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NUMERICAL SOLUTION OF A CONTROL PROBLEM FOR A PARABOLIC SYSTEM WITH DISTURBANCES

A controlled parabolic system that describes the heating of a given number of rods is considered. The density functions of the internal heat sources of the rods are not known exactly, and only the segments of their change are given. At the ends of the rods there are controlled heat sources and disturbances. The goal of the choice of control is to lead the vector of average temperatures of the rods at a fixed time to a given compact for any admissible functions of the density of internal heat sources and any admissible realizations of disturbances. After replacing variables, the problem of controlling a system of ordinary differential equations in the presence of uncertainty is obtained. Using a numerical method, a solvability set is constructed for this problem. Model calculations are carried out.

Keywords: control, disturbance, parabolic system.

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Introduction

Problems of controlling parabolic equations arise in the mathematical modelling of controlled processes of thermal conductivity, diffusion, and filtration [1–5]. In practice, there are frequent problems of heat propagation in a rod, the ends of which are under the influence of variable controlled temperatures. These problems are reduced to the study of the heat equation, the boundary conditions of which contain controls [6, 7].

There may be cases when some of the system parameters are not precisely specified, and there is also influence from uncontrolled disturbances [8–11].

When solving such problems, the method of optimizing a guaranteed result can be applied [12]. Mathematical modelling of control within the framework of this method is based on an approach that prescribes realization of uncertainties and disturbances that worsens the quality indicator in accordance with which the control is modelled. This approach leads to consideration of the problem of constructing control within the framework of the theory of differential games (see, for example, [13, 14]).

In the works of L. S. Pontryagin and representatives of his scientific school, an analytical scheme for solving linear differential pursuit games, which is based on the Minkowski sum and difference operations of two sets and the procedure of alternating integration, was proposed and justified [15]. Since in the case of arbitrary sets the result of the above-mentioned Minkowski operations cannot be calculated exactly, numerical solution methods have been widely developed (see, for example, [16, 17]).

In the works of N. N. Krasovskii and representatives of his scientific school, the main results of the theory of positional differential games were obtained [18]. This theory is based on the concept of a stable bridge and the rule of extreme aiming at it. Papers [19–21] discuss grid methods for solving positional differential games. These methods are based on the transition to discrete representation of time and approximation of solvability sets by finite sets of points. Another approach to the numerical solution of such problems is based on the approximation of solvability sets by polygons (see, as example, [22]).

The present article continues the research started in [23, 24]. The article [23] considers the problem of heating a rod by controlling the rate of temperature change at its left end. The rate of temperature change at the right end of the rod is determined by the limited disturbance. The density function of the internal heat sources of the rod is not precisely known, but only a segment of its possible values is given. The goal of the control is to make the average temperature of the rod at a fixed time moment no more than one given value and no less than another for any admissible realization of disturbance and function of the density of internal heat sources. The average temperature value is calculated as the integral of the product of temperature and a given function. In [24] the problem of controlling a parabolic system describing the heating of a given number of rods using point heat sources located at the ends of the rods is considered. The goal of choosing a control is to ensure that at a fixed time moment the weighted sum of average temperatures of the rods belongs to a given segment.

This article solves a modification of problems [23, 24]. At a fixed time moment, it is required to lead the vector of average temperatures of the rods to a terminal set, which is compact (possibly non-convex). We assume that the control can be corrected only at a finite set of time moments. After changing variables, taking disturbances and uncertainties as the control of the second player, the problem is reduced to a differential game, the dynamics of which are described by a system of ordinary differential equations. An approximate algorithm is proposed to construct the solvability set of the first player in this problem. A program has been written that implements this algorithm. Using this program, model calculations have been carried out for a specific example.

§ 1. Problem statement

The heat equation

$$\frac{\partial T_i(x,t)}{\partial t} = \frac{\partial^2 T_i(x,t)}{\partial x^2} + f_i(x,t), \tag{1.1}$$

where $0 \leqslant t \leqslant p$ and $0 \leqslant x \leqslant 1$, describes the temperature distribution $T_i(x,t)$ in ith ($i = \overline{1,n}$) homogeneous rod of unit length as a function of time t. At the initial time moment $t = 0$, the temperature distributions $T_i(x,0) = g_i(x)$, $i = \overline{1,n}$, are given, where $g_i(x)$ are continuous functions. We assume that the controlled temperatures $T_i(0,t)$ and $T_i(1,t)$ at the ends of the ith rod, $i = \overline{1,n}$, change according to the equations

$$\frac{d\,T_i(0,t)}{dt} = a_i^{(1)}(t) + a_i^{(2)}(t) G_i^{(1)} \overline{\xi}(t); \tag{1.2}$$

$$\frac{d\,T_i(1,t)}{dt} = b_i^{(1)}(t) + b_i^{(2)}(t)\big(G_i^{(2)} \overline{\xi}(t) + \eta_i(t)\big). \tag{1.3}$$

Here, $a_i^{(j)}(t)$ and $b_i^{(j)}(t)$, $i = \overline{1,n}$, $j = 1,2$, are continuous functions for $0 \leqslant t \leqslant p$, where $a_i^{(2)}(t) \geqslant 0$ and $b_i^{(2)}(t) \geqslant 0$. The vector-function $\overline{\xi}(t) = (\xi_1(t), \xi_2(t), \ldots, \xi_l(t))^* \in U$, where U is convex compact in $\mathbb{R}^l$, is a control. The symbol $*$ denotes the transposition operation. The choice of the corresponding one-dimensional controls $\xi_j(t)$ for the left and right ends of each rod is given by matrices $G^{(1)}$ and $G^{(2)}$ of dimension $n \times l$, respectively. $G_i^{(1)}$ and $G_i^{(2)}$ denote the ith rows of the corresponding matrix. The functions $\eta_i(t)$ ($|\eta_i(t)| \leqslant 1$), $i = \overline{1,n}$, are disturbances.

We know the estimates of the continuous functions $f_i(x,t)$, which are the densities of internal heat sources of the rods:

$$f_i^{(1)}(x,t) \leqslant f_i(x,t) \leqslant f_i^{(2)}(x,t), \quad 0 \leqslant t \leqslant p, \quad 0 \leqslant x \leqslant 1. \tag{1.4}$$

Here, functions $f_i^{(j)}(x,t)$, $i = \overline{1,n}$, $j = 1,2$, are continuous.

Assumption 1.1. Each function $f_i\colon [0,1]\times[0,p]\to\mathbb{R}$, $i=\overline{1,n}$, is such that for any numbers $0\leqslant\tau<\nu$ and continuous functions $\varrho_j\colon[\tau,\nu]\to\mathbb{R}$, $j=1,2$, $\beta\colon[0,1]\to\mathbb{R}$ such that the matching condition $\varrho_1(\tau)=\beta(0)$, $\varrho_2(\tau)=\beta(1)$ is satisfied, the first boundary value problem

$$\frac{\partial Q(x,t)}{\partial t}=\frac{\partial^2 Q(x,t)}{\partial x^2}+f_i(x,t),$$
$$Q(0,t)=\varrho_1(t),\quad Q(1,t)=\varrho_2(t),\quad \tau\leqslant t\leqslant\nu;$$
$$Q(x,\tau)=\beta(x),\quad 0\leqslant x\leqslant 1,$$

has a unique solution $Q(x,t)$ continuous for $0\leqslant x\leqslant 1$, $\tau\leqslant t\leqslant\nu$.

Let a compact $Z\subset\mathbb{R}^n$ be given. The goal of choosing control $\overline{\xi}(t)$ in (1.2), (1.3) is to implement the inclusion

$$\left(\int_0^1 T_1(x,p)\sigma_1(x)\,dx,\ldots,\int_0^1 T_n(x,p)\sigma_n(x)\,dx\right)^*\in Z \tag{1.5}$$

for any realized disturbances $\eta_i(t)$ (1.3), $i=\overline{1,n}$, and for any continuous functions $f_i(x,t)$ (1.4), $i=\overline{1,n}$, satisfying Assumption 1.1. Here, given continuous functions $\sigma_i\colon[0,1]\to\mathbb{R}$, $i=\overline{1,n}$, satisfy the conditions

$$\sigma_i(0)=\sigma_i(1)=0. \tag{1.6}$$

§ 2. Problem formalization

Let us describe the admissible rule for choosing control $\overline{\xi}$. It means that each time $0\leqslant\nu<p$ and each admissible temperature distribution $\overline{T}(x,\nu)=(T_1(x,\nu),T_2(x,\nu),\ldots,T_n(x,\nu))$ at this time, a measurable vector-function $\overline{\xi}(t)$ such that $\overline{\xi}\colon[\nu,p]\to U$ is put in correspondence. We shall denote such a rule as

$$\overline{\xi}(t)=N\big(t,\overline{T}(\cdot,\nu)\big),\quad t\in[\nu,p]. \tag{2.1}$$

Fix a partition

$$\omega\colon 0=t_0<t_1<\ldots<t_j<t_{j+1}<\ldots<t_{m+1}=p \tag{2.2}$$

of the segment $[0,p]$. Further, we will assume that control $\overline{\xi}(t)$ can be corrected only at time moments t_j, $j=\overline{0,m}$.

Let the temperature distribution $\overline{T}^{(\omega)}(x,t_j)$, $0\leqslant x\leqslant 1$, be realized at time t_j, $j=\overline{0,m}$. Denote $\overline{\xi}^{(j)}(t)=N\big(t,\overline{T}^{(\omega)}(\cdot,t_j)\big)$, $t\in[t_j,p]$. Let measurable disturbances $\eta_i^{(j)}\colon[t_j,t_{j+1}]\to[-1,1]$, $i=\overline{1,n}$, and continuous functions $f_i(x,t)$, $i=\overline{1,n}$, are realized.

Denote by $T_i^{(\omega)}(x,t)$ for $0\leqslant x\leqslant 1$, $t_j\leqslant t\leqslant t_{j+1}$ the solution of the equation (1.1) with the following initial and boundary conditions:

$$T_i(x,t)=T_i^{(\omega)}(x,t_j),\quad x\in[0,1];\quad T_i(0,t)=T_i^{(\omega)}(0,t),\quad T_i(1,t)=T_i^{(\omega)}(1,t);$$
$$T_i^{(\omega)}(0,t)=T_i^{(\omega)}(0,t_j)+\int_{t_j}^t\big(a_i^{(1)}(r)+a_i^{(2)}(r)G_i^{(1)}\overline{\xi}^{(j)}(r)\big)\,dr,\quad i=\overline{1,n}, \tag{2.3}$$
$$T_i^{(\omega)}(1,t)=T_i^{(\omega)}(1,t_j)+\int_{t_j}^t\Big(b_i^{(1)}(r)+b_i^{(2)}(r)\big(G_i^{(2)}\overline{\xi}^{(j)}(r)+\eta_i^{(j)}(r)\big)\Big)\,dr,\quad i=\overline{1,n}, \tag{2.4}$$

for $t\in[t_j,t_{j+1}]$.

§ 3. Reduction to a linear problem

Let us denote by $\psi_i(x,\tau)$ for $0 \leqslant x \leqslant 1$, $0 \leqslant \tau \leqslant p$ solutions of the following first boundary value problems

$$\frac{\partial \psi_i(x,\tau)}{\partial \tau} = \frac{\partial^2 \psi_i(x,\tau)}{\partial x^2}, \quad \psi_i(x,0) = \sigma_i(x), \quad \psi_i(0,\tau) = \psi_i(1,\tau) = 0, \quad i = \overline{1,n}. \tag{3.1}$$

Equality (1.6) implies that the matching conditions at the ends of the segment in problems (3.1) are satisfied.

Using the conditions (1.4), it can be shown that [25]

$$\left\{ \int_0^1 f_i(x,t)\psi_i(x,p-t)\,dx \right\} = \left\{ c_i^{(1)}(t) + c_i^{(2)}(t) s_i(t) \right\}, \quad |s_i(t)| \leqslant 1, \quad i = \overline{1,n}, \tag{3.2}$$

where

$$c_i^{(1)}(t) = \frac{1}{2} \int_0^1 \left(f_i^{(1)}(x,t) + f_i^{(2)}(x,t) \right) \psi_i(x,p-t)\,dx,$$
$$c_i^{(2)}(t) = \frac{1}{2} \int_0^1 \left(f_i^{(2)}(x,t) - f_i^{(1)}(x,t) \right) |\psi_i(x,p-t)|\,dx.$$

Note that $c_i^{(j)}(t)$, $i = \overline{1,n}$, $j = 1,2$, are continuous functions for $t \in [0,p]$, and $c_i^{(2)}(t) \geqslant 0$.

Fix a control (2.1). Let's introduce new variables

$$\begin{aligned} y_i^{(\omega)}(t) &= \int_0^1 T_i^{(\omega)}(x,t)\psi_i(x,p-t)\,dx \\ &\quad + T_i^{(\omega)}(0,t) \int_t^p \frac{\partial \psi_i(0,p-r)}{\partial x}\,dr - T_i^{(\omega)}(1,t) \int_t^p \frac{\partial \psi_i(1,p-r)}{\partial x}\,dr \\ &\quad + \int_t^p \left(a_i^{(1)}(\tau) \int_\tau^p \frac{\partial \psi_i(0,p-r)}{\partial x}\,dr - b_i^{(1)}(\tau) \int_\tau^p \frac{\partial \psi_i(1,p-r)}{\partial x} dr + c_i^{(1)}(\tau) \right) d\tau, \\ &\quad i = \overline{1,n}. \end{aligned} \tag{3.3}$$

Then, taking into account formulas (1.1), (2.3)–(2.4), (3.1) and (3.2), we obtain

$$\begin{aligned} \dot{y}_i^{(\omega)}(t) &= \left(a_i^{(2)}(t) \int_t^p \frac{\partial \psi_i(0,p-r)}{\partial x}\,dr \right) G_i^{(1)} \overline{\xi}^{(j)}(t) \\ &\quad - \left(b_i^{(2)}(t) \int_t^p \frac{\partial \psi_i(1,p-r)}{\partial x}\,dr \right) \left(G_i^{(2)} \overline{\xi}^{(j)}(t) + \eta_i(t) \right) + c_i^{(2)}(t) s_i(t), \\ &\quad i = \overline{1,n}. \end{aligned} \tag{3.4}$$

Next, we rewrite (3.4) in the matrix form

$$\dot{\overline{y}}^{(\omega)}(t) = -A(t)\overline{\xi}^{(j)}(t) + B(t)\overline{v}(t), \quad \overline{\xi}^{(j)}(t) \in U, \quad \overline{v}(t) \in \Pi(n). \tag{3.5}$$

Here

$$\begin{aligned} \overline{y}^{(\omega)}(t) &= (y_1^{(\omega)}(t), y_2^{(\omega)}(t), \ldots, y_n^{(\omega)}(t))^*; \\ \Pi(n) &= \{\overline{s} = (s_1, s_2, \ldots, s_n)^* \in \mathbb{R}^n : |s_i| \leqslant 1,\ i = \overline{1,n}\}; \\ A_i(t) &= -\left(a_i^{(2)}(t) \int_t^p \frac{\partial \psi_i(0,p-r)}{\partial x}\,dr \right) G_i^{(1)} + \left(b_i^{(2)}(t) \int_t^p \frac{\partial \psi_i(1,p-r)}{\partial x}\,dr \right) G_i^{(2)},\ i = \overline{1,n}; \end{aligned} \tag{3.6}$$

$$B(t) = \operatorname{diag}\left\{c_1^{(2)}(t) + b_1^{(2)}(t)\left|\int_t^p \frac{\partial\psi_1(1,p-r)}{\partial x}\,dr\right|, \ldots, \right.$$
$$\left. c_n^{(2)}(t) + b_n^{(2)}(t)\left|\int_t^p \frac{\partial\psi_n(1,p-r)}{\partial x}\,dr\right|\right\}. \tag{3.7}$$

Taking into account (3.3), we obtain that a polygonal line $y^{(\omega)}(t)$ satisfies the equality

$$\overline{y}^{(\omega)}(p) = \left(\int_0^1 T_1^{(\omega)}(x,p)\sigma_1(x)\,dx, \ldots, \int_0^1 T_n^{(\omega)}(x,p)\sigma_n(x)\,dx\right)^*.$$

It follows that inclusion (1.5) takes the form

$$\overline{y}^{(\omega)}(p) \in Z. \tag{3.8}$$

Taking the uncertain function $\overline{v}$ as a control of the second player, we obtain the differential game (3.5), (3.8).

§ 4. Approximate algorithm for solving a differential game

4.1 Construction of a solvability set

We propose the following heuristic algorithm for constructing the solvability set of the first player in the differential game (3.5), (3.8). In ideological terms, this algorithm is close to the works of V. N. Ushakov [19–21].

We take a partition (2.2) with a sufficiently small constant step $\delta > 0$. Then for each t_j the solvability set can be found using the recurrent formula

$$W(p) = Z, \quad W(t_j) = \left(W(t_{j+1}) \dot{-} \int_{t_j}^{t_j+1} B(r)\Pi(n)\,dr\right) + \left(\int_{t_j}^{t_j+1} A(r)U\,dr\right), \quad j = \overline{0,m}.$$

Here «+» denotes the Minkowski sum operation, which is defined by the formula $A + B = \{a + b \colon a \in A,\ b \in B\}$; «$\dot{-}$» denotes the Minkowski difference operation, which is defined by the formula

$$A \dot{-} B = \bigcap_{b \in B}(A - b). \tag{4.1}$$

Since solvability sets cannot be calculated exactly, we will calculate them approximately, replacing $W(t_j)$ with the sets

$$W^{(\delta)}(p) = Z^{(\delta)}, \quad W^{(\delta)}(t_j) = \left(W^{(\delta)}(t_{j+1}) \dot{-} \delta B(t_{j+1})\Pi^{(\delta)}(n)\right) + \left(\delta A(t_{j+1})U^{(\delta)}\right), \quad j = \overline{0,m}.$$

Here, following the ideas of grid methods [19–21], we replaced the sets Z, U, $\Pi(n)$ with their finite subsets $Z^{(\delta)}$, $U^{(\delta)}$, $\Pi^{(\delta)}(n)$, respectively.

For finite sets, we redefine the intersection operation in (4.1) as follows

$$A \bigcap B = \\ = \{a \in A \colon \|a - b\|_\infty < \varepsilon \text{ for some } b \in B\} \bigcup \{b \in B \colon \|a - b\|_\infty < \varepsilon \text{ for some } a \in A\}, \tag{4.2}$$

where $\|a - b\|_\infty = \max_{i=\overline{1,n}} |a_i - b_i|$; $\varepsilon > 0$ is a sufficiently small given number.

At some iteration, the number of points in the set $W^{(\delta)}(t_j)$ may become so large that the resources of the computer on which the calculations take place may not be enough to calculate the set $W^{(\delta)}(t_{j-1})$. Thus, if the number of points in the set $W^{(\delta)}(t_j)$ exceeds a certain specified threshold value β, then it is necessary to carry out a reduction procedure. Next, we use a reduction procedure with a constant step $h>0$, which is defined as follows.

1. Let us construct an estimated n-dimensional parallelepiped containing $W^{(\delta)}(t_j)$ which is defined as

$$[y_1^{\min}, y_1^{\max}] \times [y_2^{\min}, y_2^{\max}] \times \ldots \times [y_n^{\min}, y_n^{\max}].$$

2. Then, taking into account the geometry of this estimated parallelepiped, we calculate

$$\tilde{y}_i^{\min} = \left\lfloor \frac{y_i^{\min}}{h} \right\rfloor h, \quad \tilde{y}_i^{\max} = \left\lceil \frac{y_i^{\max}}{h} \right\rceil h, \quad i = \overline{1, n},$$

and partition the segment $[\tilde{y}_i^{\min}, \tilde{y}_i^{\max}]$ with the step $h > 0$. Here $\lfloor \cdot \rfloor$ is the operation of rounding down to the nearest integer; $\lceil \cdot \rceil$ is the operation of rounding up to the nearest integer.

Thus we obtain a constant mesh contained in parallelepiped

$$[\tilde{y}_1^{\min}, \tilde{y}_1^{\max}] \times [\tilde{y}_2^{\min}, \tilde{y}_2^{\max}] \times \ldots \times [\tilde{y}_n^{\min}, \tilde{y}_n^{\max}].$$

3. For each grid node, which is the vertex of the «exposed» cell (containing points of the set $W^{(\delta)}(t_j)$), we choose the point of set $W^{(\delta)}(t_j)$ closest to it. We denote the resulting set as $\widehat{W}^{(\delta)}(t_j)$.

4. Replace $W^{(\delta)}(t_j)$ with the set $\widehat{W}^{(\delta)}(t_j)$.

Finally, we note that in the proposed algorithm, the choice of parameters h, ε and the rules for discretizing the set Z must be consistent with each other. We propose to define the set $Z^{(\delta)}$ as the intersection of the set Z and a constant grid with step h. In addition, following a heuristic consideration, we require that ε be close to $0.5\,h$, but $\varepsilon < 0.5\,h$, otherwise the ε-neighborhoods of the grid nodes intersect with each other.

4.2 Constructing the first player's control

Let the initial temperature distributions $T_i(x, t_0) = g_i(x)$, $i = \overline{1, n}$, be such that $y(t_0) \in W^{(\delta)}(t_0)$. Let us construct a guaranteeing control of the first player $\overline{\xi}$ using the rule of extreme aiming [18].

1. Find the point $\overline{y}^*(t_1) \in W^{(\delta)}(t_1)$, which is closest to the point $\overline{y}(t_0)$.
2. Construct the vector $\overline{y}^*(t_1) - \overline{y}(t_0)$.
3. If $\overline{y}^*(t_1) - \overline{y}(t_0) \neq 0$, then we find the control $\overline{\xi}^{(0)}(t)$ as a solution to the problem

$$\max_{\overline{\xi} \in U} \big\langle -A(t_0)\overline{\xi}, \overline{y}^*(t_1) - \overline{y}(t_0) \big\rangle.$$

Otherwise we take any $\overline{\xi} \in U$.

4. Substitute the constructed control $\overline{\xi}^{(0)}(t)$ into the equation (3.5) and for some function $\overline{v}(t) \in \Pi(n)$, realized on $[t_0, t_1]$, let's move to point $\overline{y}(t_1)$.
5. Let us continue the described procedure for $\overline{y}(t_j)$, $j = \overline{1, m}$.

§ 5. Example

As an example, we consider the problem of heating a system consisting of two homogeneous rods of unit length. The heating of the left and right ends of the first rod is determined by the control $\xi_1(t)$ and the disturbance $\eta_1(t)$, respectively. The heating of the left end of the second rod is determined by the control $\xi_1(t)$, and the heating of the right end of this rod is determined by the control $\xi_2(t)$ and the disturbance $\eta_2(t)$. Let us define the rule for selecting controls $\xi_1(t)$ and $\xi_2(t)$ using matrices

$$G^{(1)} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad G^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.1}$$

Next, $\overline{\xi}(t) = \big(\xi_1(t), \xi_2(t)\big)^*$ such that $\xi_1^2(t) + \xi_2^2(t) \leqslant 1$, and $|\eta_j(t)| \leqslant 1$, $j = \overline{1,2}$.

Substituting the matrices (5.1) into the formulas (3.6)–(3.7), we write the matrices $A(t)$ and $B(t)$ for this example:

$$A(t) = \begin{pmatrix} -a_1^{(2)}(t)\displaystyle\int_t^p \frac{\partial\psi_1(0,p-r)}{\partial x}\,dr & 0 \\ -a_2^{(2)}(t)\displaystyle\int_t^p \frac{\partial\psi_2(0,p-r)}{\partial x}\,dr & b_2^{(2)}(t)\displaystyle\int_t^p \frac{\partial\psi_2(1,p-r)}{\partial x}\,dr \end{pmatrix}; \tag{5.2}$$

$$B(t) = \operatorname{diag}\left\{c_1^{(2)}(t) + b_1^{(2)}(t)\left|\int_t^p \frac{\partial\psi_1(1,p-r)}{\partial x}\,dr\right|, c_2^{(2)}(t) + b_2^{(2)}(t)\left|\int_t^p \frac{\partial\psi_2(1,p-r)}{\partial x}\,dr\right|\right\}. \tag{5.3}$$

Let $\sigma_i(x) = \gamma_i \sin\pi x$, $0 \leqslant x \leqslant 1$, $\gamma_i > 0$, $i = 1, 2$. Then the solution to problem (3.1) is the functions

$$\psi_i(x,\tau) = \gamma_i e^{-\pi^2\tau}\sin\pi x, \quad 0 \leqslant x \leqslant 1, \quad \tau \geqslant 0, \quad i = 1, 2.$$

Therefore

$$\int_t^p \frac{\partial\psi_i(0,p-r)}{\partial x}\,dr = -\int_t^p \frac{\partial\psi_i(1,p-r)}{\partial x}\,dr = \gamma_i\left(\frac{1 - e^{-\pi^2(p-t)}}{\pi}\right), \quad i = 1, 2. \tag{5.4}$$

Let the functions $f_i^{(j)}(x,t)$, $i = 1, 2$, $j = 1, 2$, be constant in this example. Then

$$c_i^{(2)}(t) = \frac{\gamma_i}{\pi}\big(f_i^{(2)} - f_i^{(1)}\big)e^{-\pi^2(p-t)}, \quad i = 1, 2.$$

Taking this and (5.4) into account, we rewrite the matrices (5.2) and (5.3) in the following form:

$$A(t) = \frac{1 - e^{-\pi^2(p-t)}}{\pi}\begin{pmatrix} -a_1^{(2)}(t)\gamma_1 & 0 \\ -a_2^{(2)}(t)\gamma_2 & -b_2^{(2)}(t)\gamma_2 \end{pmatrix}; \tag{5.5}$$

$$\begin{aligned} B(t) = \operatorname{diag}\Bigg\{ & \frac{\gamma_1}{\pi}\big(f_1^{(2)} - f_1^{(1)}\big)e^{-\pi^2(p-t)} + b_1^{(2)}(t)\gamma_1\left(\frac{1 - e^{-\pi^2(p-t)}}{\pi}\right), \\ & \frac{\gamma_2}{\pi}\big(f_2^{(2)} - f_2^{(1)}\big)e^{-\pi^2(p-t)} + b_2^{(2)}(t)\gamma_2\left(\frac{1 - e^{-\pi^2(p-t)}}{\pi}\right)\Bigg\}. \end{aligned} \tag{5.6}$$

Let the numbers $0 < \varepsilon_1 < \varepsilon_2$ and the temperatures $\widehat{T}_1$ and $\widehat{T}_2$ desired at time p be given. We define a cross-shaped terminal set as follows:

$$\begin{gathered} Z(\widehat{T}_1, \widehat{T}_2, \varepsilon_1, \varepsilon_2) = \\ = \{z = (z_1, z_2)^* \in \mathbb{R}^2 \colon |\widehat{T}_1 - z_1| \leqslant \varepsilon_1,\ |\widehat{T}_2 - z_2| \leqslant \varepsilon_2 \ \text{ or } \ |\widehat{T}_1 - z_1| \leqslant \varepsilon_2,\ |\widehat{T}_2 - z_2| \leqslant \varepsilon_1\}. \end{gathered}$$

5.1 Model calculations

The algorithm from § 4 is implemented in the C++ programming language using OpenMP parallel computing technology. Using this program, calculations have been carried out for the example under consideration. The program was run on the supercomputer node of the South Ural State University «Tornado SUSU», equipped with two Intel Xeon X5680 3.33 GHz processors (12 cores/24 threads per node) with 24 GB of RAM.

Let's consider an example with specific parameter values and given functions in $A(t)$ (5.5) and $B(t)$ (5.6):

$$t_0 = 0, \quad p = 2, \quad \gamma_1 = \pi, \quad \gamma_2 = \pi, \quad f_1^{(1)} = 1, \quad f_1^{(2)} = 3, \quad f_2^{(1)} = 0, \quad f_2^{(2)} = 2;$$
$$a_1^{(2)}(t) = 4, \quad a_2^{(2)}(t) = 3, \quad b_1^{(2)}(t) = 2, \quad b_2^{(2)}(t) = 2 \quad \text{for} \quad t \in [0, p].$$

Note that with the selected parameters, matrix $B(t)$ (5.6) becomes a constant matrix.

Let us take the following parameters of the algorithm for calculating the solvability set:

$$\delta = 0.05, \quad h = 0.1, \quad \beta = 30000, \quad \varepsilon = 0.0425.$$

Let $\widehat{T}_1 = 0$, $\widehat{T}_2 = 0$, $\varepsilon_1 = 1$, $\varepsilon_2 = 3$. We define finite subsets $Z^{(\delta)}(0,0,1,3)$ (see Fig. 1), $U^{(\delta)}$, $\Pi^{(\delta)}(2)$ as follows:

$$U^{(\delta)} = \{(ih, jh)^* \in \mathbb{R}^2 \colon (ih)^2 + (jh)^2 \leqslant 1, \quad i,j = \overline{-10,10}\}.$$
$$\Pi^{(\delta)}(2) = \{(i,j)^* \in \mathbb{R}^2 \colon i,j = \overline{-1,1}\},$$
$$Z^{(\delta)}(0,0,1,3) = \{(ih, jh)^* \in \mathbb{R}^2 \colon (ih, jh)^* \in Z(0,0,1,3), \quad i,j = \overline{-30,30}\}.$$

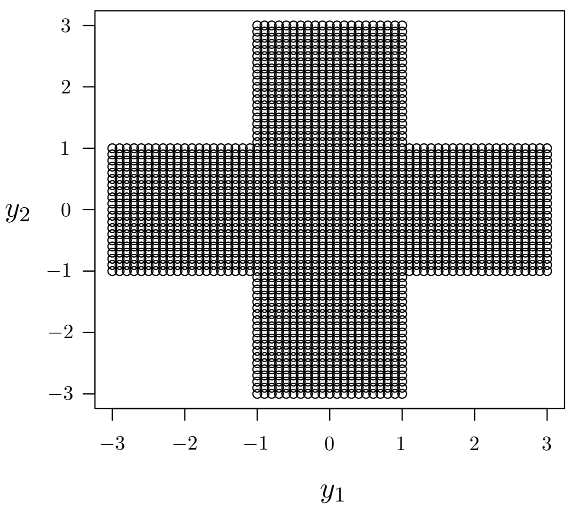

Fig. 1. Set $W^{(\delta)}(p) = Z^{(\delta)}(0,0,1,3)$

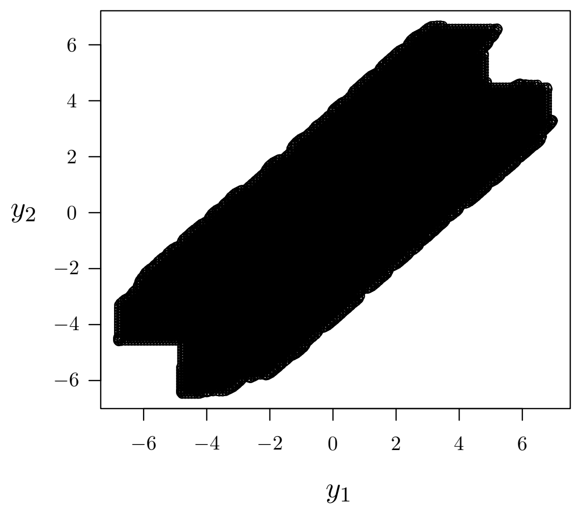

Fig. 2. Constructed set $W^{(\delta)}(t_0)$

The Figure 2 shows set $W^{(\delta)}(t_0)$ calculated by the approximate algorithm from Subsection 4.1, for the parameters specified above.

All calculated sets $W^{(\delta)}(t_j)$, $j = \overline{0, m+1}$, we save in the computer memory. Let's take the point $(-6.78, -3.905)^* \in W^{(\delta)}(t_0)$ and construct the first player's control $\overline{\xi}^{(\omega)}(t)$ leading from this point to a small neighborhood of $Z^{(\delta)}(0,0,1,3)$ using the algorithm described in Subsection 4.2. We assume that the control of the second player is chosen randomly from $\Pi(2)$ and it is constant at each half-interval $[t_j, t_{j+1})$, $j = \overline{0,m}$. Figures 3 and 4 show the coordinates of the constructed control $\overline{\xi}^{(\omega)}(t)$.

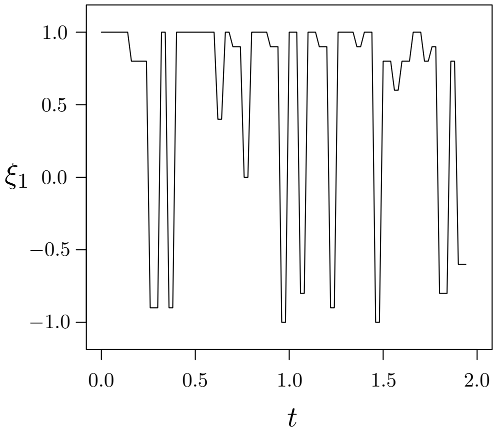

Fig. 3. $\xi_1^{(\omega)}(t)$

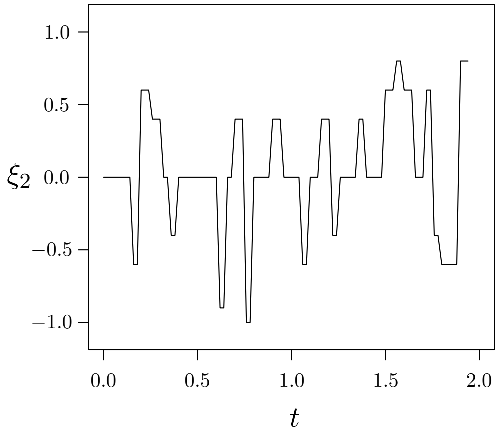

Fig. 4. $\xi_2^{(\omega)}(t)$

Figures 5 and 6 show the coordinates of the trajectory $\overline{y}^{(\omega)}(t)$ generated by the found control $\overline{\xi}^{(\omega)}(t)$ of the first player and the corresponding random piecewise constant control $\overline{v}(t)$ of the second player.

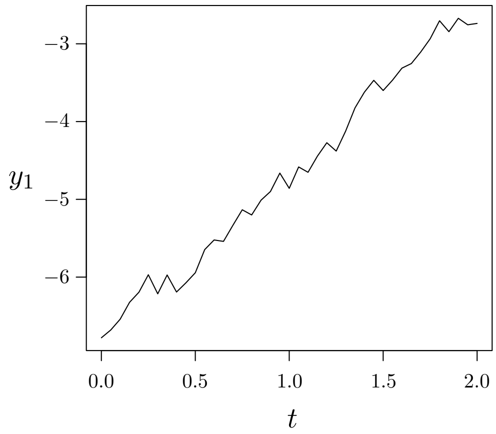

Fig. 5. $y_1^{(\omega)}(t)$

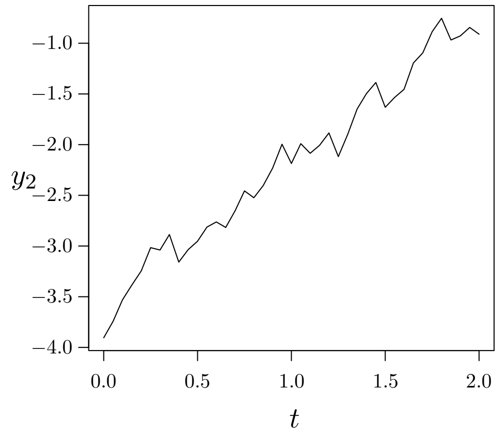

Fig. 6. $y_2^{(\omega)}(t)$

Note that for the constructed trajectory $\overline{y}^{(\omega)}(p) = (-2.73813, -0.910883)^* \in Z(0,0,1,3)$.

§ 6. Conclusion

This paper considers the problem of controlling a parabolic system that describes the heating of a given number of rods under conditions of uncertainty and the influence of external disturbances. A terminal set is compact (possibly non-convex). After changing variables the problem is reduced to an antagonistic differential game (3.5), (3.8). For this differential game, an approximate algorithm for constructing the solvability set of the first player is proposed. This algorithm is implemented in the C++ programming language using OpenMP parallel computing technology. Model calculations of the solvability set for specific values of the numerical parameters of the system have been carried out on the supercomputer node of the South Ural State University «Tornado SUSU». Using the extreme aiming method, a guaranteeing control of the first player leading to the terminal set has been found.

In the future, we plan to consider a version of this problem in which the dynamics of the system in (1.1) are described by more complex parabolic equations. In addition, the procedure for approximate calculation of the Minkowski difference for finite sets will be improved.

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И. В. Изместьев, В. И. Ухоботов, К. Н. Кудрявцев

Численное решение задачи управления параболической системой с помехами

Ключевые слова: управление, помеха, параболическая система.

УДК 517.977

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Рассматривается управляемая параболическая система, которая описывает нагрев заданного количества стержней. Функции плотности внутренних источников тепла стержней точно неизвестны, а заданы только отрезки их изменения. На концах стержней находятся управляемые источники тепла и помехи. Цель выбора управления заключается в том, чтобы привести вектор средних температур стержней в фиксированный момент времени на заданный компакт при любых допустимых функциях плотности внутренних источников тепла и любых допустимых реализациях помех. После замены переменных получена задача управления системой обыкновенных дифференциальных уравнений при наличии неопределенности. Используя численный метод, для этой задачи построено множество разрешимости. Выполнены модельные расчеты.

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