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THE STABILITY OF COMPLETELY CONTROLLABLE SYSTEMS

The subject of this paper is the stability of completely controllable systems defined on a smooth manifold. It is known that the controllability sets of symmetric systems generate singular foliations. In the case when the controllability sets have the same dimension, a regular foliation arises. Thus, the possibility of applying the methods of foliation theory to control theory problems arises. This paper presents some of the authors' results on the possibility of applying the theorems on the stability of leaves to the problems on the stability of completely controllable systems and on the geometry of attainability sets. Smoothness throughout the work will mean smoothness of class C^∞.

Keywords: control systems, controllability sets, orbit of vector fields, singular foliation.

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Introduction

This paper contains some results of the first author on the stability of control systems, which were published in papers [1–7]; some examples of control systems were suggested by the second author.

The paper deals with the control system

$$\dot{x} = f(x, u), \quad x \in M, \quad u \in U \subset \mathbb{R}^m, \tag{0.1}$$

where M is a connected smooth (of class C^∞) manifold of dimension n, for each $u \in U$ the vector field $f(\cdot, u)$ is smooth of class C^∞, and the mapping $f\colon M \times U \to TM$ is continuous, U is a compact set.

Admissible controls for the system (0.1) are piecewise constant functions $u\colon [0, T] \to U$, where $0 < T < \infty$. The trajectory of system (0.1) is a piecewise smooth mapping $x\colon [0, T] \to M$ satisfying the equality $\dot{x} = f(x(t), u(t))$ for all $t \in [0, T] \backslash E$, where $u\colon [0, T] \to U$ is some admissible control with a set E of discontinuity points, which consists of a finite number of points.

We say that from a point $x_1 \in M$ one can get to a point $x_2 \in M$ in time T if there exists a trajectory $x\colon [0, T] \to M$ of system (0.1) such that $x(0) = x_1$ and $x(T) = x_2$. The set of points M from which one can get to $\eta \in M$ will be called the controllability set with the target point η and will be denoted by G_η. By definition, we put $\eta \in G_\eta$ for $\eta \in M$.

Let M be a smooth manifold of dimension n, D be the family of smooth vector fields defined on the manifold M. The family D may contain a finite or infinite number of smooth vector fields.

For a point $x \in M$, by $t \to X^t(x)$ we denote the integral curve of the vector field X passing through the point x at $t = 0$. The mapping $t \to X^t(x)$ is defined on some domain $I(x)$, which in the general case depends not only on the field X, but also on the initial point x.

Definition 1 (see [8, 9]). The *orbit* $L(x)$ of a family D of vector fields passing through a point x is defined as the set of points y from M for which there are real numbers $t_1, t_2, \ldots, t_k$ and vector fields $X_1, X_2, \ldots, X_k$ from D (where k is an arbitrary natural number) such that

$$y = X_k^{t_k}(X_{k-1}^{t_{k-1}}(\ldots(X_1^{t_1}(x))\ldots)).$$

It is clear that an orbit is a smooth curve (one-dimensional manifold) if D consists of one vector field.

It was proved in [8, 9] that each orbit of a family of smooth vector fields has a differential structure with respect to which it is a smooth submanifold immersed in M (see also [10–12]). Recall that a submanifold $N \subset M$ is called immersed in M if the canonical injection $i\colon N \to M$ is a differentiable map of maximum rank.

Each orbit has two topologies: its own topology as an immersed submanifold and the induced topology from M. The proper topology of an orbit is stronger than the topology induced from M. Indeed, if $x \in L(x_0)$, where $x \in M$, $V(x)$ is an open set in M containing x, then $L(x_0) \cap V(x)$ is an open set in the induced topology $L(x_0)$. For each point $y \in L(x_0) \cap V(x)$ the image of the point y under $i\colon L(x) \to M$ is contained in $V(x)$, and by virtue of the continuity of the mapping i, there exists a neighborhood $U(y)$ of a point in the topology $L(x_0)$ such that $U(y) \subset V(x)$. This implies that $L(x_0) \cap V(x)$ is open in the topology $L(x_0)$.

As examples show, even when D consists of one vector field, these two topologies do not always coincide. For example, for an irrational winding of a torus, these topologies are different for all trajectories.

Numerous studies have been devoted to the study of the geometry and topology of the orbits of vector fields [1–14].

Definition 2. An *orbit* L is called *proper* if the canonical injection $i\colon L \to M$ is an injection, that is, when the topology of the leaf coincides with the induced topology from M.

The orbits of the family of C^r-vector fields generate a singular C^r-foliation for $r \geq 1$. This follows from the works of P. Stefan [8] and H. Sussmann [9].

Let $V(M)$ be the set of all smooth (of class C^∞) vector fields defined on M. The set $V(M)$ is a Lie algebra in which the commutator of two vector fields $X, Y \in V(M)$ is their Lie bracket $[X, Y]$.

Let $A(D)$ denote the smallest Lie subalgebra containing D and $A_x(D) = \{X(x)\colon X \in A(D)\}$ for all $x \in M$. The resulting distribution $x \to A_x(D)$ is involutive, and if $\dim A_x(D) = \text{const} = k$ for all $x \in M$, then by the Frobenius theorem it is completely integrable. In this case, each orbit is a leaf of the k-dimensional foliation F [11, 12].

The study of the geometry and topology of the orbits of vector fields is important in control theory and in the theory of differential games [13, 14].

Control system (0.1) generates a family of vector fields

$$D = \{f(\cdot, u)\colon u \in U\}. \tag{0.2}$$

If $u\colon [0, T] \to U$ is an admissible control with discontinuity points $t_1, t_2, \ldots, t_m$, where $0 < t_1 < t_2 < \ldots < t_m < T$, and $x\colon [0, T] \to M$ is the corresponding trajectory of system (0.1), then the restriction of x to $[t_i, t_{i+1}]$, where $i = 0, 1, \ldots, m$, $t_0 = 0$, $t_{m+1} = T$, is the integral curve of some vector field X_{i+1} from D. Therefore, if $x(0) = x_1$, $x(T) = x_2$, then it holds the equality

$$x_2 = X_m^{t_m}(X_{m-1}^{t_{m-1}}(\ldots(X_1^{t_1}(x_1))\ldots)),$$

where $\tau_i = t_i - t_{i-1}$, $i = 1, 2, \ldots, m$.

Consequently, the set G_η is a subset of the orbit $L(\eta)$ of the family D for every $\eta \in M$.

Definition 3. *System* (0.1) is called *symmetric* if, for every $u \in U$ there exists $v \in U$ such that $f(x, v) = -f(x, u)$ for all $x \in M$.

Obviously, if system (0.1) is symmetric, then the set G_η coincides with the orbit $L(\eta)$ of the family vector fields D from (0.2). Thus, the controllability sets of symmetric control system (0.1) generate a singular foliation F. If $\dim A_x(D) = k$ for all $x \in M$, then each orbit is a leaf of the k-dimensional foliation F [7].

Definition 4. We say that *system* (0.1) is *controllable from the point* η if $G_\eta = L(\eta)$.

Definition 5. We say that *system* (0.1) is *completely controllable on* $L(\eta_0)$ if $G_\eta = L(\eta_0)$ for every $\eta \in L(\eta_0)$ where $\eta_0 \in M$.

By the definition of an orbit, for each $\eta \in M$ the set $L(\eta)$ is an invariant set of system (0.1), that is, every trajectory of system (0.1) starting on it stays on it. Therefore, if the purpose of control is to bring system (0.1) to a point $\eta \in M$ then it is sufficient to consider the system only on $L(\eta)$, since it is impossible to get to the point η from the points $M \setminus L(\eta)$.

Let $L = L(\eta_0)$ be some orbit of the family D, and system (0.1) is completely controllable on L. Consider the question of under what conditions completely controllable system (0.1) on $L(\eta_0)$ will be completely controllable on orbits $L(\eta_0)$ if the point η is sufficiently close to η_0.

In the case when the orbit L is a compact set, this question was solved in [13]. In [13] it was proved that if the foliation F is regular and L is a compact leaf with a finite holonomy group, then the complete controllability of system (0.1) on L implies that system (0.1) is completely controllable on all leaves sufficiently close to L.

In this paper, we will consider this question when the orbit L is not a compact set. It turns out that this question is closely related to the question of the stability of the leaf $L(\eta_0)$ of the foliation F, as well as the continuity of the map $\eta \to L(\eta)$.

In § 1, stability theorems are discussed for the leaves of the foliation F. In § 2, using the theorems of the first section, sufficient conditions for the “stability” of a completely controllable system (0.1) are obtained. In the § 3, the control system under consideration, the right-hand side of which continuously depends on a certain parameter, and the question of sufficient conditions for the stability of a completely controllable system with respect to a parameter, are studied.

§ 1. Stability theorems for foliations

Let F be a k-dimensional foliation. J. Reeb proved the following theorem on the stability of a compact leaf in 1944 [14].

Theorem 1. *Let L be a compact leaf of a foliation F. If the holonomic group of the leaf L is finite, then for every open set V containing L there exists an open invariant set V_0 such that $L \subset V_0 \subset V$, every leaf from V_0 is compact, and has a finite holonomy group.*

In 1976, at an international conference in Rio de Janeiro, Hector raised the question of the possibility of generalizing Reeb theorem to non-compact leaves [13]. In 1977, the Japanese mathematician T. Inaba constructed an example that showing that when the codimension of a foliation is greater than one, then Reeb theorem cannot be generalized for non-compact proper leaves [16]. Thus, Hector’s question about a generalization of Reeb theorem to non-compact leaves needs to be considered only for foliations of codimension one.

Let $\dim A_x(D) = n-1$ for all $x \in M$. Then F is a foliation of dimension $n-1$ (codimension one). Suppose that the foliation F is transversally orientable, that is, there is a nondegenerate smooth vector field X on M that is transversal to the leaves of F.

Let L_0 be a proper leaf of the foliation F, and $r > 0$. We put

$$U_r = \{y \in M \colon \rho(y, L_0) < r\},$$

where $\rho(y, L_0)$ is the distance from the point y to the leaf L_0. Obviously, for every $r > 0$, the set U_r is an open set.

In 1977, T. Inaba in the same paper [16] proved the following theorem.

Theorem 2. *Let M be a compact manifold of dimension 3, F be a transversally orientable foliation of codimension one, L_0 is a proper leaf of F. Then, if the fundamental group of the leaf L_0 is finitely generated and the holonomy group $H(L_0)$ is trivial, then for every $r > 0$ there exists an open invariant set V such that $L \subset V \subset U_r$, each leaf of V is diffeomorphic to the leaf L_0, and the restriction of F to V is a bundle with fiber L_0.*

Let L_0 be a proper leaf of F. Suppose that for each point $x \in L_0$ there exists a number $r = r_x > 0$ such that for each horizontal curve

$$h\colon [0,1] \to U_r = \{y \in M\colon \rho(y, L_0) < r\}$$

starting at x, and for each path $v\colon [0,1] \to L_0$ starting at $x = h(0)$ (vertical path), there exists a continuous mapping (vertical-horizontal homotopy) $\psi\colon [0,1]\times[0,1] \to M$ such that $\psi(t,0) = v(t)$ for $t \in [0,1]$, $\psi(0,s) = h(s)$ for $s \in [0,1]$. A smooth curve $h\colon [0,1] \to M$ is called horizontal if $\dfrac{dh(s)}{ds} \in H(h(s))$, where $H(x) = \{\lambda X(x)\colon \lambda \in R\}$, X-transverse to F vector field on M. Under this condition, the following generalization of the theorem of J. Reeb takes place [7].

Theorem 3. *Let F be a transversally orientable foliation of codimension one on M, L_0 be a relatively compact proper leaf with a finitely generated fundamental group. Then, if the holonomy group of the leaf L_0 is trivial, then for every $r > 0$ there exists an open invariant set V such that $L_0 \subset V \subset U_r$, every leaf from V is diffeomeorphic to the leaf L_0, and the restriction of F to V is a bundle over R with fiber L_0.*

Remark 1. A compact leaf is always an immersed submanifold of M, i. e., it is a proper leaf. It is known that the fundamental group of a compact manifold is a finitely generated group.

In what follows, assume that the foliation is Riemannian.

Definition 6. A *foliation F* is said to be *Riemannian* if every geodesic orthogonal at some point to a leaf of F remains orthogonal to all leaves F at all its points.

Regular Riemannian foliations were introduced by Reinhart in [19] and studied by many authors, in particular, in [5, 17, 18, 20]. P. Molino introduced singular Riemannian foliations in his monograph [18]. Singular Riemannian foliations arise in the classical problem of Riemannian geometry on the action of the group of isometries. If the set D consists of Killing vector fields orbits of the family D generate singular Riemannian foliations.

Let us recall the notion of a Killing vector field [22].

Definition 7. The *vector field X on M* is called a *Killing vector field* if the one-parameter group of local transformations $x \to X^t(x)$, generated by the field X, consists of isometries.

Remark 2. A vector field X on Riemannian manifold (M, g) is a Killing vector field if and only if $L_X g = 0$, where $L_X g$ denotes the Lie derivative of the metric g with respect to X. The equality $L_X g = 0$ is equivalent to the condition

$$Xg(Y, Z) = g([X, Y], Z) + g(Y, [X, Z]),$$

where Y, Z are arbitrary smooth vector fields, and $[X, Y]$ is a Lie bracket of vector fields X, Y [12].

The geometry of singular Riemannian foliations generated by orbits of Killing vector fields are studied in papers [21–26].

Let (M, g) be a Riemannian manifold of dimension n, F is a singular Riemannian foliation on M. In this case the following generalization of Reeb theorem holds [5].

Theorem 4. *Let (M, g) be a complete Riemannian manifold of dimension n and L is a proper leaf of F. Then, for each $r > 0$ there exists an open invariant neighborhood V of the leaf L such that $L \subset V \subset U_r$ and the restriction of F to V is a smooth bundle with base L.*

In the case when F is a regular foliation of codimension one the following theorem is true [5].

Theorem 5. *Let F be a Riemannian foliation of codimension one on a complete Riemannian manifold (M, g), L is a compact leaf. Then, for every open set V containing L, there exists an open invariant neighborhood U of the leaf F such that $L \subset U \subset V$ and U consists of compact leaves diffeomorphic to L.*

§ 2. Stability of completely controllable systems

In this section, using the results of § 1, we obtain sufficient conditions for the "stability" of a completely controllable system (0.1). Admissible controls are piecewise constant functions taking values from U.

By using Reeb stability theorem the following theorem was proved in [13].

Theorem 6. *Let L_0 be a compact leaf of a foliation F with a finite holonomy group. Then, if system (0.1) is completely controllable on L_0, then it is completely controllable on leaves sufficiently close to L_0.*

By Reeb's theorem, if L_0 is a compact leaf with a finite holonomy group, then for every open set V containing L_0 there exists an open invariant set U such that $L_0 \subset U \subset V$, U consists of compact leaves. Thus, Reeb's theorem allows one to obtain a sufficient condition for the local stability of a completely controllable system in the case when L_0 is compact. As the following examples show, this theorem is not true if L_0 is a non-compact leaf or L_0 is a compact leaf whose holonomy group is not a finite group.

Example 1. Let $M = \mathbb{R}^2 \backslash \{(0, 0)\}$ with Cartesian coordinates (x, y), the family D consists of one vector field

$$X(x, y) = \big((1-\rho)x - y\big)\frac{\partial}{\partial x} + \big(x + (1-\rho)y\big)\frac{\partial}{\partial y}$$

where $\rho = x^2 + y^2$. The circle $S^1 = \{(x, y) \colon x^2 + y^2 = 1\}$ is a limit cycle for the system

$$\dot{x} = -y + (1-\rho)x, \quad \dot{y} = x + (1-\rho)y \tag{2.1}$$

since the vector field X tangents S^1 at its each point.

If we take S^1 as the compact leaf L_0, then system (2.1) is completely controllable on L_0. Other trajectories are not closed; therefore, system (2.1) is not completely controllable on other trajectories. The holonomy group of the L_0 is a cyclic group.

Example 2. Let

$$M = \mathbb{R}^3 \backslash \{(x_1, x_2, x_3) \colon x_1 = x_2 = 0\},$$

with coordinates (x_1, x_2, x_3), $D = \{X_1, X_2, X_3\}$, where

$$X_1 = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}, \quad X_2 = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}, \quad X_3 = -\varphi(x)x_1\frac{\partial}{\partial x_1} - \varphi(x)x_2\frac{\partial}{\partial x_2},$$

$\varphi(x) = \psi\big((x_1^2 + x_2^2)x_3^2\big)$, $\psi\colon \mathbb{R}^1 \to \mathbb{R}^1$ is a smooth function of class C^∞ such that $\psi(t) > 0$ for $-1 < t < 1$ and $\psi(t) = 0$ for $|t| \geq 1$.

The orbits of the family D define a two-dimensional foliation F on the M, where for the point $\eta_0 = \big(x_1^0, x_2^0, x_3^0\big)$ the orbit $L(\eta_0)$ coincides with the intersection $M \bigcap \Pi(x_3^0)$ where $\Pi(x_3^0)$ is the plane in R^3 defined by the equation $x_3 = x_3^0$.

Consider a control system

$$\dot{x} = f(x, u), \quad u \in U = \{u_1, u_2, u_3\}, \tag{2.2}$$

where $f(x, u_i) = X_i(x)$, $i = 1, 2, 3$.

We put $B_1 = \{x \in M\colon X_3(x) \neq 0\}$, $B_2 = M \backslash B_1$. For $\eta_0 = (x_1^0, x_2^0, 0)$ the leaf $L(\eta_0)$ does not intersect with B_1, and therefore system ??(4) is completely controllable $L_0 = L(\eta_0)$. For each leaf L such that the intersection $L \cap B_2 \neq \emptyset$, it is impossible to get from the points $L \cap B_2$ to the points of the set $L \cap B_1 \neq \emptyset$.

Consequently, system (2.2) is not completely controllable on leaves L different from L_0. The holonomy group of the leaf L_0 is trivial, but it is not a compact leaf. In this example although the leaf L_0 is not relatively compact, it is locally stable.

This fact shows that if system (0.1) is completely controllable on the leaf L_0 satisfying the conclusion of Theorem 3, then this does not imply that system (0.1) is completely controllable on close leaves. Therefore, in the case when L_0 is a non-compact leaf, additional conditions are needed on system (0.1), which would guarantee the "stability" of system (0.1) that is completely controllable on L_0.

Definition 8. A control *system* (0.1) is called *normally-locally controllable* (in short, *N-locally controllable*) *near a point* $p \in L(\eta)$, if for any $T > 0$ there exists a neighborhood V of the point p in $L(\eta)$, such that from each point of the set V one can reach the point p in time less than T.

Definition 9. We say that *system* (0.1) is *completely controllable* (or *N-locally controllable*) *on an invariant set* $S \subset M$, if it is completely controllable (or N-locally controllable) on each leaf from S.

Theorem 3 of § 2 allows us to prove the following theorem [7].

Theorem 7. *Let* $\dim A_x(D) = n - 1$ *for all* $x \in M$*, and* F *is a transversally orientable foliation, the leaf* L_0 *satisfies the conditions of Theorem* 3*. If system* (0.1) *is* N*-locally controllable on* $\overline{L_0}$ (*closure in* M)*, then there exists an invariant neighborhood* V *of the leaf* L_0 *such that system* (0.1) *is completely controllable on each leaf from* V.

Now let us return to the case $\dim A_x(D) = k$ for all $x \in M$, where $0 < k < n$. In this case, F is a k-dimensional foliation. Suppose that the foliation F is a Riemannian foliation with respect to the Riemannian metric g. A necessary and sufficient condition for being Riemannian was given in [5]. This condition applies to vector fields from D and the Riemannian metric g.

The following result was obtained in [7].

Theorem 8. *Let* (M, g) *be a complete Riemannian manifold and* L_0 *be a relatively compact leaf of the foliation* F*. Then, if system* (0.1) *is* N*-locally controllable on* $\overline{L_0}$ (*the closure of* L_0) *then there exists an invariant neighborhood* V *of the leaf* L_0 *such that on each leaf of* V *the system* (0.1) *is completely controllable.*

§ 3. Stability of completely controllable systems with respect to the parameter

Consider a system of equations with a parameter

$$\dot{x} = f(x, u, \alpha), \quad x \in M, \quad u \in U, \tag{3.1}$$

where M is a smooth (of class C^{r+1}, $r \geq 1$) connected manifold of dimension n with some Riemannian metric g, U is a nonempty compact subset $\mathbb{R}^p$, the parameter α takes values from some open set $A \subset \mathbb{R}^q$.

It is assumed that, for each $\alpha \in A$ the mapping

$$f(\cdot, \cdot, \alpha)\colon M \times U \to TM$$

is continuous, and the vector fields $\{f(\cdot, u, \alpha)\colon u \in U\}$ are C^r-smooth vector fields (TM is the tangent bundle of the manifold M).

Considered admissible controls are piecewise constant functions $u\colon [0, T] \to U$, where $0 < T < \infty$.

Let $\eta \in M$, $G_\eta(\alpha_0)$ be the controllability set of system (3.1) with the target point η for $\alpha = \alpha_0$ that is, for control system

$$\dot{x} = f(x, u, \alpha_0), \quad x \in M, \quad u \in U, \tag{3.2}$$

which is obtained from (3.1) by setting $\alpha = \alpha_0$.

Recall that $G_\eta(\alpha_0)$ is the set of points M from which one can get to the point η along the trajectories of system (3.2). In what follows, we will everywhere assume that the right-hand side of system (3.1) depends continuously on α.

In [27], the following question was considered: if system (3.2) is N-locally controllable near the point η, then under what conditions will system (3.1) be N-locally controllable near the point η for α sufficiently close to α_0.

The following theorem is derived from the results of [27, 28].

Theorem 9. *Let system (3.2) be N-locally controllable near the point η. Then if the set $D^0(\eta_0) = \{f(\eta_0, u, \alpha_0)\colon u \in U\}$ contains a positive basis of the tangent space $T_{\eta_0}M$ of the manifold M at the point η_0, then there exists a neighborhood V of the point α_0 such that system (3.1) is N-locally controllable near the point η_0 for each $\alpha \in V$.*

Recall that a family of vectors $\{a_1, a_2, \ldots, a_m\}$ is called a positive basis in $\mathbb{R}^n$, if for every a there exist nonnegative numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that $a = \lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_m a_m$ [28].

It is known that for the vectors $a_1, a_2, \ldots, a_m$ to form a positive basis in $\mathbb{R}^n$, it is necessary and sufficient that for each unit vector a there is a_i, such that $(a, a_i) < 0$, where $(\cdot, \cdot)$ is the inner product [28]. Using this criterion for a positive basis and the continuity of the mapping $f(x, u, \alpha)$, it is easy to show that under the conditions of Theorem 9 there exists $\delta > 0$, such that for $\rho(\eta, \eta_0) < \delta$ and $\left|\alpha - \alpha_0\right| < \delta$ the set $D(\eta) = \{f(\eta, u, \alpha)\colon u \in U\}$ contains a positive basis for the tangent space $T_\eta M$.

Here $\rho(\eta, \eta_0)$ is the distance between the points η_0 and η defined by the Riemannian metric g on M, and $|\alpha - \alpha_0|$ is the Euclidean norm in $\mathbb{R}^q$. Since the set $D(\eta)$ contains a positive basis, system (3.1) is N-locally controllable near each point of the set $B(\delta) = \{\eta \in M\colon \rho(\eta, \eta_0) < \delta\}$ for each α, if $|\alpha - \alpha_0| < \delta$.

As noted in [27], if the set $D(\eta)$ does not contain a positive basis, then system (3.1) may not be N-locally controllable for $\alpha \neq \alpha_0$ even if α is sufficiently close to α_0.

Now we assume that system (3.2) is completely controllable and consider the question: under what conditions will system (3.1) be completely controllable for α sufficiently close to α_0.

In the case when M is a compact manifold and f smoothly depends on α, from the results of [10] we can obtain the following proposition.

Proposition 1. *There exists $\delta > 0$ such that system (3.1) is completely controllable for each α from $\{\alpha \in A\colon |\alpha - \alpha_0| < \delta\}$.*

The following theorem is a generalization of the above statement to the case when f depends continuously on α, which was proved in [7].

Theorem 10. *Let $K \subset M$ be a compact connected submanifold of dimension n and system (3.2) be completely controllable on K. Then there exists $\delta > 0$ such that system (3.1) is completely controllable on K for each α from the set $\{\beta \in A\colon |\beta - \alpha_0| < \delta\}$.*

Recall that system (3.1) is completely controllable on a subset $S \subset M$ if for each pair $(x, y) \in S \times S$ from the point x one can get to the point y along the trajectory of system (3.1).

Remark 3. The assertion of Theorem 10 is false if the dimension of K is less than n. For example, if the set U consists of one point, we have a family of vector fields $\{X_\alpha\colon \alpha \in A\}$. Suppose that the vector field X_{α_0} has a closed trajectory K. It is clear that system (3.1) for $\alpha = \alpha_0$ is completely controllable on K. But, if $\dim M > 1$ and the vector fields X_α are not tangent to K for $\alpha \neq \alpha_0$, then the system (3.1) is not completely controllable on K for $\alpha \neq \alpha_0$.

Recall that the set $V(M)$ of smooth vector fields of class C^∞ is a Lie algebra with respect to the Lie bracket $[X, Y]$ of vector fields $X, Y \in V(M)$. We put $D_\alpha = \{f(\cdot, u, \alpha)\colon u \in U\}$ and denote by P_α the smallest Lie subalgebra containing the set of vector fields D_α. Now, suppose that for every α system (3.1) is symmetric, and for every $u \in U$ a mapping $f(\cdot, u, \cdot)\colon M \times A \to TM$ of class C^∞. Symmetry means that if $X \in D_\alpha$ then $-X \in D_\alpha$.

In the paper [7] the following theorem was proved.

Theorem 11. *Let $P_\alpha(x) = \{X(x)\colon X \in P_\alpha\}$, $x \in M$. Suppose that system (3.2) is N-locally controllable near the point η. Then, if $\dim P_{\alpha_0}(\eta) = n$, then there exists a number $\delta > 0$ such that system (3.1) is N-locally controllable near the point η for each α from the set $B_\delta(\alpha_0) = \{\alpha \in A\colon |\alpha - \alpha_0| < \delta\}$.*

Now suppose that the dimension $\dim P_\alpha(x)$ does not depend on x, but depends on α. In this case, the following result was obtained in [7].

Theorem 12. *Let system (3.1) be completely controllable for $\alpha = \alpha_0$. Then there exists $\delta > 0$ such that system (3.1) is completely controllable for each α from the set $B_\delta(\alpha_0)$.*

§ 4. Geometry of the attainability set of vector fields

Consider the set $D \subset V(M)$, which can contain finite or infinite number of smooth vector fields.

Definition 10. The *point* $y \in L(x)$ such that

$$y = X_k^{t_k}(X_{k-1}^{t_{k-1}}(\ldots(X_1^{t_1}(x))\ldots))$$

is said *T-attainable from a point* $x \in M$ if $\sum_i t_i = T$.

We denote by $A_x(T)$ the set of all points that are T-attainable from the point x.

It was proved in [9] that an orbit is a smooth manifold. Based on the same idea, we proved in [6] the following theorem on the geometry of the set of T-attainable points.

Theorem 13. *For each $x \in M$ and any T, the set $A_x(T)$ is an immersed submanifold of the orbit $L(x)$ of codimension 1 or 0.*

Recall that if one additionally requires in Definition 1 of an orbit that t_1, t_2, …, t_k are nonnegative numbers, then one obtains the definition of the positive semiorbit $L^+(x)$. Another important contribution of Sussmann in the geometry of the attainability set is the following theorem [10].

Theorem 14. *Let M be a smooth connected manifold of dimension n. There exists a system D consisting of two vector fields such that $L^+(x) = M$ for each point $x \in M$.*

Using Theorem 14, we proved in [6] the following assertion.

Theorem 15. *Let M be a smooth connected manifold of dimension $n \geq 2$. There exists a system D consisting of three vector fields such that $A_x(0) = M$ for each point $x \in M$.*

For manifolds with nonzero Euler characteristic, the following result is valid.

Theorem 16 (see [6]). *Let M be a smooth, compact, connected manifold of dimension $n \geq 2$, whose Euler characteristic is nonzero. There exists a system D consisting of two vector fields such that $A_x(0) = M$ for each point $x \in M$.*

The following example from [6] shows that on a compact connected manifold with zero Euler characteristic, a system consisting of two vector fields can exist such that $A_x(0) = M$ for each point $x \in M$.

Let the three-dimensional sphere $S^3 \subset \mathbb{R}^4$ be given by equation $x^2 + y^2 + z^2 + w^2 = 1$, where x, y, z, w are Cartesian coordinates in $\mathbb{R}^4$. We consider on the sphere S^3 two Killing vector fields:

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - w\frac{\partial}{\partial z} + z\frac{\partial}{\partial w}, \qquad Y = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}.$$

The Lie bracket $[X, Y]$ of vector fields X, Y has the following form:

$$[X, Y] = -w\frac{\partial}{\partial x} - z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} + x\frac{\partial}{\partial w}.$$

As, at the point $p(1, 0, 0, 0) \in S^3$, the vectors $X(p)$, $Y(p)$, and $[X, Y](p)$ are linearly independent, the orbit $L(p)$ is a three-dimensional manifold. It follows that $L(p) = S^3$ [6].

If the sets $A_q(0)$ for $q \in S^3$ are submanifolds of the orbit $L(p)$ of codimension 1 then they generate Riemannian codimension one foliation on $L(p) = S^3$. But it is known that there do not exists two-dimensional Riemannian foliations on S^3 [6]. It follows that $A_q(0) = S^3$ for all $q \in S^3$.

For symmetric systems, the following theorem holds [6].

Theorem 17. *Let a system D be symmetric and contain a complete vector field. Then for any $T \in R$ and any point $x \in M$ the following equality holds $A_x(T) = L(x)$.*

Recall that a system D of vector fields is said to be symmetric if $X \in D$ implies $-X \in D$.

It follows from Theorem 13 that the manifolds $A_y(0)$ for points $y \in L(x)$, either coincide with $L(x)$ or generate a foliation of codimension 1 on $L(x)$. This allows one to apply the methods of foliation theory for the study of geometry of the manifolds $A_y(0)$. The following result was obtained [6].

Theorem 18. *Let $M = \mathbb{R}^n$, a system D of vector fields consist of Killing vector fields, and, for a point $x \in M$, let the orbit of $L(x)$ be a k-dimensional plane, $0 \leq k \leq n$. Then, for all $y \in L(x)$, the sets $A_y(0)$ either coincide with $L(x)$ or are parallel hyperplanes in $L(x)$.*

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МАТЕМАТИКА 2022. Т. 32. Вып. 1. С. 81–93.

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Стабильность вполне управляемых систем

Ключевые слова: системы управления, множества управляемости, орбита векторных полей, сингулярное слоение.

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Предметом настоящей работы является вопрос о стабильности вполне управляемых систем, заданных на гладком многообразии. Известно, что множества управляемости симметричных систем порождают сингулярные слоения. В случае, когда множества управляемости имеют одинаковую размерность, возникает регулярное слоение. Таким образом, возникает возможность применения методов теории слоений в задачах теории управления. В данной работе излагаются некоторые результаты авторов о возможности применения теорем о стабильности слоев для задачи о стабильности вполне управляемых систем и для изучения геометрии множества достижимости. Гладкость всюду в работе будет означать гладкость класса C^{∞}.

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