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KERNEL DETERMINATION PROBLEM IN AN INTEGRO-DIFFERENTIAL EQUATION OF PARABOLIC TYPE WITH NONLOCAL CONDITION

In this paper, an inverse problem for a one-dimensional integro-differential heat equation is investigated with nonlocal initial-boundary and integral overdetermination conditions. We use the Fourier method and the Schauder principle to investigate the solvability of the direct problem. Further, the problem is reduced to an equivalent closed system of integral equations with respect to unknown functions. Existence and uniqueness of the solution of the integral equations are proved using a contractive mapping. Finally, using the equivalency, the existence and uniqueness of the classical solution is obtained.

Keywords: integro-differential equation, nonlocal initial-boundary problem, inverse problem, integral equation, Schauder principle.

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Introduction and setting up the problem

Nowadays in the theory of mathematical physics equations, investigations devoted to the direct and inverse problems take an important place. These problems arise in situations, when the structure of the mathematical model of the studying process is known and it is necessary to set the problems of determining the parameters of the mathematical model itself. Such problems include the problems of determining various kernels, leading and lower coefficients of equations, nonlocal initial and boundary conditions, and so on (see [1]).

Problems with nonlocal conditions for partial differential equations have been studied by many authors. In the paper [2–4], two initial-boundary problems for a hyperbolic equation with nonlocal conditions are considered and a method for proving the solvability of nonlocal problems with integral conditions of the first kind is proposed. The method is based on the equivalence of a nonlocal problem with an integral condition of the first kind and a nonlocal problem with an integral condition of the second kind in a special form. As a result, the unique existence of generalized solutions of both problems is proved.

An m-point nonlocal boundary value problem was studied for quasilinear differential equations of first order on the plane by David Devadze (see [5]) and the conditions for the existence and uniqueness of a generalized solution in the space were proved.

In the articles [6–8], the unique solvability of nonlocal inverse boundary value problems for hyperbolic, elliptic equations with overdetermination conditions is considered. The existence and uniqueness theorem for the classical solution of the considered inverse coefficient problem is proved for the smaller value of time.

The inverse problem of determining the time-dependent thermal diffusivity and the temperature distribution in a parabolic equation in the case of nonlocal initial-boundary conditions containing a real parameter and integral overdetermination conditions are investigated in the works [9–15]. Sufficient conditions for the existence and uniqueness of the classical solution to the inverse problem of this equation are obtained for small time.

The problem of determining the kernel $k(t)$ of the integral term in an integro-differential heat equation was studied in many publications, [16–24], in which both one- and multidimensional inverse problems with classical initial, initial-boundary conditions were investigated. There is proven existence and uniqueness of inverse problem solutions.

In this article, we study an inverse problem in integro-differential equation for a second-order parabolic equation with nonlocal initial-boundary conditions. For the inverse problem of determining the kernel $k(t)$ function in the one-dimensional integro-differential parabolic equation, the existence and uniqueness of the solution to this problem is studied.

Let $T > 0$, $l > 0$ be fixed numbers and $D_{Tl} = \{(x,t) \mid 0 < x < l,\ 0 < t \le T\}$. Consider the inverse problem of determining the functions $u(x,t)$, $k(t)$ such that they satisfy the equation

$$u_t - u_{xx} = \int_0^t k(t-\tau)u(x,\tau)\,d\tau, \quad (x,t) \in D_{Tl}, \tag{0.1}$$

the nonlocal initial condition

$$u(x,0) + \lambda u(x,T) = \varphi(x), \quad x \in [0,l], \tag{0.2}$$

the boundary conditions

$$u|_{x=0} = u|_{x=l} = 0, \quad \varphi(0) = \varphi(l) = 0, \quad t \in [0,T], \tag{0.3}$$

and the additional condition

$$\int_0^l \omega(x)u(x,t)\,dx = h(t); \tag{0.4}$$

here $\lambda \ge 0$ is a given number, $\varphi(x)$, $\omega(x)$, $h(t)$ are given functions of $x \in [0,l]$ and $t \in [0,T]$.

It is clear that (0.1)–(0.3) constitute a nonstandard problem since the initial data $u(x,0)$ is specified in terms of the solution at $t = T$. There are a variety of physical situations that might be modeled by (0.1)–(0.3). A straightforward interpretation is that $u(x,t)$ represents the concentration of some diffusing substance, whose initial level is unspecified, although it is known that this initial level must be balanced against some weighted average of all future levels of concentration. Condition (0.4) represents a space average measurement of the temperature. This inverse formulation is significant to modeling several practical applications related to unknown potential and temperature. For instance, in the various fields of human activity, such as mineral exploration, medicine, seismology, biology, desalination of seawater, movement of liquid in a porous medium, etc.

In the direct problem, for given numbers l, T, λ and sufficiently smooth functions $k(t)$, $\varphi(x)$, it is required to find a function $u(x,t) \in C^{2,1}(D_{Tl}) \bigcap C(\overline{D}_{Tl})$ satisfying equations (0.1)–(0.3).

Let $C^m(0,l)$ be the class of m times continuously differentiable with all derivatives up to the m-th order (inclusive) in $(0,l)$ functions. In the case $m = 0$ this space coincides with the class of continuous functions. $C^{m,j}(D_{Tl})$ is the class of m times continuously differentiable with respect to x and j times continuously differentiable with respect to t all derivatives in the domain D_{Tl} functions.

§ 1. Investigation of the direct problem

The solution of equation (0.1) with the nonlocal initial condition (0.2) and the boundary conditions (0.3) satisfies the relation

$$\begin{aligned} u(x,t) = \Phi(x,t) - \lambda \int_0^T \int_0^l G_0(x,\xi,T+t-\tau) \int_0^\tau k(\alpha)u(\xi,\tau-\alpha)\,d\alpha\,d\xi\,d\tau\, + \\ + \int_0^t \int_0^l G(x,\xi,t-\tau) \int_0^\tau k(\alpha)u(\xi,\tau-\alpha)\,d\alpha\,d\xi\,d\tau, \end{aligned} \tag{1.1}$$

where

$$\Phi(x,t)=\int_0^l G_0(x,\xi,t)\varphi(\xi)\,d\xi,$$

$$G(x,\xi,t)=\frac{2}{l}\sum_{n=1}^{\infty}e^{-(\frac{\pi n}{l})^2t}\sin\frac{\pi n}{l}\xi\sin\frac{\pi n}{l}x,$$

$$G_0(x,\xi,t)=\frac{2}{l}\sum_{n=1}^{\infty}\frac{1}{1+\lambda e^{-(\frac{\pi n}{l})^2T}}e^{-(\frac{\pi n}{l})^2t}\sin\frac{\pi n}{l}\xi\sin\frac{\pi n}{l}x.$$

Denote the operator taking the function $u(x,t)$ to the right-hand side of (1.1) by L. Then (1.1) is written as the operator equation

$$u=Lu. \tag{1.2}$$

Let

$$\Phi_0=\max_{(x,t)\in\overline{D}_{Tl}}|\Phi(x,t)|,\qquad k_0=\max_{t\in[0,T]}|k(t)|,$$

let $S_d(0)=\{u\mid \|u\|\le d\}$, let d be some positive number.

We use the Schauder principle (see [20, p. 411]) to the existence of solution of the operator equation (1.2).

Lemma 1. *Suppose that the following conditions are satisfied:* $\varphi(x)\in C[0,l]$, $k(t)\in C[0,T]$, $\varphi(0)=\varphi(l)=0$. *Then for all* T *and* $d>\Phi_0$ *satisfying the estimate*

$$0<T\le\sqrt{\frac{2(d-\Phi_0)}{dk_0(1+\lambda)}}, \tag{1.3}$$

the operator L *is uniformly bounded and equicontinuous.*

Then according to the Schauder principle there exists at least one classical solution of problem (0.1)–(0.3) in the space $C^{2,1}(D_{Tl})$.

P r o o f. First, we establish the uniform boundedness of the operator L. To this end, we show that there exists a $\rho\in(0,d]$ such that $\|Lu\|\le\rho$, where $\|Lu\|=\max_{(x,t)\in\overline{D}_{Tl}}|Lu|$. For $u\in S_d(0)$ and $(x,t)\in\overline{D}_T$, we find estimate $\|Lu\|\le\Phi_0+k_0d(1+\lambda)\frac{T^2}{2}\equiv\rho$. For T that satisfies the estimate (1.3), the operator L is uniformly bounded.

Definition 1. *An operator L is said to be equicontinuous* if for each $\varepsilon>0$ there exists $\delta_0=\delta_0(\varepsilon)>0$ such that the inequality

$$\|Lu_1-Lu_2\|\le\varepsilon$$

holds for all $(x,t)\in D_{Tl}$, $u_1(x,t)$, $u_2(x,t)\in S_d(0)$ with $\|u_1(x,t)-u_2(x,t)\|\le\delta_0$.

We consider the estimates

$$\|Lu_1-Lu_2\|\le$$

$$\le\lambda\max_{(x,t)\in\overline{D}_T}\left|\int_0^T\int_0^l G_0(x,\xi,T+t-\tau)\int_0^\tau k(\alpha)(u_1(\xi,\tau-\alpha)-u_1(\xi,\tau-\alpha))\,d\alpha\,d\xi\,d\tau\right|+$$

$$+\max_{(x,t)\in\overline{D}_T}\left|\int_0^t\int_0^l G(x,\xi,t-\tau)\int_0^\tau k(\alpha)(u_1(\xi,\tau-\alpha)-u_2(\xi,\tau-\alpha))\,d\alpha\,d\xi\,d\tau\right|\le$$

$$\le(1+\lambda)k_0\frac{T^2}{2}\|u_1-u_2\|\le(1+\lambda)k_0\frac{T^2}{2}\delta.$$

Consequently, if we take $\delta_0 = \frac{2\varepsilon}{(1+\lambda)k_0T^2}$, then inequality (1.3) will hold for $\delta \in (0, \delta_0]$, the operator L is equicontinuous. Then the operator L is completely continuous on S_d, and it has at least one fixed point on S_d by the Schauder principle. The proof of the lemma is complete. □

Thus, Lemma 1 implies the following assertion on the existence of a solution of the operator equation (1.2).

Let us prove the uniqueness of this solution.

Theorem 1. *For all* $u \in S_d(0)$ *and*

$$T < \sqrt{\frac{2}{k_0(1+\lambda)}}, \tag{1.4}$$

the operator equation (1.2) *has a unique solution in the class* $C^{2,1}(D_{Tl})$.

P r o o f. Let problem (0.1)–(0.3) have two solutions u_1, u_2, $u_1 \neq u_2$. We denote their difference by $\vartheta = u_1 - u_2$. For the difference ϑ, we obtain the problem

$$\vartheta_t - \vartheta_{xx} = \int_0^t k(t-\tau)\vartheta(x,\tau)\,d\tau, \quad (x,t) \in D_{Tl},$$
$$\vartheta(x,0) + \lambda\vartheta(x,T) = 0, \quad x \in [0,l],$$
$$\vartheta|_{x=0} = \vartheta|_{x=l} = 0, \quad t \in [0,T].$$

The solution of this problem can be written as

$$\vartheta(x,t) = -\lambda \int_0^T \int_0^l G_0(x,\xi,T+t-\tau) \int_0^\tau k(\alpha)\vartheta(\xi,\tau-\alpha)\,d\alpha\,d\xi\,d\tau + \\ + \int_0^t \int_0^l G(x,\xi,t-\tau) \int_0^\tau k(\alpha)\vartheta(\xi,\tau-\alpha)\,d\alpha\,d\xi\,d\tau.$$

Estimating we have

$$\|\vartheta\|_{C(\overline{D}_T)} \leq \frac{1}{2}k_0T^2(1+\lambda)\|\vartheta\|_{C(\overline{D}_T)}.$$

Hence for T that satisfies estimate (1.4) we obtain $u_1 = u_2$. The proof of the theorem is complete. □

§ 2. Classical solvability of inverse problem

The solution of the nonlocal initial-boundary problem (0.1)–(0.4) satisfies the integral equation

$$u(x,t) = \Phi(x,t) - \lambda \int_t^{T+t} \int_0^l G_0(x,\xi,\tau) \int_0^{T+t-\tau} k(\alpha)u(\xi,T+t-\tau-\alpha)\,d\alpha\,d\xi\,d\tau + \\ + \int_0^t \int_0^l G(x,\xi,\tau) \int_0^{t-\tau} k(\alpha)u(\xi,t-\tau-\alpha)\,d\alpha\,d\xi\,d\tau. \tag{2.1}$$

Now we write a property of the Green function which will be needed in the future.

Remark 1. The integral of the Green function does not exceed 1:

$$\int_0^l G_0(x,\xi,t)\,d\xi \leq \int_0^l G(x,\xi,t)\,d\xi = 1, \quad x \in [0,l], \quad t \in [0,T].$$

We differentiate the equation (2.1) with respect to t and rewrite the result in the form

$$\begin{aligned} u_t(x,t) = \Phi_t(x,t) + \lambda \int_0^l G_0(x,\xi,t) \int_0^T k(\alpha) u(\xi, T-\alpha)\, d\alpha\, d\xi \ - \\ - \lambda \int_t^{T+t} \int_0^l G_0(x,\xi,\tau) k(T+t-\alpha) u(\xi,0)\, d\alpha\, d\xi\, d\tau - \\ - \lambda \int_t^{T+t} \int_0^l G_0(x,\xi,\tau) \int_0^{T+t-\tau} k(\alpha) u_t(\xi, T+t-\tau-\alpha)\, d\alpha\, d\xi\, d\tau + \\ + \int_0^t \int_0^l G(x,\xi,\tau) k(t-\tau) u(\xi,0)\, d\xi\, d\tau + \\ + \int_0^t \int_0^l G(x,\xi,\tau) \int_0^{t-\tau} k(\alpha) u_t(\xi, t-\tau-\alpha)\, d\alpha\, d\xi\, d\tau. \end{aligned} \tag{2.2}$$

We calculate the derivative $\Phi_t(x,t)$, using the relation

$$G_{0t}(x,\xi,t) = G_{0\xi\xi}(x,\xi,t).$$

Integrating by parts, using matching conditions $\varphi(0) = \varphi(l) = 0$, we find that

$$\Phi_t(x,t) = \frac{\partial}{\partial t}\Big(\int_0^l G_0(x,\xi,t)\varphi(\xi)\, d\xi \Big) = \int_0^l G_{0\xi\xi}(x,\xi,t)\varphi(\xi)\, d\xi = \int_0^l G_0(x,\xi,t)\varphi''(\xi)\, d\xi.$$

Suppose that the conditions $\omega(0) = \omega(l) = 0$ are satisfied. Let us multiply (0.1) by $\omega(x)$ and integrate over x from 0 to l. Taking into account conditions (0.2)–(0.4) and differentiate with respect to t, we obtain the relation

$$k(t) = \frac{h''(t)}{h(0)} - \frac{1}{h(0)} \int_0^l \omega''(x) u_t(x,t)\, dx - \frac{1}{h(0)} \int_0^t k(\tau) h_t(t-\tau)\, d\tau. \tag{2.3}$$

We represent the system of equations (2.1)–(2.3) in the form

$$g = Ag, \tag{2.4}$$

where $g = (g_1, g_2, g_3) = (u(x,t), u_t(x,t), k(t))$ is the vector-function.
$A = (A_1, A_2, A_3)$ is defined by the right sides of equations (2.1)–(2.3):

$$\begin{aligned} A_1 g = g_{01} - \lambda \int_0^T \int_0^l G_0(x,\xi,T+t-\tau) \int_0^\tau g_3(\alpha) g_1(\xi,\tau-\alpha)\, d\alpha\, d\xi\, d\tau + \\ + \int_0^t \int_0^l G(x,\xi,t-\tau) \int_0^\tau g_3(\alpha) g_1(\xi,\tau-\alpha)\, d\alpha\, d\xi\, d\tau, \end{aligned}$$

$$\begin{aligned} A_2 g = g_{02} + \lambda \int_0^l G_0(x,\xi,t) \int_0^T g_3(\alpha) g_1(\xi, T-\alpha)\, d\alpha\, d\xi \ - \\ - \lambda \int_t^{T+t} \int_0^l G_0(x,\xi,\tau) g_3(T+t-\alpha) g_1(\xi,0)\, d\alpha\, d\xi\, d\tau - \\ - \lambda \int_t^{T+t} \int_0^l G_0(x,\xi,\tau) \int_0^{T+t-\tau} g_3(\alpha) g_2(\xi, T+t-\tau-\alpha)\, d\alpha\, d\xi\, d\tau + \\ + \int_0^t \int_0^l G(x,\xi,\tau) g_3(t-\tau) g_1(\xi,0)\, d\xi\, d\tau + \\ + \int_0^t \int_0^l G(x,\xi,\tau) \int_0^{t-\tau} g_3(\alpha) g_2(\xi, t-\tau-\alpha)\, d\alpha\, d\xi\, d\tau, \end{aligned}$$

$$A_3 g = g_{03} - \frac{1}{h_0} \int_0^l \omega''(x) g_2(x,t)\, dx - \frac{1}{h_0} \int_0^t g_1(\tau) h_t(t-\tau)\, d\tau.$$

The following notations have been introduced in the equalities (2.1)–(2.3):

$$g_0(t,x) = (g_{01}(t,x), g_{02}(t,x), g_{03}(t)) = \left(\Phi(x,t), \Phi_t(x,t), \frac{h''(t)}{h(0)}\right).$$

Theorem 2. *Let conditions*

$$(\varphi(x), \omega(x)) \in C^2[0,l], \quad h(t) \in C^2[0,T],$$
$$\int_0^l \omega(x)\varphi(x)\,dx = h(0) + \lambda h(T),$$
$$\varphi(0) = \varphi(l) = 0, \quad h(0) \neq 0, \quad \lambda \geq 0, \quad \omega(0) = \omega(l) = 0$$

be satisfied. Then there exist sufficiently small numbers $T^* \in (0,T)$, $l^* \in (0,l)$ such that the solution to the integral equations (2.1)–(2.3) in the class of functions $u(x,t) \in C^{2,1}\left(\overline{D}_{Tl^*}\right)$, $k(t) \in C[0,T^*]$ exists and is unique, where $D_{Tl^*} = \{(x,t) \mid x \in (0,l^*),\ t \in (0,T^*]\}$.

P r o o f. We introduce the notations

$$\varphi_0 := \|\varphi\|_{C^2[0,l]}, \quad \omega_0 := \|\omega\|_{C^2[0,l]}, \quad h_0 := \|h\|_{C^2[0,T]}.$$

We define, for the unknown vector-function $g(x,t) \in C(D_{Tl})$, the following norm:

$$\|g\| = \max\left\{\max_{(x,t)\in\overline{D}_{Tl}} |g_1(x,t)|,\ \max_{(x,t)\in\overline{D}_{Tl}} |g_2(x,t)|,\ \max_{t\in[0,T]} |g_3(t)|\right\}.$$

We have

$$\|g_0\| = \max\left\{\max_{(x,t)\in\overline{D}_{Tl}} |g_{01}(x,t)|,\ \max_{(x,t)\in\overline{D}_{Tl}} |g_{02}(x,t)|,\ \max_{t\in[0,T]} |g_{03}(t)|\right\},$$

where

$$\max_{(x,t)\in\overline{D}_{Tl}} |g_{01}(x,t)| \leq \max_{(x,t)\in\overline{D}_{Tl}} \left|\int_0^l G_0(x,\xi,t)\varphi(\xi)\,d\xi\right| \leq \varphi_0,$$
$$\max_{(x,t)\in\overline{D}_{Tl}} |g_{02}(x,t)| \leq \max_{(x,t)\in\overline{D}_{Tl}} \left|\int_0^l G_0(x,\xi,t)\varphi''(\xi)\,d\xi\right| \leq \varphi_0,$$
$$\max_{t\in[0,T]} |g_{03}(t)| \leq \max_{t\in[0,T]} \left|\frac{h''(t)}{h(0)}\right| \leq \frac{h_0}{h(0)}.$$

Denote by $S(g_0,\rho)$ the ball of vector-functions g with center at the point g_0 and radius $\rho > 0$, i. e., $S(g_0,\rho) = \{g \mid \|g - g_0\| \leq \rho\}$.

Obviously, $\|g\| \leq 2\|g_0\|$ for $g(x,t) \in S(g_0, \|g_0\|)$. We prove that the operator A is contracting on the set $S(g_0, \|g_0\|)$ if the number T is chosen in suitable way.

Now we check the first condition of contractive mapping [19, pp. 87–97] for operator A. Let $g(x,t)$ be an element of $S(g_0, \|g_0\|)$. Then for $(x,t) \in D_{Tl}$ we have

$$\|A_1 g - g_{01}\| = \max_{(x,t)\in\overline{D}_{Tl}} |(A_1 g - g_{01})| \leq$$
$$\leq \max_{(x,t)\in\overline{D}_{Tl}} \left|\lambda \int_0^T \int_0^l G_0(x,\xi,T+t-\tau) \int_0^\tau g_3(\alpha) g_1(\xi,\tau-\alpha)\,d\alpha\,d\xi\,d\tau\right| +$$
$$+ \max_{(x,t)\in\overline{D}_{Tl}} \left|\int_0^t \int_0^l G(x,\xi,t-\tau) \int_0^\tau g_3(\alpha) g_1(\xi,\tau-\alpha)\,d\alpha\,d\xi\,d\tau\right| \leq (1+\lambda)T^2\|g_0\|^2.$$

It follows that if $T < \sqrt{\frac{1}{(1+\lambda)\|g_0\|}} = T_1$, then $A_1 g \in S(g_0, \|g_0\|)$.

$$\|A_2 g - g_{02}\| = \max_{(x,t)\in\overline{D}_{Tl}} |(A_2 g - g_{02})| \leq$$

$$\leq \lambda \max_{(x,t)\in\overline{D}_{Tl}} \left| \int_0^l G_0(x,\xi,t) \int_0^T g_3(\alpha) g_1(\xi, T-\alpha)\, d\alpha\, d\xi \right| +$$

$$+ \lambda \max_{(x,t)\in\overline{D}_{Tl}} \left| \int_t^{T+t} \int_0^l G_0(x,\xi,\tau) g_3(T+t-\alpha) g_1(\xi,0)\, d\alpha\, d\xi\, d\tau \right| +$$

$$+ \lambda \max_{(x,t)\in\overline{D}_{Tl}} \left| \int_t^{T+t} \int_0^l G_0(x,\xi,\tau) \int_0^{T+t-\tau} g_3(\alpha) g_2(\xi, T+t-\tau-\alpha)\, d\alpha\, d\xi\, d\tau \right| +$$

$$+ \max_{(x,t)\in\overline{D}_{Tl}} \left| \int_0^t \int_0^l G(x,\xi,\tau) g_3(t-\tau) g_1(\xi,0)\, d\xi\, d\tau \right| +$$

$$+ \max_{(x,t)\in\overline{D}_{Tl}} \left| \int_0^t \int_0^l G(x,\xi,\tau) \int_0^{t-\tau} g_3(\alpha) g_2(\xi, t-\tau-\alpha)\, d\alpha\, d\xi\, d\tau \right| \leq$$

$$\leq 2(1+\lambda)T^2\|g_0\|^2 + 4(2\lambda+1)T\|g_0\|^2.$$

It follows that if

$$T < \frac{\sqrt{4(2\lambda+1)^2\|g_0\|^2 + 2(1+\lambda)\|g_0\|} - 2(2\lambda+1)\|g_0\|}{2\|g_0\|(1+\lambda)} = T_2,$$

then $A_2 g \in S(g_0, \|g_0\|)$.

$$\|A_3 g - g_{03}\| = \max_{t\in[0;T]} |(A_3 g - g_{03})| \leq$$

$$\leq \frac{1}{h(0)} \max_{t\in[0;T]} \left| \int_0^l \omega''(x) g_2(x,t)\, dx - \int_0^t g_1(\tau) h_t(t-\tau)\, d\tau \right| \leq \frac{\omega_0 l + h_0 T}{h(0)} \|g_0\|.$$

If this satisfies the following conditions

$$T < \frac{h(0) - \omega_0 l}{h_0} = T_3, \quad l < \frac{h(0)}{\omega_0} = l_1,$$

then $A_3 g \in S(g_0, \|g_0\|)$.

As a result, we conclude that if T and l satisfy the conditions $T < \min\{T_i\}$, $i = 1,2,3$, $l < l_1$, then operator A maps $S(g_0, \|g_0\|)$ into itself, i. e., $Ag \in S(g_0, \|g_0\|)$.

Further we check the second condition of contractive mapping. In accordance with (2.1) for the first component of operator A we get

$$\|(Ag^1 - Ag^2)_1\| =$$

$$= \max_{(x,t)\in\overline{D}_{Tl}} \left| \lambda \int_0^T \int_0^l G_0(x,\xi,T+t-\tau) \int_0^\tau [g_3^1(\alpha) g_1^1(\xi,\tau-\alpha) - g_3^2(\alpha) g_1^2(\xi,\tau-\alpha)]\, d\alpha\, d\xi\, d\tau \right| +$$

$$+ \max_{(x,t)\in\overline{D}_{Tl}} \left| \int_0^t \int_0^l G(x,\xi,t-\tau) \int_0^\tau [g_3^1(\alpha) g_1^1(\xi,\tau-\alpha) - g_3^2(\alpha) g_1^2(\xi,\tau-\alpha)]\, d\alpha\, d\xi\, d\tau \right|.$$

Here the integrand in the last integral can be estimated as follows

$$\left\|g_2^1 g_1^1 - g_2^2 g_1^2\right\| = \left\|(g_2^1 - g_2^2) g_1^1 + g_2^2 (g_1^1 - g_1^2)\right\| \leq$$
$$\leq 2\left\|g^1 - g^2\right\| \max\left(\left\|g_1^1\right\|, \left\|g_2^2\right\|\right) \leq 4\left\|g_0\right\| \left\|g^1 - g^2\right\|.$$

We have that the first component of A can be estimated in the following form:

$$\|(Ag^1 - Ag^2)_1\| \leq 2T^2\,(1+\lambda)\,\|g_0\|\,\left\|g^1 - g^2\right\|.$$

Now we choose number T so that the expression at $\|g^1 - g^2\|$ becomes less than 1, i. e., the inequality

$$2T^2\,(1+\lambda)\,\|g_0\| < 1$$

is fulfilled. From these estimates it is clear that if T is chosen from condition $T < \frac{1}{\sqrt{2(1+\lambda)\|g_0\|}} = T_4$ then the operator A_1 satisfies the second condition of contracting mapping.

The second component of A can be estimated in the following form:

$$\|(Ag^1 - Ag^2)_2\| \leq \left(2\|g_0\|T^2(1+\lambda) + 4(1+2\lambda)\|g_0\|T\right)\left\|g^1 - g^2\right\|.$$

Now we choose number T so that the expression at $\|g^1 - g^2\|$ becomes less than 1, i. e., the inequality

$$2\|g_0\|T^2(1+\lambda) + 4(1+2\lambda)\|g_0\|T < 1$$

is fulfilled. From these estimates it is clear that if T is chosen from condition

$$T < \frac{\sqrt{4(2\lambda+1)^2\|g_0\|^2 + 2(1+\lambda)\|g_0\|} - 2(2\lambda+1)\|g_0\|}{2\|g_0\|(1+\lambda)} = T_5,$$

then the operator A_2 satisfies the second condition of contracting mapping.

Similary,

$$\|(Ag^1 - Ag^2)_3\| \leq \frac{1}{h(0)}\,(\omega_0 l + h_0 T)\,\|g^1 - g^2\|.$$

If $T < \frac{h(0)-\omega_0 l}{h_0} = T_6$, $l < \frac{h(0)}{\omega_0} := l_2$, then the operator A_3 satisfies the second condition of contracting mapping. As a result, we conclude that if T, l are taken from conditions $T \in (0, \min(T_i))$, $i = \overline{1,6}$, $l \in (0, \min(l_1, l_2))$ then the operator A carries out contracting mapping of the ball $S(g_0, \|g_0\|)$ into itself and according to Banach theorem in this ball it has a unique fixed point, i. e., there exists a unique solution of operator equation (2.4). The proof of the theorem is complete. □

Since the integral equations (2.1)–(2.3), which are the same of the vector-operator equations (2.4), are equivalent to the inverse problem (0.1)–(0.4), for small T there is a unique solution to the inverse problem (0.1)–(0.4) $k(t) \in C[0,T]$.

Conclusion

In this work, the solvability of a nonlinear inverse problem for integro-differential heat equation with nonlocal conditions was studied. Firstly we investigated solvability of the direct problem. The (0.1)–(0.3) problem was replaced by an equivalent integral equation. Existence and uniqueness of the direct problem solution was proven. The inverse problem was considered for determining the kernel $k(t)$ included in the equation (0.1) with integral observation (0.4) of the solution of this system with the initial and boundary conditions (0.2), (0.3). Conditions for given functions have been obtained, under which the inverse problem has a unique solution for a sufficiently small interval.

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Задача определения ядра в интегро-дифференциальном уравнении параболического типа с нелокальным условием

Ключевые слова: интегро-дифференциальное уравнение, нелокальная начально-краевая задача, обратная задача, интегральное уравнение, принцип Шаудера.

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В данной работе исследуется обратная задача для одномерного интегро-дифференциального уравнения теплопроводности с нелокальными начально-краевыми и интегральными условиями переопределения. Мы использовали метод Фурье и принцип Шаудера для исследования разрешимости прямой задачи. Далее задача сводится к эквивалентной замкнутой системе интегральных уравнений относительно неизвестных функций. Существование и единственность решения интегральных уравнений доказывается с помощью сжимающего отображения. Наконец, с помощью эквивалентности получается существование и единственность классического решения.

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