Интегрирование уравнения Кортевега–де Фриза с нагруженными членами и самосогласованным источником в классе быстроубывающих функций

Ключевые слова: нагруженное уравнение Кортевега–де Фриза, решения Йоста, обратная задача рассеяния, интегральное уравнение Гельфанда–Левитана–Марченко, эволюция данных рассеяния.

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В данной работе решается задача Коши для уравнения Кортевега–де Фриза с нагруженными членами и самосогласованным источником в классе быстроубывающих функций. Для решения этой задачи используется метод обратной задачи рассеяния. Получена эволюция данных рассеяния самосопряженного оператора Штурма–Лиувилля, коэффициент которого является решением уравнения Кортевега–де Фриза с нагруженными членами и самосогласованным источником. Приведены примеры, иллюстрирующие применение полученных результатов.

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Хоитметов Умид Азадович, к. ф.-м. н., доцент, кафедра прикладной математики и математической физики, Ургенчский государственный университет, 220100, Узбекистан, г. Ургенч, ул. Х. Алимджана, 14.
ORCID: https://orcid.org/0000-0002-5974-6603
E-mail: x_umid@mail.ru

Хасанов Темур Гафуржонович, PhD студент, Ургенчский государственный университет, 220100, Узбекистан, г. Ургенч, ул. Х. Алимджана, 14.
ORCID: https://orcid.org/0000-0001-7321-5764
E-mail: temur.xasanov.2018@mail.ru

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