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WEAKLY INVO-CLEAN RINGS HAVING WEAK INVOLUTION

We completely describe up to an isomorphism the structure of weakly invo-clean rings possessing weak involution. The obtained results expand two own establishments, namely those from Afrika Mat. (2017) concerning weakly invo-clean rings as well as those from Far East J. Math. Sci. (2021) concerning invo-clean rings with weak involution.

Keywords: (weakly) invo-clean rings, (weak) involution.

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Introduction and background

Everywhere in the text of the current paper, all rings R will be assumed associative, containing the identity element 1, which in general differs from the zero element 0 of R. Our standard terminology and notations are mainly in agreement with [13]. Specifically, $U(R)$ denotes the group of all units in R, $Id(R)$ the set of all idempotents in R, $Nil(R)$ the set of all nilpotents in R, and $J(R)$ the Jacobson radical of R. More unusual notions and concepts will be stated explicitly in the sequel.

Recall that an element w in a ring R is *involution* provided that $w^2 = 1$. By writing $v^2 = \pm 1$ for some element v from a ring R, we shall mean that either $v^2 = 1$ or $v^2 = -1$ and we shall call v *weak involution* (see, e.g., [1]).

In [5] we have described up to an isomorphism those rings R whose elements r satisfy the condition that $r = rvr$ (resp., $r = r^2v$) for some involution v depending on r. On the other hand, in [2–4, 7, 8] we have completely characterized three special classes of clean rings; e. g., whose elements are of the kind $v + e$, where v is an involution and e is an idempotent as the other two closely related classes have a rather more complicated structure than that of the class presented first.

The motivation of writing up this article is to promote a new insight in ring structure by replacing the existing involution with weak involution, thus obtaining a definite complication in the characterization due to the specific nature.

§ 1. Weakly invo-clean rings with weak involution

Referring to [3], a ring R is called *weakly invo-clean* if, for each $r \in R$, there are $v \in R$ with $v^2 = 1$ and $e \in Id(R)$ such that $r = v + e$ or $r = v - e$. On the other hand, mimicking [9], a ring R is said to be *invo-clean with weak involution* if, for each $r \in R$, there are $v \in R$ with $v^2 = \pm 1$ and $e \in Id(R)$ such that $r = v + e$.

Likewise, in [11] we have studied rings which are closely related to the defined above in [9] so-called invo-clean rings with weak involution, but the latter class is not well-explored there, so that the results obtained herein unambiguously contribute something to the subject.

The goal of the present paper is to unify these two concepts into a single notion as demonstrated below. Precisely, the basic concept in this section is the following one:

Definition 1. We shall say that a ring R is *weakly invo-clean with weak involution* if, for every $r \in R$, there exist $v \in R$ with $v^2 = \pm 1$ and $e \in Id(R)$ such that $r = v + e$ or $r = v - e$.

By a direct verification, pretty obvious examples of such rings are the fields $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_5$ as well as the indecomposable ring $\mathbb{Z}_4$.

Surprisingly, in contrast to [9], some more non-trivial constructions satisfying Definition 1 are the following:

- $\mathbb{Z}_3 \times \mathbb{Z}_5$

Indeed, all elements of these two fields are the idempotents $\{0, 1\}$ as well as the involutions 2 in $\mathbb{Z}_3$ and 4 in $\mathbb{Z}_5$ as well as the weak involutions 2 and 3 in $\mathbb{Z}_5$. Thus, if $w^2 = -1$ in $\mathbb{Z}_5$, then one sees that $(w + 1)^2 = \pm 1$ and $(w - 1)^2 = \pm 1$. This allows us to assert that any element from $\mathbb{Z}_3 \times \mathbb{Z}_5$ can be presented as either (u, v) or $(u, v) \pm (1, 1)$ or $(u, v) \pm (1, 0)$ or $(u, v) \pm (0, 1)$, where u, v are simultaneously either involutions or weak involutions.

- $\mathbb{Z}_5 \times \mathbb{Z}_5$

Indeed, all elements of the field $\mathbb{Z}_5$ are idempotents and weak involutions, so that the same method described in the previous point is successfully applied.

- Another valuable example of such a ring is from the theory of finite fields (compare also with [13]), especially pertaining to the finite field of characteristic 3 with 9 elements: For its construction, consider λ a root of the polynomial $x^2 + x + 2$, so that $\lambda^2 + \lambda + 2 = 0$ or, equivalently, $\lambda^2 = 2\lambda + 1$ as $3 = 0$. Now the powers of λ give us that $\lambda^3 = 2\lambda + 2$, $\lambda^4 = 2$, $\lambda^5 = 2\lambda$, $\lambda^6 = \lambda+2$, $\lambda^7 = \lambda+1$ and $\lambda^8 = 1$. So λ is a primitive element and we have represented the elements of $\mathbb{F}_9$ as the 8 powers of λ that can be written as linear combinations of the basis $\{1, \lambda\}$ over $\mathbb{F}_3$ together with 0. In other words, $\mathbb{F}_9 = \{0, 1, 2, \lambda, 2\lambda + 1, 2\lambda + 2, 2\lambda, \lambda + 2, \lambda + 1\}$. Moreover, $x^2 + 1$, $x^2 + x + 2$ and $x^2 + 2x + 2$ are the only irreducible monic quadratic (i. e., of degree 2) polynomials in $\mathbb{F}_3[x]$, i.e., over $\mathbb{F}_3$. In particular, all elements of the field $\mathbb{F}_9$ satisfy one of the equations $x^3 = x$ and $(x^3 - x)^2 = -1$.

We claim now that $\mathbb{F}_9$ is a weakly invo-clean ring with weak involution. In fact, we first observe that $\lambda^2 \neq \pm 1$, $(\lambda + 1)^2 \neq \pm 1$, $(\lambda - 1)^2 \neq 1$, but $(\lambda - 1)^2 = -1$. Therefore, one can represent all elements as follows: $\lambda = (\lambda - 1) + 1$, $\lambda + 1 = (\lambda - 1) - 1$, $\lambda + 2 = \lambda - 1$, $2\lambda = 2(\lambda-1)-1$, $2\lambda+1 = 2(\lambda-1)-1$, $2\lambda+2 = 2(\lambda-1)+1$, where $[2(\lambda-1)]^2 = (\lambda-1)^2 = -1$ as $3 = 0$.

In particular, the ring $\mathbb{Z}_3[i] = \{a + b\,i \mid a, b \in \mathbb{Z}_3\} = \{0, 1, 2, i, 1 + i, 2 + i, 2i, 1 + 2i, 2 + 2i \mid i^2 = -1\}$ of Gaussian integers modulo 3 is a field of characteristic 3 containing exactly 9 elements, that is, it is a field of the type $\mathbb{F}_9$.

Other non-trivial examples will be given later on, too. Reciprocally, two quick examples of finite rings which are *not* weakly invo-clean with weak involution are the rings $\mathbb{Z}_9$ and $\mathbb{Z}_{17}$ which verification we leave to the interested reader for a direct inspection.

The next two preliminary technicalities are pivotal for our further applications.

Lemma 1. *Let R be a ring with $q, v, e \in R$ such that $q^2 = 0$, $v^2 = 1$, $e^2 = e$ and $q = v + e$, or such that $q^2 = 0$, $v^2 = -1$, $e^2 = e$ and $q = v - e$. Then $e = 1$.*

P r o o f. By squaring the given equality $q = v + e$, one obtains that $1 + e + ve + ev = 0$. Multiplying by e on the left, we detect that $2e + eve + ev = 0$, and by e on the right that $2e + ve + eve = 0$. Comparing both sides, one infers that $ve = ev$. Consequently, $1 + e = -2ve$ and again squaring, we inspect that $1 + 3e = 4e$, i.e., $e = 1$, as expected.

Now, squaring the equality $q = v - e$, one derives that $-1 + e - ve - ev = 0$. Multiplying subsequently on the left and on the right, we get $-eve - ev = 0$ and $-ve - eve = 0$ ensuring $ev = ve$. Thus $1 - e = -2ev$. Squaring this, we find that $1 - e = -4e$, i.e., $3e = -1$. Again squaring, it follows that $-3 = 9e = 1$, that is, $4 = 0$. This, finally, yields that $1 - e = 0$, i. e., $e = 1$, as promised. □

The essence of our technical preliminaries is the following one.

Proposition 1. *Suppose that R is a ring of characteristic 5. Then the following three conditions are equivalent:*

(i) $x^3 = x$ *or* $x^4 = 1$, $\forall x \in R$;

(ii) $x^3 = x$ *or* $x^3 = -x$, $\forall x \in R$;

(iii) $x^3 = -x$ *or* $x^4 = 1$, $\forall x \in R$;

(iv) R *is isomorphic to the field* $\mathbb{Z}_5$.

P r o o f. (iii) $\iff$ (i) $\Rightarrow$ (ii). These relations follow pretty easy by using the argumentation as presented: in fact, for an arbitrary but fixed $y \in R$ satisfying $y^4 = 1$ but $y^3 \neq y$, considering the element $y^2 - 1 \in R$, it must be that $(y^2-1)^4 = 1$ or $(y^2-1)^3 = y^2 - 1$. In the first case, we receive $y^2 = -1$ and thus equivalently $y^3 = -y$, as required, while in the second one, we arrive at $y^2 = 1$ and so in an equivalent form $y^3 = y$ which is against our initial assumption.

(ii) $\iff$ (iv). Let P be the subring of R generated by 1, and thus note that $P \cong \mathbb{Z}_5$. We claim that $P = R$, so we assume in a way of contradiction that there exists $b \in R \setminus P$. With no loss of generality, we shall also assume that $b^3 = b$ since $b^3 = -b$ obviously implies that $(2b)^3 = 2b$ as $5 = 0$ and $b \notin P \iff 2b \notin P$.

Now let $(1+b)^3 = -(1+b)$. Hence $b = b^3$ together with $5 = 0$ gives $b^2 = 1$. This allows us to conclude that $(1+2b)^3 \neq \pm(1+2b)$, however. In fact, if $(1+2b)^3 = 1+2b$, then one deduces that $2b = 3 \in P$ which is manifestly untrue. If now $(1+2b)^3 = -1-2b$, then one infers that $2b = 2 \in P$ which is obviously false. That is why, only $(1+b)^3 = 1+b$ holds. This, in turn, guarantees that $b^2 = -b$. Moreover, $b^3 = b$ is equivalent to $(-b)^3 = -b$ as well as $b^3 = -b$ to $(-b)^3 = -(-b)$ and thus, by what we have proved so far applied to $-b \notin P$, it follows that $-b = b^2 = (-b)^2 = -(-b) = b$. Consequently, $2b = 0 = 6b = b \in P$ because $5 = 0$, which is the wanted contradiction. We thus conclude that $P = R$, as expected.

Conversely, it is trivial that the elements of $\mathbb{Z}_5$ are solutions of one of the equations $x^3 = x$ or $x^3 = -x$.

(iv) $\Rightarrow$ (i). It is self-evident that all elements of $\mathbb{Z}_5$ satisfy one of the equations $x^3 = x$ or $x^4 = 1$. $\square$

Imitating [10], let us recall that a ring is said to be *weakly nil-clean*, provided that each its element is the sum or the difference of a nilpotent and an idempotent.

We are now in a position to proceed by proving the main achievement of this section, stating the following theorem.

Theorem 1. *A non-zero ring R is weakly invo-clean with weak involution if and only if R is decomposable in the form $R_1 \times R_2 \times R_3$, where each direct factor R_i $(i = 1, 2, 3)$ is either $\{0\}$, but not all simultaneously, or satisfies the following conditions:*

(1) *R_1 is a weakly invo-clean ring with weak involution which is also weakly nil-clean whose nilpotents satisfy (one of) the equations $x^2 - 2x = 0$ or $x^2 - 2x + 2 = 0$ or $x^2 + 2x = 0$ or $x^2 + 2x + 2 = 0$ and, in particular, $x^2 = 0$ provided $2 = 0$, $x^4 = 0$ provided $4 = 0$, and $x^8 = 0$ provided $8 = 0$;*

(2) *R_2 is a ring which is a subdirect product of a family of copies of the fields $\mathbb{Z}_3$ and $\mathbb{F}_9$;*

(3) *R_3 is a ring which is a subdirect product of a family of copies of the field $\mathbb{Z}_5$.*

P r o o f. ($\Rightarrow$). Write $3 = v + e$ or $3 = v - e$ for some $e \in R$ with $e^2 = e$ and $v \in R$ with $v^2 = 1$ or $v^2 = -1$.

Then, writing $3 - v = e$, we square this equality and obtain that either $7 = 5v$ or $5 = 5v$, provided $v^2 = 1$ or $v^2 = -1$, respectively. Again by next squaring these two equalities, we detect that $24 = 2^3 \cdot 3 = 0$ or that $50 = 2 \cdot 5^2 = 0$. So, one writes in both situations that $2^3 \cdot 3 \cdot 5^2 = 0$.

Assume now we have written that $v - 3 = e$. As above, by subsequent double squaring, we inspect that either $120 = 2^3 \cdot 3 \cdot 5 = 0$ or $170 = 2 \cdot 5 \cdot 17 = 0$. So, one writes in both situations that $2^3 \cdot 3 \cdot 5 \cdot 17 = 0$.

Furthermore, combining all four different cases, one establishes that $2^3 \cdot 3 \cdot 5^2 \cdot 17 = 0$. Therefore, the Chinese Remainder Theorem works to conclude that $R \cong R_1 \times R_2 \times R_3 \times R_4$ for some four rings R_1, R_2, R_3, R_4 which could be either zero rings (not, however, simultaneously as $R \neq \{0\}$) or are weakly invo-clean rings with weak involution such that $8 = 0$ in R_1, $3 = 0$ in R_2, $5 = 0$ in R_3 and $17 = 0$ in R_4, respectively, claiming also that $R_4 = \{0\}$ must hold.

We will now classify these four rings separately and in more details:

About R_1: Since $2 \in Nil(R_1)$, one easily obtains that $v + 1$ and $v - 1$ are always nilpotents, provided $v^2 = 1$ or $v^2 = -1$. In fact, concerning $v + 1$, in the first situation it must be that $(v + 1)^2 = 2(v + 1)$, while, in the second situation, $(v + 1)^2 = 2v$. Dealing now with $v - 1$, we calculate that $(v - 1)^2 = 2(1 - v)$ when $v^2 = 1$, and that $(v - 1)^2 = -2v$ when $v^2 = -1$. Furthermore, one simply writes for any $r \in R_1$ that $r = v + e = (v + 1) - (1 - e) \in \in Nil(R_1) - Id(R_1)$ and that $r = v - e = (v - 1) + (1 - e) \in Nil(R_1) + Id(R_1)$, where $e \in R_1$ with $e^2 = e$, as required.

Moreover, for an arbitrary $q \in Nil(R_1)$, we write in the notations above that $q = v + e$ or $q = v - e$ with $v^2 = \pm 1$. Thus we may distinguish four cases like these:

- $q = v + e$ with $v^2 = 1$. Claim that $e = 1$. In fact, one deduces after squaring the equivalent equality $-v = -q + e$ and multiplying subsequently the result on the left and on the right by e that e commutes with the nilpotent $q(q-1) = q^2 - q$; thus this nilpotent is commuting also with v. Furthermore, setting $f := 1 - e$, one verifies that $f = fv^2 = fqv = fq(q - 1)(q - 1)^{-1}v = = f(q - 1)^{-1}q(q - 1)v = f(q - 1)^{-1}vq(q - 1) \in Nil(R_1) \cap Id(R_1) = \{0\}$ whence $f = 0$ giving us that $e = 1$ and hence that $q = v + 1$. After squaring that, it follows that $q^2 - 2q = 0$, as needed. This guarantees also that $q^2 = 0$ when $2 = 0$, that $q^4 = 0$ when $4 = 0$ and that $q^8 = 0$ when $8 = 0$.
- $q = v + e$ with $v^2 = -1$. Assert that $e = 1$. Indeed, we may process in the same manner as above to find that e commutes with $q^2 - q$, and so imitating the same idea $-f = fv^2$ will be a nilpotent ensuring its immediate zeroing. Thus, as $f = 0$, we obtain at once that $e = 1$ and that $q = v + 1$. By squaring, one sees that $q^2 - 2q + 2 = 0$, as needed. A plain check shows that the other three equalities about q remain valid too.
- $q = v - e$ with $v^2 = 1$. Thus $-q = (-v) + e$ and as $-q$ is still nilpotent and $-v$ is still involution, arguing as in the first bullet point, we deduce that $e = 1$. Hence $v = q + 1$ and, by squaring, one obtains that $q^2 + 2q = 0$, as required. A straightforward check shows that the other three equalities about q remain also fulfilled.
- $q = v - e$ with $v^2 = -1$. As above, $-q = (-v) + e$ where $-q$ is still nilpotent and $-v$ is still weak involution, so we may argue as in the second bullet point to get that $e = 1$. Hence $v = q + 1$ and, by squaring, one receives that $q^2 + 2q + 2 = 0$, as stated. A routine check shows that the other three equalities about q remain true as well.

About R_2: Here $3 = 0$. In the light of subsequent results in [6,12,14], it suffices to prove that the equation $x^9 = x$ holds for all elements of R_2. In fact, R_2 will then be a subdirect product of copies of the fields $\mathbb{F}_q$, where $q = 3^k$ for some positive integer k such that $(3^k - 1)/8$ and hence $k = 1$ or $k = 2$ arises only, thus showing that the fields $\mathbb{F}_3 = \mathbb{Z}_3$ and $\mathbb{F}_9$ occur in the subdirect decomposition. To that aim, we will foremost detect that R_2 is reduced, i. e., there are no non-trivial nilpotent elements. In fact, given $q \in R_2$ with $q^2 = 0$, we write that $q = v + e$ and, respectively $q = v - e$, with $e^2 = e$ and $v^2 = 1$ or $v^2 = -1$. In the first possibility when $q = v + e$

and $v^2 = 1$, Lemma 1 implies that $e = 1$. So, squaring $q = v+1$, one detects that $v = -1$ because $3 = 0$ is fulfilled here. Therefore, $q = 0$, as expected. We shall illustrate now that the second possibility when $q = v + e$ and $v^2 = -1$ is nonsense. Indeed, squaring $q = v + e$, it follows that $0 = -1 + e + ve + ev$, that is, $1 - e = ve + ev$. Multiplying this on the left, we obtain that $0 = eve + ev$, whereas by a multiplication on the right, we obtain that $0 = ve + eve$. This means that $ev = ve$ and, consequently, that $qv = vq$. Thus $q - v = e \in U(R_2) \cap Id(R_2) = \{1\}$. So, $q = v + 1$ and $2v = -v = 0$ which is absurd, as promised. If we write now $q = v - e$ with either $v^2 = 1$ or $v^2 = -1$, we can process like this: In the case where $v^2 = 1$, after squaring $q = v - e$ we obtain that $ve+ev = 1+e$ and thus, multiplying by e on the left and on the right, respectively, one derives that $eve + ev = 2e = eve + ve$. Hence $ev = ve$ and $1 + e = 2ev$. Squaring this, one sees that $1 + 3e = 4e$, i. e., $e = 1$ and so $2v = 2$, that is, $v = 1$ as $3 = 0$. Finally, $q = 0$, as promised. If, however, $q = v - e$ with $v^2 = -1$, we may write $(-q) = (-v) + e$, where still $(-q)^2 = 0$ and $(-v)^2 = -1$. But this was already done above, so this substantiates, in all cases, that R_2 has to be abelian, being reduced.

Returning now to the initial case, for any $r \in R_2$ we write that $r = w + f$ for some weak involution w and an idempotent f. We, therefore, find that $r^3 = (w + f)^3 = w^3 + f$ as $wf = fw$. But $w^2 = 1$ yields that $w^3 = w$, while $w^2 = -1$ yields that $w^3 = -w$. That is why, $r^3 = w+f = r$ whence $r^9 = r$, or $r^3 = -w + f = r + w$ since $3 = 0$. Then we have in the second situation that $(r^3 - r)^3 = r - r^3$, yielding $r^9 - r^3 = r - r^3$, i.e., $r^9 = r$. Finally, in both cases, $r^9 = r$ is fulfilled for all elements r of R_2, as required.

About R_3: Here $5^2 = 0$. However, we claim that $5 = 0$. In fact, since $5^2 = 0$, it must be that $5 \in J(R_3)$. We claim that $J(R_3) = \{0\}$, which will substantiate our desired equality. Indeed, for any $z \in J(R_3)$, writing $z = w + f$ or $z = w - f$ for some weak involution w and idempotent f, one infers that $z - w = f \in Id(R_3) \cap U(R_3) = \{1\}$ whence $z = w + 1$ or that $w-z = f \in Id(R_3)\cap U(R_3) = \{1\}$ whence $z = w-1$. Besides, when $z = w+1$, we deduce that $(z-1)^2 = 1$ or $(z-1)^2 = -1$, giving up that $z(z-2) = 0$ or that $z(z-2) = -2$. Since $2 \in U(R_3)$, it follows that $z - 2 \in U(R_3)$ and hence either $z = 0$, as required, or $z = -2(z - 2)^{-1} \in U(R_3)$ which is false. When $z = w - 1$, we obtain by similarity that $z(z + 2) = 0$ or $z(z + 2) = -2$. Hence, once again, $z = 0$ or $z \in U(R_3)$, and so we are set.

We also assert that R_3 is reduced and therefore abelian. In fact, by a slight modification of the technique demonstrated in the previous case of the ring R_2, we will come to this conclusion without any difficulty. Indeed, under the same notations as above, the case when $q = v + e$ can be processed step by step in the same way. In the other case where $q = v - e$, as observed above, we have $2v = 2$ which means that $6v = 6$, i. e., $v = 1$, because $5 = 0$, and thus we are set.

Next, with the latter in mind, we observe that the equality $x^5 = x$ holds for all elements in R_3. To that goal, with $r = v + e$ and $ve = ev$ at hand, we will obtain that $r^5 = (v + e)^5 = v^5 + e^5 =$ $= v + e = r$, because both $v^2 = 1$ and $v^2 = -1$ imply $v^4 = 1$ and so $v^5 = v$. Henceforth, the corresponding result from [13] can be applied to get the asserted structural description.

About R_4: We assert that $R_4 = \{0\}$ is mandatory to be fulfilled. In proving that, as we showed in the previous cases, R_4 is a subdirect product of copies of the field $\mathbb{F}_{17}$, but *not* so hard verification shows that this field is definitely *not* weakly invo-clean with weak involution, so that R_4 must really be zero, as claimed.

($\Leftarrow$). We are now planning to show that the direct product $R_1 \times R_2 \times R_3$, with the rings R_1, R_2, R_3 described as in the text, is a weakly invo-clean ring with weak involution. In fact, in order to illustrate that, one sees that the elements of the direct product $R_1 \times R_2 \times R_3$ are the triples (x, y, z) with coordinates $x \in R_1, y \in R_2, z \in R_3$. Since subdirect products of the fields $\mathbb{Z}_3$, $\mathbb{F}_9$ and $\mathbb{Z}_5$ obviously retain the pursued "weakly invo-clean property with weak involution" by using the traditional elementary coordinate-wise manipulations, and hence so does the direct product $R_2 \times R_3$, what suffices to show is that the first direct component R_1 will have the same behavior. To that goal, it is only necessary to take into account that, by what we have established in point (1), R_1 is a weakly nil-clean ring, which completes our argumentation after all. $\square$

The next commentaries could be helpful to simplify the used machinery in [9] and also to have a scheme for further investigations.

Remark 1. We now will manage to show that some things of the proofs of the results presented in [9] could be simplified substantially. Indeed, [9, Proposition 3.3], which is actually Proposition 1 here, can be confirmed as follows: All three conditions (i), (ii) and (iii) are clearly satisfied in $\mathbb{Z}_5$, thus point (iv) implies all these three conditions. On the other hand, any of these three conditions implies the identity $x^5 = x$, thus R is a subdirect product of copies of $\mathbb{Z}_5$. If, however, we assume for a moment that R is not isomorphic/equal to $\mathbb{Z}_5$, then it will have $\mathbb{Z}_5 \times \mathbb{Z}_5$ as a homomorphic image, which is nonsense because this direct product satisfies none of the presumed conditions (i), (ii), (iii).

Likewise, an omnibus in the given there arguments is that every subdirect product of fields obviously has no non-zero nilpotent elements. Also, to exclude the field $\mathbb{F}_9$ from the decomposition in the main theorem, it is just enough to show that this ring is *not* invo-clean with weak involution, but this is a pretty elementary exercise.

As a culmination of our explorations, we finish off our work with the following two challenging questions of some interest and importance, which immediately arise.

Problem 1. Describe the structure of those rings R whose elements can be written as $r = v + e$ such that $v^2 = \pm 1$ and $e^2 = \pm e$.

Are these rings very close to the rings as defined in Definition 1 or they are totally different? In any case, we may view them as a non-trivial generalization of the invo-clean rings with weak involution presented in [9].

Problem 2. Classify the structure of *feebly invo-clean rings with weak involution* as defined in the sense that, for every $r \in R$, there are two commuting idempotents e, f and a weak involution v such that $r = v + e - f$.

Notice that this notion generalizes in a rather non-trivial way both concepts of weakly invo-clean rings with weak involution (see Definition 1) and feebly invo-clean rings (see [4]). However, this will be the theme of some other research investigation, where a new approach might work.

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Слабо инволютивно-чистые кольца со слабой инволюцией

Ключевые слова: (слабо) инволютивно чистые кольца, (слабая) инволюция.

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Мы полностью описываем с точностью до изоморфизма структуру слабо инволютивно-чистых колец, обладающих слабой инволюцией. Полученные результаты расширяют две собственные работы, а именно работы из Afrika Mat. (2017), касающиеся слабо инволютивно-чистых колец, а также результаты из Far East J. Math. Sci. (2021), касающиеся инволютивно-чистых колец со слабой инволюцией.

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