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ON SHIMODA'S THEOREM

The present work is devoted to Shimoda's Theorem on the holomorphicity of a function $f(z,w)$ which is holomorphic by $w \in V$ for each fixed $z \in U$ and is holomorphic by $z \in U$ for each fixed $w \in E$, where $E \subset V$ is a countable set with at least one limit point in V. Shimoda proves that in this case $f(z,w)$ is holomorphic in $U \times V$ except for a nowhere dense closed subset of $U \times V$. We prove the converse of this result, that is for an arbitrary given nowhere dense closed subset of U, $S \subset U$, there exists a holomorphic function, satisfying Shimoda's Theorem on $U \times V \subset \mathbb{C}^2$, that is not holomorphic on $S \times V$. Moreover, we observe conditions which imply empty exception sets on Shimoda's Theorem and prove generalizations of Shimoda's Theorem.

Keywords: Hartogs's phenomena, Shimoda's Theorem, separately holomorphic functions, power series.

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Introduction

Well-known Hartogs's Theorem [1] states that if a function f is holomorphic at any point of the domain $D \subset \mathbb{C}^n$ with respect to each of the variables z_ν, then it is holomorphic in D with respect to the set of all variables. This fundamental result laid the foundation of the theory of analytic functions of several variables. Proving this fact, he discovered aspects of forced nature holomorphic continuation of functions in multidimensional complex analysis. Hartogs's Theorem has different variations and generalizations in the works of several authors. Significant results in this direction were obtained in the works of M. Hukuhara [2], I. Shimoda [3], T. Terada [4], J. Siciak [5], V. P. Zaharjuta [6], A. A. Gonchar [7], A. S. Sadullaev and E. M. Chirka [8], M. Jarnicki and P. Pflug [9–11], A. S. Sadullaev and S. A. Imomkulov [12], Pham Hien Bang [13], A. S. Sadullaev and T. T. Tuichiev [14], L. Baracco [15], J. Boman [16], S. Krantz [17], J. Bochnak and W. Kucharz [18], Y. L. Cho and K. T. Kim [19], A. Boggess, R. Dwilewicz and E. Porten [20]. More new approaches to the question are considered in the works of M. G. Lawrence [21] (CR separate analytic variations of Hartogs's Theorem), A. Lewandowski [22] (described envelopes of holomorphy of cross-like objects such as the so-called A-cross). In the works of Thai Thuan Quang and Nguyen Van Dai [23, 24], Thai Thuan Quang and Lien Vuong Lam [25] authors considered Hartogs type Theorem for vector-valued functions, $(\cdot, W)$-holomorphic and $(\cdot, W)$-meromorphic functions.

Properties of separately analytic functions have found important applications in the theory of elliptic differential equations. In the work of S. A. Imomkulov and S. M. Abdikadirov [26] it is applied to the removing thin singularities of separately harmonic functions and it is applied to a system of elliptic equations in the work of V. P. Palamodov [27].

In this paper, we again turn to one of the important classical cases to clarify some questions.

In 1942, Hukuhara posed the next problem: is a function $f(z,w)$ holomorphic or not in a domain $U \times V \subset \mathbb{C}_z \times \mathbb{C}_w$, when $f(z,w)$ is holomorphic with respect to w for each fixed $z \in U$ and holomorphic with respect to z for all w_m with one limit point $w_0 \in V$?

In 1957, Shimoda [3] gave the next solution to this problem.

Theorem 1 (Shimoda's Theorem). *Let $f(z,w)$ be a function defined in a polydisc $U \times V = \{|z| < 1\} \times \{|w| < 1\} \subset \mathbb{C}^2$ and let $E \subset V$ be a countable subset having at least one limit point belonging to V. If*

(1) *for each fixed $z^0 \in U$, $f(z^0, w) \in \mathcal{O}(V)$,*

(2) *for each fixed $w^0 \in E$, $f(z, w^0) \in \mathcal{O}(U)$,*

then there exists a nowhere-dense and relatively closed subset $S \subset U$ such that $f(z,w) \in \mathcal{O}((U \backslash S) \times V)$.

This Theorem presents special interest, because it stays independent of the rest of the categories of results of authors mentioned above, which include and generalize one another.

In this paper, we will discuss some questions depending on Shimoda's Theorem. Mainly, we prove the converse of Shimoda's Theorem that states the existence of a separately holomorphic function with a predefined set of singularities and satisfying all conditions of Shimoda's Theorem. Moreover, we study conditions of ensuring the nonexistence of singularities. First, discussing Shimoda's Theorem, we give a simplified proof of it.

§ 1. Proof of Shimoda's Theorem

General idea of the proof is the same and it goes back to Osgood's method. Let U' be an arbitrary open subset of U and $V' = \{|w| < r\}$, $r < 1$, be a ball compactly belonging to V. Without loss of generality, we may assume that $w = 0$ is a limit point of the set E.

For a fixed $z \in \overline{U}'$ we put

$$M(z) = \max_{w \in \overline{V}'} |f(z,w)|$$

and consider the sets

$$E_m = \left\{ z \in \overline{U}' : M(z) \le m \right\}, \quad m = 1, 2, \dots .$$

First, we show that these sets are closed. Assume that the sequence of points $\{z_n\}$ converging to z_0, is contained in E_m. If we put $f(z_n, w) = f_n(w)$, then $f_n(w)$ is holomorphic in V' and satisfies $|f_n(w)| \le m$. Therefore we get sequence of holomorphic functions $\{f_n(w)\}$ which is uniformly bounded on V' and consequently, it is compact there.

On the other side, we take the sequence $\{w_\nu\}$ in E, which tends to $w = 0$. Since, for a fixed w_ν, the function $f(z, w_\nu)$ is holomorphic with respect to z, then $\lim\limits_{n\to\infty} f(z_n, w_\nu) = f(z_0, w_\nu)$, i. e., $\lim\limits_{n\to\infty} f_n(w_\nu) = f(z_0, w_\nu)$, where $m = 1, 2, 3, \dots$.

By Vitali's Theorem (see [28]), the sequence of holomorphic functions $\{f_n(w)\}$ in V' converges uniformly to $f_0(w)$. Since $z_0 \in U$, the function $f(z_0, w)$ is holomorphic with respect to w. In addition, the equality $f_0(w_\nu) = f(z_0, w_\nu)$ holds for $\nu = 1, 2, 3, \dots$. Since the sequence of points w_ν converges to a limit point $0 \in V'$, according to the uniqueness Theorem for holomorphic functions we get $f_0(w) \equiv f(z_0, w)$ on V'. Since $|f(z_n, w)| \le m$, it follows that $|f(z_0, w)| = \lim\limits_{n\to\infty} |f_n(z_n, w)| \le m$. Thus, the sets E_m are closed.

So we have the increasing sequence of sets E_m, $m = 1, 2, 3, \dots$, and any point $z \in \overline{U}'$ belongs to all E_m, starting from some of them, and $\overline{U}' = \bigcup_m E_m$. According to Baire category Theorem, there exists a number m_0, such that E_{m_0} contains some ball $\mathcal{B}$. Then the inequality $|f(z,w)| \le m_0$ holds for all $(z,w) \in (\mathcal{B} \times V')$.

Now we fix the point $z \in \mathcal{B}$ and expand the function $f(z,w)$ into the Taylor series

$$f(z,w) = \sum_{k=0}^{\infty} c_k(z) w^k. \tag{1}$$

For all $z \in \mathcal{B}$ the series (1) converges at $|w| < r$ and by Cauchy's inequality for the coefficients we have $|c_k(z)| \leq \frac{m_0}{r^k}$. On the other hand, for $w = w_\nu$ the function

$$f(z,w_\nu) = \sum_{k=0}^{\infty} c_k(z) w_\nu^k$$

is holomorphic with respect to z on $\mathcal{B}$. Then, using the Vitali's Theorem again, we get that the limit function $f(z,0)$ is holomorphic in $\mathcal{B}$, i. e., $c_0(z) = f(z,0) \in \mathcal{O}(\mathcal{B})$. Since $\lim_{\nu\to\infty} w_\nu = 0$, we can assume that $|w_\nu| < \frac{r}{2}$ (for this it is enough to skip a finite number of elements of the sequence for which this inequality does not hold). Then from inequality

$$\left|\sum_{k=1}^{\infty} c_k(z)(w_\nu)^{k-1}\right| \leq \sum_{k=1}^{\infty} |c_k(z)|\,|w_\nu|^{k-1} \leq \sum_{k=1}^{\infty} \frac{m_0}{2^{k-1}r} \leq \frac{m_0}{r}$$

we obtain that the sequence of holomorphic functions $g_\nu(z) = \frac{1}{w_\nu}\left(f(z,w_\nu) - c_0(z)\right)$ is uniformly bounded at $\mathcal{B}$ and has a finite limit $c_1(z)$ at every point $z \in \mathcal{B}$. Hence, again by the Vitali's Theorem, the sequence $\{g_\nu(z)\}$ converges uniformly to $c_1(z)$ in $\mathcal{B}$ and $c_1(z) \in \mathcal{O}(\mathcal{B})$. Continuing this process, we will establish that all coefficients $c_k(z) \in \mathcal{O}(\mathcal{B})$, $k = 0, 1, 2, \dots$.

Thus, series (1) is a Hartogs series with holomorphic coefficients $c_k(z)$ on $\mathcal{B}$. Now, we show that this series (1) converges uniformly on compact subsets of the domain $\mathcal{B} \times V'$. For this purpose, we consider a sequence of subharmonic functions $\frac{1}{k}\ln|c_k(z)|$ in $\mathcal{B}$. Since for each fixed $z \in \mathcal{B}$ it holds true that $|c_k(z)|r^k \to 0$ as $k \to \infty$, then for any $z \in \mathcal{B}$ there exists a number k_0, such that starting from this number the inequality $|c_k(z)|r^k \leq 1$ holds. Therefore from $\frac{1}{k}\ln|c_k(z)| \leq \ln\frac{1}{r}$, the inequality follows

$$\overline{\lim_{k\to\infty}} \frac{1}{k}\ln|c_k(z)| \leq \ln\frac{1}{r}.$$

On the other hand, the Cauchy inequality implies

$$\frac{1}{k}\ln|c_k(z)| \leq \frac{\ln m_0}{k} - \ln r \leq A.$$

Thus, the considered sequence of subharmonic functions satisfies the conditions of Hartogs's lemma and therefore, for a fixed compact set $K \subset\subset \mathcal{B}$ and an arbitrary positive number $\rho < r$, there exists a number k_0 such that for all $k > k_0$ and for any $z \in K$ the inequality holds

$$\frac{1}{k}\ln|c_k(z)| \leq \ln\frac{1}{\rho},$$

i. e.,

$$|c_k(z)|\rho^k \leq 1.$$

This implies that for any $(z, w) \in K \times \{|w| < \sigma\}$, $0 < \sigma < \rho$, the estimation

$$|c_k(z) \cdot w^k| = |c_k(z)| \cdot |w^k| = |c_k(z)| \cdot \rho^k \cdot \frac{|w|^k}{\rho^k} \leq \left(\frac{\sigma}{\rho}\right)^k$$

holds and the series (1) converges uniformly on the compact set $K \times \{|w| \leq \sigma\}$. From the arbitrariness of the compact set K and the numbers $\sigma < \rho < r < 1$, we obtain that the series (1) converges uniformly on compact subsets of the domain $\mathcal{B} \times V$ and consequently, its sum $f(z, w)$ is holomorphic on this domain.

Thus, we have proved that for any open subset $U' \subset U$, there exists such a ball $\mathcal{B} \subset U'$, that the function $f(z, w)$ is holomorphic in $\mathcal{B} \times V$. Let's denote by Ω the union of all balls $\mathcal{B} \subset U$ for which the function $f(z, w)$ is holomorphic on $\mathcal{B} \times V$. Then, the complement $S = U \backslash \Omega$ is a closed and nowhere dense subset of U. Otherwise, if we assume that S contains some ball $\mathcal{B} \subset S$, then there is another ball $\mathcal{B}' \subset \mathcal{B}$ for which $f(z, w) \in \mathcal{O}(\mathcal{B}' \times V)$, and this situation contradicts to the definition of the set Ω. □

§ 2. Converse of Shimoda's Theorem

Theorem 2. *Let $U \times V = \{|z| < 1\} \times \{|w| < 1\}$ be a unit polydisc in space $\mathbb{C}^2$. If $S \subset U$ is an arbitrary nowhere dense, relatively closed subset of U, such that the union $\partial U \cup S$ is connected and $E \subset V$ is a countable subset having at least one limit point belonging to V, then there exists a function $f(z, w)$ in $U \times V$, which satisfies the following conditions:*

(1) *for each fixed $z^0 \in U$, $f(z^0, w) \in \mathcal{O}(V)$;*

(2) *for each fixed $w^0 \in E$, $f(z, w^0) \in \mathcal{O}(U)$;*

(3) *$f(z, w) \in \mathcal{O}((U \backslash S) \times V)$ and $f(z, w)$ doesn't extend holomorphically to any point of the set $S \times V$.*

P r o o f. First, we consider a compact set S for which the complement $\mathbb{C} \backslash S$ is connected. We fix a strictly decreasing sequence of positive real numbers ε_n that is vanishing. Let $B_{11}, B_{12}, \ldots, B_{1j_1}$ be the balls with radii not exceeding ε_1, which cover the set S and we take points $z_{11}, z_{12}, \ldots, z_{1j_1}$ from these balls respectively and not belonging to S.

After that, we take balls $B_{21}, B_{22}, \ldots, B_{2j_2}$ covering the set S with radii not exceeding ε_2 and satisfying the following conditions:

(1) $\bigcup_{k=1}^{j_1} B_{1k} \supset \bigcup_{k=1}^{j_2} B_{2k}$,

(2) $z_{11}, z_{12}, \ldots, z_{1j_1} \notin B_{2k}$, $k = 1, 2, \ldots j_2$.

We take points $z_{21}, z_{22}, \ldots, z_{2j_2} \notin S$ from these balls respectively. And so on continuing this process, we get a sequence of compact sets of the form $K_m = \overline{U} \backslash \bigcup_{k=1}^{j_m} B_{mk}$, $m = 1, 2, \ldots$, with connected complement and a countable set of points $\{z_{mj}\} \subset U \backslash S$, such that the set of all limit points $\{z_{mj}\}'$ coincides with S. We put $G_m = K_m \cup S$.

For each fixed $m \in \mathbb{N}$ we construct polynomials $p_{k_m}(z)$ of a degree k_m on the complex plane $\mathbb{C}_z$ satisfying conditions $|p_{k_m}(z_{m\nu})| \geq m$, $m = 1, 2, \ldots$, $\nu = \overline{1, j_m}$, and $\|p_{k_m}(z)\|_{G_m} \leq \frac{1}{3}$. By Mergelyan's Theorem (see [29]) such a polynomial always exists. Let's put $\|p_{k_m}\|_{\overline{U}} = A_m$.

Let $E \subset V$ be a polar compact (as every countable subset of V is polar we consider generally polar compacts). We denote by $t_m(w)$ Chebyshev polynomials (see [28]) for the compact set E, all zeroes of which also lie on E. Since the capacity is $C(E) = 0$, we have $\|t_m(w)\|_E^{\frac{1}{m}} \to 0$

as $m \to \infty$. This implies that there is a sequence of numbers s_m ($s_m < s_{m+1}$) such that the inequality $\|t_{s_m}(w)\|_E^{\frac{1}{s_m}} \leq \frac{1}{2A_m}$ holds for all $m \in \mathbb{N}$. Let's put $P_m(z) = [p_{k_m}(z)]^{s_m}$. Since all the roots of the polynomials $t_m(w)$ lie on the unit disc V, the inequality $\|t_m(w)\|_{\overline{V}} \leq 2^m$ holds.

We consider the series

$$f(z,w) = \sum_{m=1}^{\infty} P_m(z) \cdot t_{s_m}(w). \tag{2}$$

Series (2) has the following properties.

1. For each fixed $z^0 \in U$,

a) if $z^0 \in S$, then since

$$\left|P_m(z^0) \cdot t_{s_m}(w)\right| \leq \frac{1}{3^{s_m}} \cdot 2^{s_m} = \left(\frac{2}{3}\right)^{s_m},$$

the series

$$f(z^0,w) = \sum_{m=1}^{\infty} P_m(z^0) \cdot t_{s_m}(w)$$

converges uniformly in V, and hence, $f(z^0,w) \in \mathcal{O}(V)$,

b) if $z^0 \in U \backslash S$, then there exists a number m_0 such that, for all $m > m_0$, $z^0 \in K_m$, and therefore, from the expression

$$f(z^0,w) = \sum_{m=1}^{m_0} P_m(z^0) \cdot t_{s_m}(w) + \sum_{m=m_0+1}^{\infty} P_m(z^0) \cdot t_{s_m}(w),$$

using the arguments of the previous case a) again we get $f(z^0,w) \in \mathcal{O}(V)$.

2. For each fixed $w^0 \in E$, we have

$$|P_m(z)| \cdot \left|t_{s_n}(w^0)\right| \leq A_m^{s_m} \cdot \left(\frac{1}{2A_m}\right)^{s_m} = \left(\frac{1}{2}\right)^{s_m},$$

and therefore, we get $f(z,w^0) \in \mathcal{O}(U)$.

Thus, for the functions $f(z,w)$ defined by series (2), conditions (1) and (2) of Theorem 2 are satisfied.

However, since there is a lower bound

$$|f(z_{\nu\mu},w)| = \left|P_\nu(z_{\nu\mu}) \cdot t_{s_\nu}(w) + \sum_{\substack{m=1\\ m\neq\nu}}^{\infty} P_m(z_{\nu\mu}) \cdot t_{s_m}(w)\right| \geq \nu^{s_\nu} \cdot |t_{s_\nu}(w)| - \sum_{m=1}^{\infty}\left(\frac{2}{3}\right)^{s_m}$$

and when $w \notin E$, then $d = \operatorname{dist}(w,E) = \inf_{\xi \in E} |w - \xi| > 0$ and for any ν, there holds lower estimation $|t_{s_\nu}(w)| > d^{s_\nu}$. This implies $|f(z_{\nu\mu},w)| \geq (\nu \cdot d)^{s_\nu} - \sum_{m=1}^{\infty}\left(\frac{2}{3}\right)^{s_m} \to \infty$ when $\nu \to \infty$, i. e., the function $f(z,w)$ is not holomorphic at any point of the set $S \times V$.

Now we prove that the function $f(z,w)$ is holomorphic in the domain $(U \backslash S) \times V$. Let $z^0 \in U \backslash S$, then we can show that there exists a neighborhood $U(z^0,r)$, such that the series (2) converges uniformly in the domain $U(z^0,r) \times V$.

If $z^0 \in U\backslash S$, then there exists a number m_0 and a neighborhood $U(z^0, r)$ such that for all $m > m_0$, it holds $U(z^0, r) \subset K_m$. It follows that for any $z \in U(z^0, r)$ and for any $w \in V$ we have the representation

$$f(z,w) = \sum_{m=1}^{m_0} P_m(z) \cdot t_{s_m}(w) + \sum_{m=m_0+1}^{\infty} P_m(z) \cdot t_{s_m}(w),$$

where the common terms of the series are estimated as

$$|P_m(z) \cdot t_{s_m}(w)| \leq \left(\frac{2}{3}\right)^{s_m}.$$

Therefore, in this case we get $f(z,w) \in \mathcal{O}\left(U(z^0, r) \times V\right)$.

Thus, the function $f(z,w)$ defined by the series (2) is holomorphic in the domain $(U\backslash S) \times V$.

Generally, we can represent the set S as a union (at most countable number of components and they may intersect if it is necessary) of sets with connected complements and for each part we construct the function as above. Summing these functions we get the corresponding function. □

Remark 1. If a nowhere dense closed subset S compactly lies in the unit disc U then the Theorem 2 is not true, i. e., there is no function satisfying conditions of Shimoda's Theorem, holomorphic on $(U\backslash S) \times V$ and doesn't extend holomorphically to any point of the set $S \times V$. All of such functions are holomorphic on $U \times V$.

I n d e e d, let $S \subset\subset U$ be a compact nowhere dense closed subset. And let a function $f(z,w)$ be holomorphic on $(U\backslash S) \times V$ and for each fixed $z^0 \in U$ the function $f(z^0, w) \in \mathcal{O}(V)$ and for each fixed w_j from the set $E \subset V$ with at least one limit point in V, the function $f(z, w_j) \in \mathcal{O}(U)$. For a subdomain D: $S \subset\subset D \subset\subset U$ with smooth boundary ∂D we consider the Cauchy integral

$$F(z,w) = \int_{\partial D} \frac{f(\xi, w)}{\xi - z}\, d\xi$$

for fixed $z \in U\backslash \bar{D}$. As $f(z, w_j) \in \mathcal{O}(U)$, we have

$$F(z, w_j) = \int_{\partial D} \frac{f(\xi, w_j)}{\xi - z}\, d\xi = 0.$$

And consequently, from the uniqueness Theorem we get $F(z,w) \equiv 0$ for all $z \in U\backslash \bar{D}$.

On expansion of the function $f(z,w)$

$$f(z,w) = \sum_{k=0}^{\infty} c_k(z) w^k$$

it is known that $c_k(z) \in \mathcal{O}\left(U\backslash S\right)$. Therefore from equality

$$0 = \int_{\partial D} \frac{f(\xi, w)}{\xi - z}\, d\xi = \sum_{k=0}^{\infty} \int_{\partial D} \frac{c_k(\xi)}{\xi - z}\, d\xi \cdot w^k$$

we get

$$\int_{\partial D} \frac{c_k(\xi)}{\xi - z}\, d\xi = 0$$

for all $z \in U \backslash \bar{D}$. On the other hand, if we take a disc $U_r = \{|z| < r\} \supset\supset D$ with the radius $r < 1$ and close enough to 1 then from the Cauchy integral formula the equality

$$c_k(z) = \frac{1}{2\pi i}\int_{|\xi|=r} \frac{c_k(\xi)}{\xi - z}\,d\xi - \frac{1}{2\pi i}\int_{\partial D} \frac{c_k(\xi)}{\xi - z}\,d\xi$$

holds for all $z \in U_r \backslash \bar{D}$. As the second integral is zero and it is independent of the chosen subdomain D we get the integral formula

$$c_k(z) = \frac{1}{2\pi i}\int_{|\xi|=r} \frac{c_k(\xi)}{\xi - z}\,d\xi, \quad z \in U_r \backslash \bar{D}.$$

It means that coefficients holomorphically extend to U_r and we denote these extensions by

$$\hat{c}_k(z) = \frac{1}{2\pi i}\int_{|\xi|=r} \frac{c_k(\xi)}{\xi - z}\,d\xi, \quad z \in U_r, \quad \hat{c}_k(z) \in \mathcal{O}\left(U_r\right).$$

As r is arbitrary we get $\hat{c}_k(z) = c_k(z)$, $z \in U \backslash S$, and now we show that $\hat{c}_k(z) \equiv c_k(z)$, $z \in U$.

Thus we have a holomorphic function $\hat{f}(z, w) \in \mathcal{O}(U \times V)$ defined by the following Hartogs series

$$\hat{f}(z, w) = \sum_{k=0}^{\infty} \hat{c}_k(z) w^k, \quad z \in U.$$

If we fix $z \in S$, then for $w = w_j$ we have the equality

$$\sum_{k=0}^{\infty} \hat{c}_k(z) w_j^k = \sum_{k=0}^{\infty} c_k(z) w_j^k,$$

therefore,

$$\sum_{k=0}^{\infty} \left(\hat{c}_k(z) - c_k(z)\right) w_j^k = 0.$$

And the uniqueness Theorem shows that

$$\sum_{k=0}^{\infty} \left(\hat{c}_k(z) - c_k(z)\right) w_j^k \equiv 0.$$

Consequently, $\hat{c}_k(z) \equiv c_k(z)$, $z \in U$ and

$$f(z, w) = \sum_{k=0}^{\infty} \hat{c}_k(z) w^k. \qquad \square$$

§ 3. Separately holomorphic functions without singularities

In this section, we consider questions about additional conditions to Shimoda's Theorem that provide an empty exceptional set S, i. e., whether a function satisfying the conditions of the Theorem is holomorphic on $U \times V$ under additional requirements?

When the set E is not polar, T. Terada [4] proved that the exceptional set S is empty. Later J. Siciak [5] and V. P. Zaharyuta [6] described the envelope $\widehat{X}$ of holomorphy of so called separately holomorphic functions on special (cross) sets $X = (D' \times G) \cup (D \times G')$, $D' \subset D$, $G' \subset G$. They showed that if a function $f(z, w)$ defined on the cross set X satisfies the next two conditions:

(1) for each fixed $z^0 \in D'$ the function $f(z^0, w)$ is holomorphic in G;

(2) for each fixed $w^0 \in G'$ the function $f(z, w^0)$ is holomorphic in D,

then the function $f(z,w)$ is called separately holomorphic on X and holomorphically continues to the domain

$$\widehat{X} = \{(z,w) \mid \omega^*(z, D', D) + \omega^*(w, G', G) < 1\}.$$

For the cases $D = U$, $G = V$, $D' = U$ and $G' = E$ the nonpolar Terada's result is a strictly consequence of Siciak–Zaharyuta Theorem and the function f is holomorphic on

$$\widehat{X} = \{(z,w) \mid \omega^*(z, U, U) + \omega^*(w, E, V) < 1\} = U \times V,$$

i. e., $S = \emptyset$.

We can get similar result without expanding the set E, by imposing another condition on the function $f(z,w)$, i. e., the following Theorem is true.

Theorem 3. *Let $f(z,w)$ be a function defined in the domain $U \times V = \{|z| < 1\} \times \{|w| < 1\} \subset \subset \mathbb{C}^n_z \times \mathbb{C}_w$ and let $E \subset V$ be a countable subset having one limit point $a \in V$. If*

(1) *for each fixed $z^0 \in U$, $f(z^0, w) \in \mathcal{O}(V)$,*

(2) *for each fixed $w^0 \in E$, $f(z, w^0) \in \mathcal{O}(U)$,*

(3) *the function $f(z,w)$ is locally bounded in some neighborhood of the set $U \times \{a\}$,*

then $f(z,w) \in \mathcal{O}(U \times V)$.

P r o o f. Without loss of generality, we may assume that the point $a = 0$ is the limit point of the set E. According to the third condition of the Theorem 3, the function $f(z,w)$ is locally bounded on a neighborhood of the set $U \times \{0\}$. If we fix an arbitrary subdomain $U' \subset\subset U$, then there exists a circle $V_\sigma = \{|w| < \sigma\}$, such that the function $f(z,w)$ is bounded on the set $U' \times V_\sigma$. On the other hand, starting from some number j_0, all points w_j belong to $V_\sigma = \{|w| < \sigma\}$. According to the arguments used in the proof of Shimoda's Theorem, the function $f(z,w)$ expands into a Hartogs series

$$f(z,w) = \sum_{k=0}^{\infty} c_k(z) w^k$$

with holomorphic coefficients $c_k(z) \in \mathcal{O}(U')$ and the series converges uniformly on compact subsets $U' \times V_\sigma$. Therefore, we get that $f(z,w) \in \mathcal{O}(U' \times V_\sigma)$. Using now Hartogs's lemma (see [1]), we obtain that the function $f(z,w)$ is holomorphic on $U' \times V$. Since $U' \subset\subset U$ is an arbitrary subdomain, $f(z,w) \in \mathcal{O}(U \times V)$. □

This Theorem implies the following result proved by Hukuhara.

Corollary 1. *Let $f(z,w)$ be a function defined in a domain $U \times V = \{|z| < 1\} \times \{|w| < 1\} \subset \subset \mathbb{C}^n_z \times \mathbb{C}_w$ and let $E \subset V$ be a countable subset having at least one limit point belonging to V. If*

(1) *for each fixed $z^0 \in U$, $f(z^0, w) \in \mathcal{O}(V)$,*

(2) *for each fixed $w^0 \in E$, $f(z, w^0) \in \mathcal{O}(U)$,*

(3) *$f(z,w)$ is locally bounded to $U \times V$,*

then $f(z,w) \in \mathcal{O}(U \times V)$.

§ 4. Some generalizations and applications

In [14], A. Sadullaev and T. Tuichiev studied the holomorphicity of the sum of the Hartogs series.

Theorem 4. *Let a series*

$$f(z,w)=\sum_{k=0}^{\infty} c_k(z)w^k$$

be given where the coefficients $c_k(z)$ are holomorphic in the domain $D\subset\mathbb{C}^n$. Then there exists a nowhere dense closed subset $S\subset D$ and a lower semicontinuous function $R_*(z)$ on the set $D\backslash S$ such that the sum of the series $f(z,w)$ is holomorphic with respect to the set of variables in $(D\backslash S)\times\{|w|<R_*(z)\}$.

Using Theorem 4, we can generalize Theorem 1 as follows.

Theorem 5. *Let $f(z,w)$ be defined in a domain $U\times V=\{|z|<1\}\times\{|w|<1\}\subset\mathbb{C}^n_z\times\mathbb{C}_w$ and let $\{w_j\}$ be a sequence of points from V that converges to origin. If there exists a positive real-valued function $r(z)$ that is lower semicontinuous on U and the following conditions are satisfied:*

(1) *for each fixed $z^0\in U$, $f(z^0,w)\in\mathcal{O}(|w|<r(z^0))$,*
(2) *for each fixed $j=1,2,\ldots$, $f(z,w_j)\in\mathcal{O}(U)$,*

then there exists a nowhere dense set S on U and a lower semicontinuous function $R_*(z)$ on $U\backslash S$ such that $f(z,w)\in\mathcal{O}\left((U\backslash S)\times\{|w|<R_*(z)\}\right)$.

P r o o f. We fix an arbitrary open subset $G\subset\subset U$ and consider the following sets

$$E_m=\left\{z\in U\colon r(z)>\frac{1}{m}\right\},\quad m=1,2,3,\ldots.$$

Obviously $E_m\subset E_{m+1}$ and $G\subset\bigcup_{m=1}^{\infty}E_m$. Moreover, for every fixed $m\in\mathbb{N}$ the set E_m is an open set. Thus, for each point $z\in G$ there is some number $m_z\in\mathbb{N}$ and a neighborhood Ω_z, such that $r(\xi)>\frac{1}{m_z}$ holds for all $\xi\in\Omega_z$. Consider now an open cover

$$\bar{G}\subset\bigcup_{z\in\bar{G}}\Omega_z,$$

which contains a finite sub covering $\bar{G}\subset\Omega_{z_1}\cup\Omega_{z_2}\cup\ldots\cup\Omega_{z_s}$.

If we put

$$\delta=\min_{1\le\nu\le s}\frac{1}{m_{z_\nu}}$$

then, by construction, we get $\delta>0$ and therefore, the function $f(z,w)$ satisfies the following conditions:

(1) for each fixed $z^0\in G$, the function $f(z^0,w)$ is holomorphic in the circle $V_\delta=\{|w|<\delta\}$,

(2) for each fixed $j\colon|w_j|<\delta$, $f(z,w_j)\in\mathcal{O}(G)$,

i. e., the function $f(z,w)$ satisfies the conditions of Shimoda's Theorem on $G \times V_\delta$. Hence, there exists a nowhere dense closed set S such that the function $f(z,w)$ is holomorphic in $(G \backslash S) \times V_\delta$. We now expand the function $f(z,w)$ into a Hartogs series by powers of w

$$f(z,w) = \sum_{k=0}^{\infty} c_k(z) w^k,$$

where $c_k(z) \in \mathcal{O}\,(G \backslash S)$.

Then from Theorem 4 there exists a function $R_*(z)\colon\ -\ln R_*(z) = \left(\overline{\lim_{k\to\infty}} \frac{1}{k} \ln |c_k(z)| \right)^*$ such that $f(z,w) \in \mathcal{O}\,((G \backslash S) \times \{|w| < R_*(z)\})$. Since the subset $G \subset\subset U$ is arbitrary, there exists a nowhere dense closed subset $S \subset U$ and a lower semicontinuous function $R_*(z)$ on $U \backslash S$ such that $f(z,w) \in \mathcal{O}\,((U \backslash S) \times \{|w| < R_*(z)\})$. □

Question. Can a function with the properties of Shimoda's Theorem be represented by a Hartogs series, i. e., is the question posed by Hukuhara equivalent to the holomorphicity of the sum of the Hartogs series?

We can give negative answer to the question with counterexample constructed by using arguments from the proof of Theorem 2 and this shows that these two questions are different.

Example. Let $S = [0, i) \subset U = \{|z| < 1\}$ and $E = \left\{ \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots \right\} \subset V = \{|w| < 1\}$. We construct polynomials $p_{k_m}(z)$ as in the proof of the Theorem 2 and consider the series

$$f(z,w) = \sum_{m=1}^{\infty} [p_{k_m}(z)]^m \cdot \left(w - \frac{1}{2}\right) \cdot \left(w - \frac{1}{3}\right) \cdot \ldots \cdot \left(w - \frac{1}{m+1}\right).$$

This function satisfies the conditions of Shimoda's theorem and it is holomorphic in $(U \backslash [0,i)) \times V$. On the other hand, if we decompose series by the powers of w then we get

$$f(z,w) = \left(\sum_{m=1}^{\infty} (-1)^m \frac{1}{(m+1)!} [p_{k_m}(z)]^m \right) + \\ + \left(\sum_{m=1}^{\infty} (-1)^{m+1} \frac{2+3+\ldots+(m+1)}{(m+1)!} [p_{k_m}(z)]^m \right) w + \ldots.$$

Thus, we get Hartogs series with the coefficients

$$c_0(z) = \sum_{m=1}^{\infty} (-1)^m \frac{1}{(m+1)!} [p_{k_m}(z)]^m,$$

$$c_1(z) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{2+3+\ldots+(m+1)}{(m+1)!} [p_{k_m}(z)]^m, \quad \ldots.$$

Therefore, for each $k \in \mathbb{N}$ we get that $c_k(z) \in \mathcal{O}\,(U \backslash [0,i))$.

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О теореме Шимоды

Ключевые слова: теорема Гартогса, теорема Шимоды, сепаратно голоморфные функции, степенные ряды.

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Настоящая работа посвящена теореме Шимоды о голоморфности функции $f(z,w)$, которая является голоморфной по $w \in V$ при фиксированном $z \in U$ и голоморфна по $z \in U$ при фиксированном $w \in E$, где $E \subset V$ — счетное множество, по крайней мере, с одной предельной точкой в V. Шимода доказывает, что в этом случае $f(z,w)$ голоморфно в $U \times V$, за исключением нигде не плотного замкнутого подмножества $U \times V$. Рассматривается обратная задача и доказывается, что для любого заранее заданного нигде не плотного замкнутого подмножества $S \subset U$ существует голоморфная функция, удовлетворяющая теореме Шимоды на $U \times V \subset \mathbb{C}^2$, не голоморфная на $S \times V$. Кроме того, исследованы дополнительные условия, которые влекут за собой пустые множества особенностей в теореме Шимоды. Доказывается обобщение в случае, когда функция имеет переменный радиус голоморфности по одному из направлений.

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