MSC2020: 40A05, 40A30, 41A10, 41A17, 42A10, 42A24

© *S. Z. Jafarov*

APPROXIMATION BY NÖRLUND TYPE MEANS IN THE GRAND LEBESGUE SPACES WITH VARIABLE EXPONENT

In the present paper the approximation of functions by Nörlund type means in the generalized grand Lebesgue spaces with variable exponent is studied.

Keywords: grand variable exponent Lebesgue spaces, modulus of smoothness, Lipschitz classes, trigonometric approximation, Nörlund means.

DOI: 10.35634/vm240102

§ 1. Introduction and auxiliary results

Let $\mathbb{T}$ denote the interval $[0, 2\pi]$. We denote by $L^p(\mathbb{T})$, $1 \leq p < \infty$, the Lebesgue space of all measurable 2π-periodic functions, for which the norm

$$\|f\|_p = \left(\int_{\mathbb{T}} |f(x)|^p \, dx\right)^{1/p} < \infty.$$

Let us denote by $\wp$ the class of Lebesgue measurable functions $p(\cdot)\colon \mathbb{T} \to [0, \infty)$ such that $1 \leq p_* := \operatorname*{ess\,inf}_{x\in\mathbb{T}} p(x) \leq \operatorname*{ess\,sup}_{x\in\mathbb{T}} p(x) =: p^* < \infty$. The conjugate exponent of $p(x)$ is shown by $p'(x) := \dfrac{p(x)}{p(x)-1}$. For $p \in \wp$, we define a class $L^{p(\cdot)}(\mathbb{T})$ of 2π-periodic measurable functions $f\colon \mathbb{T} \to \mathbb{C}$ satisfying the condition

$$\int_{\mathbb{T}} |f(x)|^{p(x)} \, dx < \infty.$$

This class $L^{p(\cdot)}(\mathbb{T})$ is a Banach space with respect to the norm

$$\|f\|_{p(\cdot)} := \inf\left\{\lambda > 0\colon \int_{\mathbb{T}} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}. \tag{1.1}$$

The class $L^{p(\cdot)}(\mathbb{T})$ with the norm (1.1) is called the *Lebesgue space with variable exponent*. Information on the properties of this space can be found in [11, 27, 28, 44, 45]. We say that a variable exponent $p(x)$ satisfies the *local log-continuity condition*, if there is a positive constant c such that

$$|p(x) - p(y)| \ln\left(\frac{1}{\log|x-y|}\right) \leq c, \tag{1.2}$$

for all $x, y \in [0, 2\pi]$, $|x - y| \leq \dfrac{1}{2}$, $x \neq y$.

We denote by $\wp^{\log}(\mathbb{T})$ the class of 2π-periodic functions satisfying the condition (1.2). We also define $\wp_0(\mathbb{T}) := \{p(\cdot) \in \wp^{\log}(\mathbb{T})\colon 1 < p_*\}$.

Let $\theta \geq 0$ and $p \in \wp_0(\mathbb{T})$. We denote by the *generalized grand Lebesgue space with variable exponent* $L^{p(\cdot),\theta}$ the class of all 2π-periodic measurable functions f such that

$$\|f\|_{p(\cdot),\theta} = \sup_{0<\varepsilon<p_*-1} \varepsilon^{\frac{\theta}{p_*-\varepsilon}} \|f\|_{p(\cdot)-\varepsilon} < \infty.$$

The space $L^{p(\cdot),\theta}$ was introduced in [30]. Note that when p is a constant and $\theta > 0$, these spaces coincide with the grand Lebesgue spaces introduced by Iwaniec and Sbordone in [20] (for $\theta = 1$) and by Greco, Iwaniec and Sbordone in [16] (for $\theta > 1$). If $p \in \wp$, the embeddings

$$L^{p(\cdot)} \subset L^{p(\cdot),\theta} \subset L^{p(\cdot)-\varepsilon}, \qquad 0 < \varepsilon < p_* - 1,$$

hold.

Note [50] that the generalized grand Lebesgue space with variable exponent $L^{p(\cdot),\theta}$ has important applications in different areas of mathematics, physics and mechanics. In particular, the variable exponent Lebesgue spaces have considerable applications in fluid dynamic, especially, for modeling of electrorheological fluids; the grand and generalized grand Lebesgue spaces have been applied in various fields, in particular, in the theory of PDE [21,42,43], they are right spaces for the investigations of some nonlinear equations. There are sufficient investigations, relating the fundamental problems of these spaces in view of potential theory, maximal and singular operator theory, where the analogues of the classical results existing in the classical Lebesgue spaces were studied. The detailed information about these investigations can be found in the monographs [10, 11, 31, 32, 46].

Note that the closure of the space $L^{p(\cdot)}(\mathbb{T})$ in $L^{p(\cdot),\theta}(\mathbb{T})$, $\theta > 0$, does not coincide with $L^{p(\cdot),\theta}(\mathbb{T})$ [30]. We denote this closure by $L_*^{p(\cdot),\theta}(\mathbb{T})$. This space is a subspace of $L^{p(\cdot),\theta}(\mathbb{T})$. According to [28] for the functions belonging to this space

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{\theta}{p_* - 1}} \|f\|_{p(\cdot)-\varepsilon} = 0$$

holds.

We suppose that $p(\cdot) \in \wp_0(\mathbb{T})$ and $\theta > 0$. For $f \in L^{p(\cdot),\theta}(\mathbb{T})$, we set [50]

$$(\nu_h f)(x) := \frac{1}{h} \int_0^h f(x+t)\, dt, \qquad 0 < h < \pi, \quad x \in \mathbb{T}.$$

If $p(\cdot) \in \wp_0(\mathbb{T})$, $\theta > 0$, and $f \in L^{p(\cdot),\theta}(\mathbb{T})$, then the shift operator ν_{h_i} is a bounded linear operator on $L^{p(\cdot),\theta}(\mathbb{T})$ [50]:

$$\|\nu_{h_i}(f)\|_{L^{p(\cdot),\theta}(\mathbb{T})} \le c_1 \|f\|_{L^{p(\cdot),\theta}(\mathbb{T})}.$$

Let $p \in \wp_0(\mathbb{T})$, $\theta > 0$, and $f \in L^{p(\cdot),\theta}(\mathbb{T})$. The function

$$\Omega_{p(\cdot),\theta}(f,\delta) := \sup_{0<h\le\delta} \|f - (\nu_h f)\|_{L^{p(\cdot),\theta}(\mathbb{T})}, \qquad \delta > 0,$$

is called the *modulus of continuity* of $f \in L^{p(\cdot),\theta}(\mathbb{T})$.

It can easily be shown that $\Omega_{p(\cdot),\theta}(f,\cdot)$ is a continuous, nonnegative and nondecreasing function satisfying the conditions

$$\lim_{\delta \to 0} \Omega_{p(\cdot),\theta}(f,\delta) = 0, \qquad \Omega_{p(\cdot),\theta}(f+g,\delta) \le \Omega_{p(\cdot),\theta}(f,\delta) + \Omega_{p(\cdot),\theta}(g,\delta), \qquad \delta > 0,$$

for $f, g \in L^{p(\cdot),\theta}(\mathbb{T})$. Note that a detailed information about properties of the modulus of continuity $\Omega_{p(\cdot),\theta}(f,\cdot)$ can be found in the paper [50].

We use the constants $c, c_1, c_2, \ldots$ (in general, different in different relations) which depend only on the quantities that are not important for the questions of interest. We also will use the relation $f = O(g)$ which means that $f \le cg$ for a constant c independent of f and g. Let $0 < \alpha \le 1$. The set of functions $f \in L_*^{p(\cdot),\theta}(\mathbb{T})$ such that

$$\Omega_{p(\cdot),\theta}(f,\delta) = O(\delta^\alpha), \qquad \delta > 0,$$

is called the *Lipschitz class* $\mathrm{Lip}(\alpha, p(\cdot), \theta)$.

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} Q_k(x, f) \tag{1.3}$$

be the Fourier series of the function $f \in L_1(T)$, where $Q_k(x, f) := (a_k(f) \cos kx + b_k(f) \sin kx)$, $\alpha_k(f)$ and $b_k(f)$ are Fourier coefficients of the function $f \in L_1(\mathbb{T})$. The *n-th partial sum* of the series (1.3) is defined by

$$S_n(x, f) = \sum_{k=0}^{n} Q_k(x, f),$$

where

$$Q_0(x, f) := \frac{a_0}{2}; \qquad Q_k(x, f) := (a_k(f) \cos kx + b_k(f) \sin kx), \qquad k = 1, 2, \ldots.$$

Let $\{p_n\}_0^\infty$ be a sequence of positive real numbers. The sequence $\{p_n\}_0^\infty$ is called *almost monotone decreasing* (*increasing*), denoted by $\{p_n\}_0^\infty \in$ AMDS ($\{p_n\}_0^\infty \in$ AMIS), if there exists a constant c, depending only on the sequence $\{p_n\}_0^\infty$ such that for all $n \geq m$ the following inequality holds:

$$p_n \leq c p_m \quad (p_m \leq c p_n),$$

In the proof of the main result we will use the notations

$$\Delta \beta_n := \beta_n - \beta_{n+1}, \qquad \Delta_m \beta(n, m) := \beta(n, m) - \beta(n, m+1).$$

As in [38] we suppose that $\mathbb{F}$ is an infinite subset of $\mathbb{N}$ and consider $\mathbb{F}$ as a range of strictly increasing sequence of positive integers, say $\mathbb{F} = \{\lambda(n)\}_1^\infty$. Following [5] and [40], the Cesàro submethod C_λ is defined as

$$(C_\lambda x)_n = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x_k, \quad n = 1, 2, \ldots,$$

where $\{x_k\}$ is a sequence of real or complex numbers. Therefore, the C_λ-method yields a subsequence of the Cesàro method C_1, and hence it is regular for any λ. C_λ is obtained by deleting a set of rows from Cesàro matrix.

We suppose that $\{p_n\}_0^\infty$ is a sequence of positive real numbers. We define the mean of the series (1.3) as

$$N_n^\lambda(x, f) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} S_m(x; f),$$

where $P_{\lambda(n)} := \sum\limits_{m=0}^{\lambda(n)} p_m \neq 0$, $n \geq 0$, $p_{-1} := P_{-1} := 0$. Note that in the case $p_n = 1$, $n \geq 0$, $N_n^\lambda(x, f)$ is equal to the mean

$$\sigma_n^\lambda(x, f) = \frac{1}{\lambda(n) + 1} \sum_{m=0}^{\lambda(n)} S_m(x, f).$$

Using [33, 46] we introduce two new classes of numerical sequences.

Let $R_{\lambda(n),k} = \dfrac{1}{(k+1) P_{\lambda(n)}} \sum\limits_{s=\lambda(n)-k}^{\lambda(n)} p_s$. If $(R_{\lambda(n),k}) \in$ AMDS $\big((R_{\lambda(n),k}) \in$ AMIS$\big)$, then it is said that (p_k) is a *λ-almost monotone decreasing* (*increasing*) *upper mean sequence*, briefly

$(p_k) \in \lambda$-AMDUMS $\big((p_k) \in \lambda$-AMIUMS$\big)$. Note that the classes λ-AMDUMS and λ-AMIUMS are generalizations of the classes AMDUMS and AMIUMS respectively. It is clear that if $\lambda(n) = n$, $n = 1, 2, \ldots$, we obtain λ-AMDUMS = AMDUMS and λ-AMIUMS = AMIUMS defined in [48].

The best approximation of $f \in L_*^{p(\cdot),\theta}$ in the class $\prod_n$ of trigonometric polynomials of degree not exceeding n is defined by

$$E_n(f)_{p(\cdot),\theta} := \inf\Big\{\|f - T_n\|_{p(\cdot),\theta} \colon T_n \in \prod\nolimits_n\Big\}.$$

In the proof of the main result we need the following Lemmas.

Lemma 1.1 (see [50]). *Let $p \in \wp_0(\mathbb{T})$, $\theta > 0$. Then for $f \in \mathrm{Lip}(\alpha, p(\cdot), \theta)$, $0 < \alpha \le 1$, and $n = 1, 2, 3, \ldots$ the following estimate holds:*

$$\|f - S_n(\cdot, f)\|_{L^{p(\cdot),\theta}(\mathbb{T})} = O(n^{-\alpha}).$$

Lemma 1.2 (see [50]). *Let $p \in \wp_0(\mathbb{T})$, $\theta > 0$. Then for $f \in \mathrm{Lip}(1, p(\cdot), \theta)$ and $n = 1, 2, 3, \ldots$ the following estimate holds:*

$$\|S_n(\cdot, f) - \sigma_n(\cdot, f)\|_{L^{p(\cdot),\theta}(\mathbb{T})} = O(n^{-1}).$$

Lemma 1.3 (see [33]). *Let $\{p_n\}$ be a positive sequence. Let following conditions hold:*

(1) *$(p_n) \in \lambda$-AMDUMS or*

(2) *$(p_n) \in \lambda$-AMIUMS and $(\lambda(n)+1)p_{\lambda(n)} = O(P_{\lambda(n)})$.*

Then

$$\Lambda := \sum_{m=0}^{\lambda(n)} \frac{p_{\lambda(n)-m}}{(m+1)^\alpha} = O_\alpha\left(\frac{P_{\lambda(n)}}{(\lambda(n)+1)^\alpha}\right).$$

for $0 < \alpha < 1$.

Theorem 1.1 (see [33]). *The following properties are valid:*

(1) *if $(p_n) \in$ AMDS, then $(p_n) \in \lambda$-AMIUMS;*

(2) *if $(p_n) \in$ AMIS, then $(p_n) \in \lambda$-AMDUMS;*

(3) *if $\sum\limits_{s=0}^{\lambda(n)-1} \left|\Delta\left(\frac{p_s}{P_{\lambda(n)}}\right)\right| = O\left((\lambda(n))^{-1}\right)$, then $\sum\limits_{s=0}^{\lambda(n)-1} \left|\Delta(R_{\lambda(n),s})\right| = O\left((\lambda(n))^{-1}\right)$;*

(4) *if $\sum\limits_{s=1}^{\lambda(n)-1} s\left|\Delta\left(\frac{p_s}{P_{\lambda(n)}}\right)\right| = O(1)$, then $\sum\limits_{s=0}^{\lambda(n)-2} \left|\Delta(R_{\lambda(n),s})\right| = O\left((\lambda(n))^{-1}\right)$.*

§ 2. Main Results

The problems of approximation theory in variable and grand variable exponent Lebesgue spaces have been investigated by several authors (see, for example, [1–4, 12–15, 17, 22, 26, 29, 47, 50–52, 54, 55]). In the present paper we study the approximation of functions by Nörlund type means in the generalized grand Lebesgue space with variable exponent $L^{p(\cdot),\theta}$, $\theta > 0$. The results obtained in this work are generalization of the results [17, 32] to the generalized grand Lebesgue space with variable exponent. Similar approximation problems in different spaces have been investigated in [6–9, 9, 17–19, 22–26, 33–41, 48–54, 56–58].

Note that, in the proof of the main results we use the method as in the proofs of [17, 33].

Our main results are the following.

Theorem 2.1. *Let $p \in \wp_0(\mathbb{T})$, $\theta > 0$, and $\{p_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers. Also, let the following conditions hold:*

$$\{p_n\}_0^{\infty} \in \lambda\text{-}AMDUMS$$

or

$$\{p_n\}_0^{\infty} \in \lambda\text{-}AMIUMS, \text{ and } (\lambda(n)+1)p_{\lambda(n)} = O(P_{\lambda(n)}). \tag{2.1}$$

Then, for $f \in \operatorname{Lip}(\alpha, p(\cdot), \theta)$, $0 < \alpha < 1$, the relation

$$\|f - N_n^{\lambda}(\cdot, f)\|_{L^{p(\cdot,\theta)}(\mathbb{T})} = O\left((\lambda(n)+1)^{-\alpha}\right), \quad n \in \mathbb{N} \cup \{0\},$$

holds.

Theorem 2.2. *Let $p \in \wp_0(\mathbb{T})$, $\theta > 0$, and $\{p_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers. Also, let the following condition holds:*

$$\sum_{m=0}^{\lambda(n)-2} |R_{\lambda(n),m} - R_{\lambda(n),m+1}| = O\left((\lambda(n))^{-1}\right).$$

Then, for $f \in \operatorname{Lip}(1, p(\cdot), \theta)$, the relation

$$\|f - N_n^{\lambda}(\cdot, f)\| L^{p(\cdot,\theta)}(\mathbb{T}) = O\left((\lambda(n))^{-1}\right), \quad n = 1, 2, \ldots.$$

holds.

Remark 2.1. Theorem 2.2 gives the same degree of approximation with conditions different from those of Theorem 1.1, considering the case $\alpha = 1$.

Remark 2.2. If $\theta = 0$ and $\lambda(n) = n$, $n = 1, 2, \ldots$, then from Theorems 2.1 and 2.2 we obtain results of [17].

Note that, since $(\lambda(n))^{-\alpha} \leq n^{-\alpha}$, $0 < \alpha \leq 1$, the results obtained in [33] give sharper estimates than those of results in [17].

§ 3. Proofs of the main results

Proof of Theorem 2.1. It is clear that

$$N_n^{\lambda}(x, f) - f(x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m}\{f(x) - S_m(x, f)\}. \tag{3.1}$$

Then using Lemma 1.1 and Lemma 1.3 and (2.1) we have

$$\begin{aligned}
\|N_n^{\lambda}(\cdot, f) - f\|_{L^{p(\cdot),\theta}(\mathbb{T})} &\leq \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} \|f - S_m(\cdot, f)\|_{L^{p(\cdot),\theta}(\mathbb{T})} \\
&= \frac{1}{P_{\lambda(n)}} \sum_{m=1}^{\lambda(n)} p_{\lambda(n)-m} \|f - S_m(\cdot, f)\|_{L^{p(\cdot),\theta}(\mathbb{T})} + \|f - S_0(\cdot, f)\|_{L^{p(\cdot),\theta}(\mathbb{T})} \\
&= \frac{1}{P_{\lambda(n)}} O\left(\sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m}(m+1)^{-\alpha}\right) \\
&= \frac{1}{P_{\lambda(n)}} O\left(P_{\lambda(n)}(\lambda(n)+1)^{-\alpha}\right) = O\left((\lambda(n)+1)^{-\alpha}\right).
\end{aligned}$$

The proof of the Theorem is completed. □

Proof of Theorem 2.2. We can write the following equality:

$$F_n^{\lambda}(x,f) := N_n^{\lambda}(x,f) - f(x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m}\{S_m(x,f) - f(x)\}.$$

Using Abel's transformation, we find that [33]

$$\begin{aligned} F_n^{\lambda}(x,f) &= \sum_{m=1}^{\lambda(n)-1} \big(S_m(x,f) - S_{m+1}(x,f)\big) \frac{1}{P_{\lambda(n)}} \sum_{s=0}^{m} p_{\lambda(n)-s} + S_{\lambda(n)}(x,f) - f(x) \\ &= - \sum_{m=0}^{\lambda(n)-1} (m+1) Q_{m+1}(x,f) R_{\lambda(n),m} + S_{\lambda(n)}(x,f) - f(x) \\ &= - \sum_{m=0}^{\lambda(n)-2} (R_{\lambda(n),m} - R_{\lambda(n),m+1}) \sum_{s=0}^{m} (s+1) Q_{s+1}(x,f) \\ &\quad - \big((\lambda(n) P_{\lambda(n)})^{-1}\big) \sum_{s=1}^{\lambda(n)} p_s \sum_{s=0}^{\lambda(n)-1} (s+1) Q_{s+1}(x,f) + S_{\lambda(n)}(x,f) - f(x) \end{aligned} \tag{3.2}$$

Using (3.2), we obtain

$$\begin{aligned} \|F_n^{\lambda}(\cdot,f)\|_{L^{p(\cdot),\theta}(\mathbb{T})} \leq & \sum_{m=0}^{\lambda(n)-2} |R_{\lambda(n),m} - R_{\lambda(n),m+1}| \left\| \sum_{s=1}^{m-1} s Q_s(\cdot,f) \right\|_{L^{p(\cdot),\theta}(\mathbb{T})} \\ & + \big((\lambda(n))^{-1}\big) \left\| \sum_{s=1}^{\lambda(n)} s Q_s(\cdot,f) \right\|_{L^{p(\cdot),\theta}(\mathbb{T})} + \|S_{\lambda(n)}(\cdot,f) - f\|_{L^{p(\cdot),\theta}(\mathbb{T})}. \end{aligned}$$

It is clear that the equality

$$\sum_{s=1}^{\lambda(n)} s Q_s(f;x) = \big(\lambda(n)+1\big) S_{\lambda(n)}(x,f) - \sigma_{\lambda(n)}(x,f)$$

$$S_n(x,f) - \sigma_n(x,f) = \frac{1}{n+1} \sum_{k=1}^{n} k Q_k(x,f). \tag{3.3}$$

holds. Then, from Lemma 1.2 and (3.3), we have

$$\left\| \sum_{s=1}^{\lambda(n)} s Q_s(\cdot,f) \right\|_{L^{p(\cdot),\theta}(\mathbb{T})} = O(1). \tag{3.4}$$

Thus, use of Lemma 1.2 and (3.4) gives us

$$\|F_n^{\lambda}(\cdot,f)\|_{L^{p(\cdot),\theta}(\mathbb{T})} = O\left(\sum_{m=0}^{\lambda(n)-2} |R_{\lambda(n),m} - R_{\lambda(n),m+1}| \right) + O((\lambda(n))^{-1}). \tag{3.5}$$

Now we suppose that the condition

$$\sum_{m=0}^{\lambda(n)-2} |R_{\lambda(n),m} - R_{\lambda(n),m+1}| = O\left(\lambda(n)^{-1}\right)$$

is satisfied. Then the last relation and (3.5) imply that

$$\|f - N_n^{\lambda}(\cdot, f)\|_{L^{p(\cdot),\theta}(\mathbb{T})} = O\left((\lambda(n))^{-1}\right), \quad n = 1, 2, \ldots.$$

The proof of Theorem 2.2 is completed. □

REFERENCES

1. Akgün R., Kokilashvili V. On converse theorems of trigonometric approximation in weighted variable exponent Lebesgue spaces, *Banach Journal of Mathematical Analysis*, 2011, vol. 5, no. 1, pp. 70–82. https://doi.org/10.15352/bjma/1313362981
2. Akgün R. Trigonometric approximation of functions in generalized Lebesgue spaces with variable exponent, *Ukrainian Mathematical Journal*, 2011, vol. 63, no. 1, pp. 1–26. https://doi.org/10.1007/s11253-011-0485-0
3. Akgün R., Kokilashvili V. The refined direct and converse inequalities of trigonometric approximation in weighted variable exponent Lebesgue spaces, *Georgian Mathematical Journal*, 2011, vol. 18, issue 3, pp. 399–423. https://doi.org/10.1515/GMJ.2011.0037
4. Akgün R., Kokilashvili V. Some approximation problems for (α, ψ)-differentiable functions in weighted variable exponent Lebesgue spaces, *Journal of Mathematical Sciences*, 2012, vol. 186, no. 2, pp. 139–152. https://doi.org/10.1007/s10958-012-0980-3
5. Armitage D. H., Maddox I. J. A new type of Cesàro mean, *Analysis*, 1989, vol. 9, issues 1–2, pp. 195–204. https://doi.org/10.1524/anly.1989.9.12.195
6. Chandra P. Approximation bu Nörlund operators, *Matematički Vesnik*, 1986, vol. 38, issue 3, pp. 263–269.
7. Chandra P. Functions of classes L_p and $\mathrm{Lip}(\alpha, p)$ their Riesz means, *Rivista di Matematica della Università di Parma. Serie 4*, 1986, vol. 12, pp. 275–282. http://rivista.math.unipr.it/fulltext/1986-12/1986-12-275.pdf
8. Chandra P. A note on degree of approximation by Nörlund and Riesz operators, *Matematički Vesnik*, 1990, vol. 42, issue 1, pp. 9–10.
9. Chandra P. Trigonometric approximation of functions in L_p-norm, *Journal of Mathematical Analysis and Applications*, 2002, vol. 275, issue 1, pp. 13–26. https://doi.org/10.1016/S0022-247X(02)00211-1
10. Cruz-Uribe D. V., Fiorenza A. *Variable Lebesgue spaces. Foundation and harmonic analysis*, Basel: Birkhäuser, 2013. https://doi.org/10.1007/978-3-0348-0548-3
11. Diening L., Harjulehto P., Hästö P., Růžička M. *Lebesgue and Sobolev spaces with variable exponents*, Heidelberg: Springer, 2011. https://doi.org/10.1007/978-3-642-18363-8
12. Danelia N., Kokilashvili V. Approximation by trigonometric polynomials in subspace of weighted grand Lebesgue spaces, *Bulletin of the Georgian National Academy of Sciences*, 2013, vol. 7, no. 1, pp. 11–15. http://science.org.ge/old/moambe/7-1/Danelia%2011-15.pdf
13. Danelia N., Kokilashvili V. Approximation of periodic functions in grand variable exponent Lebesgue spaces, *Proceedings of A. Razmadze Mathematical Institute*, 2014, vol. 164, pp. 100–103. https://zbmath.org/1296.41004
14. Danelia V., Kokilashvili V., Tsanava Ts. Some approximation results in subspace of weighted grand Lebesgue spaces, *Proceedings of A. Razmadze Mathematical Institute*, 2014, vol. 164, pp. 104–108. https://zbmath.org/1297.42003
15. Danelia N., Kokilashvili V. Approximation by trigonometric polynomials in the framework of variable exponent grand Lebesgue spaces, *Georgian Mathematical Journal*, 2016, vol. 23, issue 1, pp. 43–53. https://doi.org/10.1515/gmj-2015-0059
16. Sbordone C., Greco L., Iwaniec T. Inverting the p-harmonic operators, *Manuscripta Mathematica*, 1997, vol. 92, issue 2, pp. 249–258. https://eudml.org/doc/156263
17. Guven A., Israfilov D. M. Trigonometric approximation in generalized Lebesgue spaces $L^{p(x)}$, *Journal of Mathematical Inequalities*, 2010, vol. 4, no. 2, pp. 285–299. https://doi.org/10.7153/jmi-04-25

18. Güven A., Israfilov D. M. Approximation by means of Fourier trigonometric series in weighted Orlicz spaces, *Advanced Studies in Contemporary Mathematics* (*Kyungshang*), 2009, vol. 19, issue 2, pp. 283–295.
19. Il'yasov N. A. Approximation of periodic functions by Zygmund means, *Mathematical Notes of the Academy of Sciences of the USSR*, 1986, vol. 39, issue 3, pp. 200–209. https://doi.org/10.1007/BF01170248
20. Iwaniec T., Sbordone C. On the integrability of the Jacobian under minimal hypotheses, *Archive for Rational Mechanics and Analysis*, 1992, vol. 119, no. 2, pp. 129–143. https://doi.org/10.1007/BF00375119
21. Iwaniec T., Sbordone C. Riesz transform and elliptic PDEs with VMO coefficients, *Journal d'Analyse Mathématique*, 1998, vol. 74, no. 1, pp. 183–212. https://doi.org/10.1007/BF02819450
22. Jafarov S. Z. Linear methods for summing Fourier series and approximation in weighted Lebesgue spaces with variable exponents, *Ukrainian Mathematical Journal*, 2015, vol. 66, no. 10, pp. 1509–1518. https://doi.org/10.1007/s11253-015-1027-y
23. Jafarov S. Z. Linear methods of summing Fourier series and approximation in weighted Orlicz spaces, *Turkish Journal of Mathematics*, 2018, vol. 42, no. 6, article 6, pp. 2916–2925. https://doi.org/10.3906/mat-1804-31
24. Jafarov S. Z. Approximation by linear means of Fourier series in weighted Orlicz spaces, *Proceedings of the Institute of Mathematics and Mechanics*, 2017, vol. 43, no. 2, pp. 175–187. https://proc.imm.az/volumes/43-2/43-02-01.pdf
25. Jafarov S. Z. On approximation of a weighted Lipschitz class functions by means $t_n(f;x)$, $N_n^{\beta}(f;x)$ and $R_n^{\beta}(f,x)$ of Fourier series, *Transactions Issue Mathematics. National Academy of Sciences of Azerbaijan. Series of Physical-Technical and Mathematical Sciences*, 2020, vol. 40, no. 4, pp. 118–129. https://doi.org/10.29228/proc.50
26. Jafarov S. Z. On approximation of functions by means of Fourier trigonometric series in weighted generalized grand Lebesgue spaces, *Advanced Studies: Euro-Tbilisi Mathematical Journal*, 2022, special issue 10, pp. 277–291. https://tcms.org.ge/Journals/ASETMJ/Special issue/10/PDF/asetmj_SpIssue_10_20.pdf
27. Kováčik O., Rákosník J. On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czechoslovak Mathematical Journal*, 1991, vol. 41, issue 4, pp. 592–618. https://doi.org/10.21136/CMJ.1991.102493
28. Kokilashvili V. On a progress in the theory of integral operators in weighted Banach function spaces, *Function Spaces, Differential Operators and Nonlinear Analysis: Proceedings of the Conference held in Milovy, Bohemian-Moravian Uplands, May 28–June 2, 2004*, Praha: Mathematical Institute of the Academy of Sciences of the Czech Republic, 2005, pp. 152–174.
29. Kokilashvili V. M., Samko S. G. Operators of harmonic analysis in weighted spaces with non-standard growth, *Journal of Mathematical Analysis and Applications*, 2009, vol. 352, issue 1, pp. 15–34. https://doi.org/10.1016/j.jmaa.2008.06.056
30. Kokilashvili V., Meskhi A. Maximal and Calderón–Zygmund operators in grand variable exponent Lebesgue spaces, *Georgian Mathematical Journal*, 2014, vol. 21, issue 4, pp. 447–461. https://doi.org/10.1515/gmj-2014-0047
31. Kokilashvili V., Meskhi A., Rafeiro H., Samko S. *Integral operators in non-standard function spaces. Volume 1: Variable exponent Lebesgue and amalgam spaces*, Cham: Birkhäuser, 2016. https://doi.org/10.1007/978-3-319-21015-5
32. Kokilashvili V., Meskhi A., Rafeiro H., Samko S. *Integral operators in non-standard function spaces. Volume 2: Variable exponent Hölder, Morrey–Campanato and grand spaces*, Cham: Birkhäuser, 2016. https://doi.org/10.1007/978-3-319-21018-6
33. Krasniqi X. Z. Trigonometric approximation of (signals) functions by Nörlund type means in the variable space $L^{p(x)}$, *Palestine Journal of Mathematics*, 2017, vol. 6, no. 1, pp. 84–93. https://zbmath.org/1352.42004

34. Krasniqi X. Z. Approximation of periodic functions by sub-matrix means of their Fourier series, *TWMS Journal of Applied and Engineering Mathematics*, 2020, vol. 10, no. 1, pp. 279–287. http://jaem.isikun.edu.tr/web/index.php/archive/104-vol10no1/513-approximation-of-periodic-functions-by-sub-matrix-means-of-their-fourier-series
35. Krasniqi X. Z. On degree of approximation of continuous functions by a linear transformation of their Fourier series, *Communications in Mathematics*, 2022, vol. 30, issue 1, pp. 37–46. https://doi.org/10.46298/cm.9273
36. Krasniqi X. Z., Lenski W., Szal B. Seminormed approximation by deferred matrix means of integrable functions in $H_P^{(\omega)}$ space, *Results in Mathematics*, 2022, vol. 77, issue 4, article number: 145. https://doi.org/10.1007/s00025-022-01696-3
37. Leindler L. Trigonometric approximation in L_p-norm, *Journal of Mathematical Analysis and Applications*, 2005, vol. 302, issue 1, pp. 129–136. https://doi.org/10.1016/J.JMAA.2004.07.049
38. Mittal M. L., Mradul Veer Singh. Approximation of signals (functions) by trigonometric polynomials in L_p-norm, *International Journal of Mathematics and Mathematical Sciences*, 2014, vol. 2014, article ID: 267383. https://doi.org/10.1155/2014/267383
39. Mohaparta R. N., Russell D. C. Some direct and inverse theorems in approximation of functions, *Journal of the Australian Mathematical Society. Series A. Pure Mathematics and Statistics*, 1983, vol. 34, issue 2, pp. 143–154. https://doi.org/10.1017/S144678870002317X
40. Osikiewicz J. A. Equivalence results for Cesàro submethods, *Analysis*, 2000, vol. 20, issue 1, pp. 35–43. https://doi.org/10.1524/anly.2000.20.1.35
41. Quade E. S. Trigonometric approximation in the mean, *Duke Mathematical Journal*, 1937, vol. 3, no. 3, pp. 529–543. https://doi.org/10.1215/S0012-7094-37-00342-9
42. Sbordone C. Grand Sobolev spaces and their applications to variational problems, *Le Matematiche*, 1996, vol. 51, no. 2, pp. 335–347. https://lematematiche.dmi.unict.it/index.php/lematematiche/article/view/443
43. Sbordone C. Nonlinear elliptic equations with right hand side in nonstandard spaces, *Atti del Seminario Matematico e Fisico dell'Università di Modena*, 1998, vol. 46 suppl., pp. 361–368. https://zbmath.org/0913.35050
44. Samko S. G. Convolution type operators in $L^{p(x)}$, *Integral Transforms and Special Functions*, 1998, vol. 7, issues 1–2, pp. 123–144. https://doi.org/10.1080/10652469808819191
45. Sharapudinov I. I. The topology of the space $\mathscr{L}^{p(t)}([0,1])$, *Mathematical Notes of the Academy of Sciences of the USSR*, 1979, vol. 26, issue 4, pp. 796–806. https://doi.org/10.1007/BF01159546
46. Sharapudinov I. I. *Nekotorye voprosy teorii priblizhenii v prostranstvakh Lebega s peremennym pokazatelem* (Some questions of approximation theory in the Lebesgue spaces with variable exponent), Vladikavkaz: Southern Mathematical Institute of the Vladikavkaz Scientific Center of the Russian Academy of Sciences and the Government of the Republic of North Ossetia–Alania, 2012. https://www.elibrary.ru/item.asp?id=22887342
47. Sharapudinov I. I. Approximation of functions in variable-exponent Lebesgue and Sobolev spaces by finite Fourier–Haar series, *Sbornik: Mathematics*, 2014, vol. 205, issue 2, pp. 291–306. https://doi.org/10.1070/SM2014v205n02ABEH004376
48. Szal B. Trigonometric approximation by Nörlund type means in L^p-norm, *Commentationes Mathematicae Universitatis Carolinae*, 2009, vol. 50, issue 4, pp. 575–589. http://dml.cz/dmlcz/137448
49. Sonker S., Singh U. Approximation of signals (functions) belonging to $\mathrm{Lip}(\alpha, p, \omega)$-class using trigonometric polynomials, *Procedia Engineering*, 2012, vol. 38, pp. 1575–1585. https://doi.org/10.1016/j.proeng.2012.06.193
50. Testici A., Israfilov D. M. Approximation by matrix transforms in generalized grand Lebesgue spaces with variable exponent, *Applicable Analysis*, 2021, vol. 100, issue 4, pp. 819–834. https://doi.org/10.1080/00036811.2019.1622680
51. Testici A. Approximation by Nörlund and Riesz means in weighted Lebesgue spaces with variable exponent, *Communications Faculty Of Science University of Ankara Series A1 Mathematics and Statistics*, 2019, vol. 68, no. 2, pp. 2014–2025. https://doi.org/10.31801/cfsuasmas.460449

52. Testici A., Israfilzade D. M. Linear methods of approximation in weighted Lebesgue spaces with variable exponent, *Hacettepe Journal of Mathematics and Statistics*, 2021, vol. 50, issue 3, pp. 744–753. https://doi.org/10.15672/hujms.798028
53. Testici A., Israfilov D. M. Approximation by matrix transforms in Morrey spaces, *Problemy Analiza — Issues of Analysis*, 2021, vol. 10 (28), issue 2, pp. 79–98. https://doi.org/10.15393/j3.art.2021.9635
54. Testici A., Israfilov D. M. Approximation by matrix transforms in generalized grand Lebesgue spaces with variable exponent, *Journal of Numerical Analysis and Approximation Theory*, 2021, vol. 50, no. 1, pp. 60–72. https://doi.org/10.33993/jnaat501-1234
55. Volosivets S. S. Approximation of functions and their conjugate in variable Lebesgue spaces, *Sbornik: Mathematics*, 2017, vol. 208, issue 1, pp. 44–59. https://doi.org/10.1070/SM8636
56. Volosivets S. S. Modified modulus of smoothness and approximation in weighted Lorentz spaces by Borel and Euler means, *Problemy Analiza — Issues of Analysis*, 2021, vol. 10 (28), issue 1, pp. 87–100. https://doi.org/10.15393/j3.art.2021.8950
57. Volosivets S. S. Approximation by Vilenkin polynomials in weighted Orlizc spaces, *Analysis Mathematica*, 2021, vol. 47, issue 2, pp. 437–449. https://doi.org/10.1007/s10476-021-0086-6
58. Zygmund A. *Trigonometric series*, Cambridge: Cambridge University Press, 2003. https://doi.org/10.1017/CBO9781316036587

Received 17.03.2023

Accepted 30.10.2023

Sadulla Z. Jafarov, PhD, Professor, Faculty of Education, Muş Alparslan University, Muş, 49250, Turkey; Institute of Mathematics and Mechanics, Azerbaijan National Academy of Sciences, ul. B. Vahabzade, 9, Baku, Az-1141, Azerbaijan.
ORCID: https://orcid.org/0000-0001-7443-5478
E-mail: s.jafarov@alparslan.edu.tr

Citation: S. Z. Jafarov. Approximation by Nörlund type means in the grand Lebesgue spaces with variable exponent, *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki*, 2024, vol. 34, issue 1, pp. 19–32.

МАТЕМАТИКА 2024. Т. 34. Вып. 1. С. 19–32.

С. З. Джафаров

Аппроксимация средними Нёрлунда в гранд-пространствах Лебега с переменным показателем

Ключевые слова: гранд-пространства Лебега с переменным показателем, модуль гладкости, классы Липшица, тригонометрическая аппроксимация, средние Нёрлунда.

УДК 517.518.8

DOI: 10.35634/vm240102

В настоящей работе исследуется аппроксимация функций средними Нёрлунда в обобщенных гранд-пространствах Лебега с переменным показателем.

СПИСОК ЛИТЕРАТУРЫ

1. Akgün R., Kokilashvili V. On converse theorems of trigonometric approximation in weighted variable exponent Lebesgue spaces // Banach Journal of Mathematical Analysis. 2011. Vol. 5. No. 1. P. 70–82. https://doi.org/10.15352/bjma/1313362981
2. Akgün R. Trigonometric approximation of functions in generalized Lebesgue spaces with variable exponent // Ukrainian Mathematical Journal. 2011. Vol. 63. No. 1. P. 1–26. https://doi.org/10.1007/s11253-011-0485-0
3. Akgün R., Kokilashvili V. The refined direct and converse inequalities of trigonometric approximation in weighted variable exponent Lebesgue spaces // Georgian Mathematical Journal. 2011. Vol. 18. Issue 3. P. 399–423. https://doi.org/10.1515/GMJ.2011.0037
4. Akgün R., Kokilashvili V. Some approximation problems for (α,ψ)-differentiable functions in weighted variable exponent Lebesgue spaces // Journal of Mathematical Sciences. 2012. Vol. 186. No. 2. P. 139–152. https://doi.org/10.1007/s10958-012-0980-3
5. Armitage D. H., Maddox I. J. A new type of Cesàro mean // Analysis. 1989. Vol. 9. Issues 1–2. P. 195–204. https://doi.org/10.1524/anly.1989.9.12.195
6. Chandra P. Approximation bu Nörlund operators // Matematički Vesnik. 1986. Vol. 38. Issue 3. P. 263–269.
7. Chandra P. Functions of classes L_p and $\mathrm{Lip}(\alpha,p)$ their Riesz means // Rivista di Matematica della Università di Parma. Serie 4. 1986. Vol. 12. P. 275–282. http://rivista.math.unipr.it/fulltext/1986-12/1986-12-275.pdf
8. Chandra P. A note on degree of approximation by Nörlund and Riesz operators // Matematički Vesnik. 1990. Vol. 42. Issue 1. P. 9–10.
9. Chandra P. Trigonometric approximation of functions in L_p-norm // Journal of Mathematical Analysis and Applications. 2002. Vol. 275. Issue 1. P. 13–26. https://doi.org/10.1016/S0022-247X(02)00211-1
10. Cruz-Uribe D. V., Fiorenza A. Variable Lebesgue spaces. Foundations and harmonic analysis. Basel: Birkhäuser, 2013. https://doi.org/10.1007/978-3-0348-0548-3
11. Diening L., Harjulehto P., Hästö P., Růžička M. Lebesgue and Sobolev spaces with variable exponents. Heidelberg: Springer, 2011. https://doi.org/10.1007/978-3-642-18363-8
12. Danelia N., Kokilashvili V. Approximation by trigonometric polynomials in subspace of weighted grand Lebesgue spaces // Bulletin of the Georgian National Academy of Sciences. 2013. Vol. 7. No. 1. P. 11–15. http://science.org.ge/old/moambe/7-1/Danelia%2011-15.pdf
13. Danelia N., Kokilashvili V. Approximation of periodic functions in grand variable exponent Lebesgue spaces // Proceedings of A. Razmadze Mathematical Institute. 2014. Vol. 164. P. 100–103. https://zbmath.org/1296.41004
14. Danelia V., Kokilashvili V., Tsanava Ts. Some approximation results in subspace of weighted grand Lebesgue spaces // Proceedings of A. Razmadze Mathematical Institute. 2014. Vol. 164. P. 104–108. https://zbmath.org/1297.42003

15. Danelia N., Kokilashvili V. Approximation by trigonometric polynomials in the framework of variable exponent grand Lebesgue spaces // Georgian Mathematical Journal. 2016. Vol. 23. Issue 1. P. 43–53. https://doi.org/10.1515/gmj-2015-0059
16. Sbordone C., Greco L., Iwaniec T. Inverting the p-harmonic operators // Manuscripta Mathematica. 1997. Vol. 92. Issue 2. P. 249–258. https://eudml.org/doc/156263
17. Guven A., Israfilov D. M. Trigonometric approximation in generalized Lebesgue spaces $L^{p(x)}$ // Journal of Mathematical Inequalities. 2010. Vol. 4. No. 2. P. 285–290. https://doi.org/10.7153/jmi-04-25
18. Güven A., Israfilov D. M. Approximation by means of Fourier trigonometric series in weighted Orlicz spaces // Advanced Studies in Contemporary Mathematics (Kyungshang). 2009. Vol. 19. Issue 2. P. 283–295.
19. Ильясов Н. А. Приближение периодических функций средними Зигмунда // Математические заметки. 1986. Т. 39. Вып. 3. С. 367–382. https://www.mathnet.ru/rus/mzm5056
20. Iwaniec T., Sbordone C. On the integrability of the Jacobian under minimal hypotheses // Archive for Rational Mechanics and Analysis. 1992. Vol. 119. No. 2. P. 129–143. https://doi.org/10.1007/BF00375119
21. Iwaniec T., Sbordone C. Riesz transform and elliptic PDEs with VMO coefficients // Journal d'Analyse Mathématique. 1998. Vol. 74. No. 1. P. 183–212. https://doi.org/10.1007/BF02819450
22. Jafarov S. Z. Linear methods for summing Fourier series and approximation in weighted Lebesgue spaces with variable exponents // Украинский математический журнал. 2014. Т. 66. № 10. С. 1348–1356. https://umj.imath.kiev.ua/index.php/umj/article/view/2226
23. Jafarov S. Z. Linear methods of summing Fourier series and approximation in weighted Orlicz spaces // Turkish Journal of Mathematics. 2018. Vol. 42. No. 6. Article 6. P. 2916–2925. https://doi.org/10.3906/mat-1804-31
24. Jafarov S. Z. Approximation by linear means of Fourier series in weighted Orlicz spaces // Proceedings of the Institute of Mathematics and Mechanics. National Academy of Sciences of Azerbaijan. 2017. Vol. 43. No. 2. P. 175–187. https://proc.imm.az/volumes/43-2/43-02-01.pdf
25. Jafarov S. Z. On approximation of a weighted Lipschitz class functions by means $t_n(f;x)$, $N_n^{\beta}(f;x)$ and $R_n^{\beta}(f,x)$ of Fourier series // Transactions Issue Mathematics. National Academy of Sciences of Azerbaijan. Series of Physical-Technical and Mathematical Sciences. 2020. Vol. 40. No. 4. P. 118–129. https://doi.org/10.29228/proc.50
26. Jafarov S. Z. On approximation of functions by means of Fourier trigonometric series in weighted generalized grand Lebesgue spaces // Advanced Studies: Euro-Tbilisi Mathematical Journal. 2022. Special Issue 10. P. 277–291. https://tcms.org.ge/Journals/ASETMJ/Special issue/10/PDF/asetmj_SpIssue_10_20.pdf
27. Kováčik O., Rákosník J. On spaces $L^{p(x)}$ and $W^{k,p(x)}$ // Czechoslovak Mathematical Journal. 1991. Vol. 41. Issue 4. P. 592–618. https://doi.org/10.21136/CMJ.1991.102493
28. Kokilashvili V. On a progress in the theory of integral operators in weighted Banach function spaces // Function Spaces, Differential Operators and Nonlinear Analysis: Proceedings of the Conference held in Milovy, Bohemian-Moravian Uplands, May 28–June 2, 2004. Praha: Mathematical Institute of the Academy of Sciences of the Czech Republic, 2005. P. 152–174.
29. Kokilashvili V. M., Samko S. G. Operators of harmonic analysis in weighted spaces with non-standard growth // Journal of Mathematical Analysis and Applications. 2009. Vol. 352. Issue 1. P. 15–34. https://doi.org/10.1016/j.jmaa.2008.06.056
30. Kokilashvili V., Meskhi A. Maximal and Calderón–Zygmund operators in grand variable exponent Lebesgue spaces // Georgian Mathematical Journal. 2014. Vol. 21. Issue 4. P. 447–461. https://doi.org/10.1515/gmj-2014-0047
31. Kokilashvili V., Meskhi A., Rafeiro H., Samko S. Integral operators in non-standard function spaces. Volume 1: Variable exponent Lebesgue and amalgam spaces. Cham: Birkhäuser, 2016. https://doi.org/10.1007/978-3-319-21015-5

32. Kokilashvili V., Meskhi A., Rafeiro H., Samko S. Integral operators in non-standard function spaces. Volume 2: Variable exponent Hölder, Morrey–Campanato and grand spaces. Cham: Birkhäuser, 2016. https://doi.org/10.1007/978-3-319-21018-6
33. Krasniqi X. Z. Trigonometric approximation of (signals) functions by Nörlund type means in the variable space $L^{p(x)}$ // Palestine Journal of Mathematics. 2017. Vol. 6. No. 1. P. 84–93. https://zbmath.org/1352.42004
34. Krasniqi X. Z. Approximation of periodic functions by sub-matrix means of their Fourier series // TWMS Journal of Applied and Engineering Mathematics. 2020. Vol. 10. No. 1. P. 279–287. http://jaem.isikun.edu.tr/web/index.php/archive/104-vol10no1/513-approximation-of-periodic-functions-by-sub-matrix-means-of-their-fourier-series
35. Krasniqi X. Z. On degree of approximation of continuous functions by a linear transformation of their Fourier series // Communications in Mathematics. 2022. Vol. 30. Issue 1. P. 37–46. https://doi.org/10.46298/cm.9273
36. Krasniqi X. Z., Lenski W., Szal B. Seminormed approximation by deferred matrix means of integrable functions in $H_P^{(\omega)}$ space // Results in Mathematics. 2022. Vol. 77. Issue 4. Article number: 145. https://doi.org/10.1007/s00025-022-01696-3
37. Leindler L. Trigonometric approximation in L_p-norm // Journal of Mathematical Analysis and Applications. 2005. Vol. 302. Issue 1. P. 129–136. https://doi.org/10.1016/J.JMAA.2004.07.049
38. Mittal M. L., Mradul Veer Singh. Approximation of signals (functions) by trigonometric polynomials in L_p-norm // International Journal of Mathematics and Mathematical Sciences. 2014. Vol. 2014. Article ID: 267383. https://doi.org/10.1155/2014/267383
39. Mohaparta R. N., Russell D. C. Some direct and inverse theorems in approximation of functions // Journal of the Australian Mathematical Society. Series A. Pure Mathematics and Statistics. 1983. Vol. 34. Issue 2. P. 143–154. https://doi.org/10.1017/S144678870002317X
40. Osikiewicz J. A. Equivalence results for Cesàro submethods // Analysis. 2000. Vol. 20. Issue 1. P. 35–43. https://doi.org/10.1524/anly.2000.20.1.35
41. Quade E. S. Trigonometric approximation in the mean // Duke Mathematical Journal. 1937. Vol. 3. No. 3. P. 529–543. https://doi.org/10.1215/S0012-7094-37-00342-9
42. Sbordone C. Grand Sobolev spaces and their applications to variational problems // Le Matematiche. 1996. Vol. 51. No. 2. P. 335–347. https://lematematiche.dmi.unict.it/index.php/lematematiche/article/view/443
43. Sbordone C. Nonlinear elliptic equations with right hand side in nonstandard spaces // Atti del Seminario Matematico e Fisico dell'Università di Modena. 1998. Vol. 46 Suppl. P. 361–368. https://zbmath.org/0913.35050
44. Samko S. G. Convolution type operators in $L^{p(x)}$ // Integral Transforms and Special Functions. 1998. Vol. 7. Issues 1–2. P. 123–144. https://doi.org/10.1080/10652469808819191
45. Шарапудинов И. И. О топологии пространства $\mathscr{L}^{p(t)}([0,1])$ // Математические заметки. 1979. Т. 26. Вып. 4. С. 613–632. https://www.mathnet.ru/rus/mzm8442
46. Шарапудинов И. И. Некоторые вопросы теории приближений в пространствах Лебега с переменным показателем. Владикавказ: Южный математический институт Владикавказского научного центра Российской академии наук и Правительства Республики Северная Осетия–Алания, 2012. https://www.elibrary.ru/item.asp?id=22887342
47. Шарапудинов И. И. Приближение функций из пространств Лебега и Соболева с переменным показателем суммами Фурье–Хаара // Математический сборник. 2014. Т. 205. № 2. С. 145–160. https://doi.org/10.4213/sm8274
48. Szal B. Trigonometric approximation by Nörlund type means in L^p-norm // Commentationes Mathematicae Universitatis Carolinae. 2009. Vol. 50. Issue 4. P. 575–589. http://dml.cz/dmlcz/137448
49. Sonker S., Singh U. Approximation of signals (functions) belonging to $\mathrm{Lip}(\alpha,p,\omega)$-class using trigonometric polynomials // Procedia Engineering. 2012. Vol. 38. P. 1575–1585. https://doi.org/10.1016/j.proeng.2012.06.193

50. Testici A., Israfilov D. M. Approximation by matrix transforms in generalized grand Lebesgue spaces with variable exponent // Applicable Analysis. 2021. Vol. 100. Issue 4. P. 819–834. https://doi.org/10.1080/00036811.2019.1622680
51. Testici A. Approximation by Nörlund and Riesz means in weighted Lebesgue spaces with variable exponent // Communications Faculty Of Science University of Ankara Series A1 Mathematics and Statistics. 2019. Vol. 68. No. 2. P. 2014–2025. https://doi.org/10.31801/cfsuasmas.460449
52. Testici A., Israfilzade D. M. Linear methods of approximation in weighted Lebesgue spaces with variable exponent // Hacettepe Journal of Mathematics and Statistics. 2021. Vol. 50. Issue 3. P. 744–753. https://doi.org/10.15672/hujms.798028
53. Testici A., Israfilov D. M. Approximation by matrix transforms in Morrey spaces // Проблемы анализа — Issues of Analysis. 2021. Т. 10 (28). Вып 2. С. 79–98. https://doi.org/10.15393/j3.art.2021.9635
54. Testici A., Israfilov D. M. Approximation by matrix transforms in generalized grand Lebesgue spaces with variable exponent // Journal of Numerical Analysis and Approximation Theory. 2021. Vol. 50. No. 1. P. 60–72. https://doi.org/10.33993/jnaat501-1234
55. Волосивец С. С. Приближение функций и их сопряженных в пространствах Лебега с переменным показателем // Математический сборник. 2017. Т. 208. № 1. С. 48–64. https://doi.org/10.4213/sm8636
56. Volosivets S. S. Modified modulus of smoothness and approximation in weighted Lorentz spaces by Borel and Euler means // Проблемы анализа — Issues of Analysis. 2021. Т. 10 (28). Вып. 1. С. 87–100. https://doi.org/10.15393/j3.art.2021.8950
57. Volosivets S. S. Approximation by Vilenkin polynomials in weighted Orlizc spaces // Analysis Mathematica. 2021. Vol. 47. Issue 2. P. 437–449. https://doi.org/10.1007/s10476-021-0086-6
58. Zygmund A. Trigonometric series. Cambridge: Cambridge University Press, 2003. https://doi.org/10.1017/CBO9781316036587

Поступила в редакцию 17.03.2023

Принята к публикации 30.10.2023

Джафаров Садулла, профессор, Университет Муш Алпарслан, 49250, Турция, г. Муш;
Институт математики и механики НАН Азербайджана, AZ-1141, Азербайджан, г. Баку, ул. Б. Вахабзаде, 9.
ORCID: https://orcid.org/0000-0001-7443-5478
E-mail: s.jafarov@alparslan.edu.tr

Цитирование: С. З. Джафаров. Аппроксимация средними Нёрлунда в гранд-пространствах Лебега с переменным показателем // Вестник Удмуртского университета. Математика. Механика. Компьютерные науки. 2024. Т. 34. Вып. 1. С. 19–32.