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ON THE GROUP OF DIFFEOMORPHISMS OF FOLIATED MANIFOLDS

Now the foliations theory is intensively developing branch of modern differential geometry, there are numerous researches on the foliation theory. The purpose of our paper is study the structure of the group $Diff_F(M)$ of diffeomorphisms and the group $Iso_F(M)$ of isometries of foliated manifold (M, F) . It is shown the group $Diff_F(M)$ is closed subgroup of the group $Diff(M)$ of diffeomorphisms of the manifold M in compact-open topology and also it is proven the group $Iso_F(M)$ is Lie group. It is introduced new topology on $Diff_F(M)$ which depends on foliation F and called F -compact open topology. It's proven that some subgroups of the group $Diff_F(M)$ are topological groups with F -compact open topology.

Keywords: manifold, foliation, group of diffeomorphisms, compact open topology.

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Introduction

The set $Diff(M)$ of all diffeomorphisms of manifold M onto itself is the group related to composition and inverse mapping. The diffeomorphism group of smooth manifolds are of great importance in differential geometry and in analysis. The fundamental works in this area are the studies of V. I. Arnold, A. M. Lukatsky, H. Omori [1, 6, 7, 15, 16]. Intensive development of the theory of groups of diffeomorphisms began after the work of V. I. Arnold, in which it was shown that the motions of an ideal incompressible fluid are geodesic on a group of diffeomorphisms that preserve volume element.

When M is a finite dimensional manifold in 1939 was proven that the group of isometries $I(M)$ of riemannian manifold M is Lie group [9].

H. Omori defined the differential structure on group of diffeomorphisms compact manifold which is weaker than that structure of Lie group in the classical sense. This differentiable structure is called ILH (inverse limit of Hilbert) Lie group. H. Omori showed that ILH Lie group of diffeomorphisms acting on compact manifold transitively and primitively can only be the following: the whole group of diffeomorphisms, the group of diffeomorphisms preserving the volume element, the group of symplectic diffeomorphisms, the group of contact diffeomorphisms. The questions of finite generation of diffeomorphism groups and curvature properties of ILH – Lie groups of diffeomorphisms are investigated by A. M. Lukatsky [6, 7].

It is known that the group $Diff(M)$ is topological group in compact open topology [18, p. 270], [13]. In this paper we investigate the groups $Diff_F(M)$ of diffeomorphisms and the group $Iso_F(M)$ of isometries of foliated manifold (M, F) with compact open topology. Also it is introduced new topology on $Diff_F(M)$ called F -compact open topology and we will show that some subgroups of $Diff_F(M)$ are topological groups with F -compact open topology.

§ 1. Main part

Let M be a connected Riemannian C^∞ -manifold of dimension n .

Definition 1. A foliation F on M of dimension k (codimension $n - k$) is a partition of M into arcwise connected subsets L_α with the following properties:

- (1) $M = \bigcup L_\alpha$;
- (2) $L_\alpha \cap L_\beta = \emptyset$ if $\alpha \neq \beta$;
- (3) for every point $p \in M$ there is an open neighborhood U of p and a chart $x = (x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_{n-k})$ such that for each leaf L_α the connected components of $L_\alpha \cap U$ are defined by the equations $y_1 = \text{const}, y_2 = \text{const}, \dots, y_{n-k} = \text{const}$.

Such a chart is a distinguished chart.

The connected components of the sets $y_1 = \text{const}, y_2 = \text{const}, \dots, y_{n-k} = \text{const}$ in a distinguished chart are called plaques (plates) of F . Fixing $y_1 = \text{const}, y_2 = \text{const}, \dots, y_{n-k} = \text{const}$, the map $x \rightarrow (x, y)$ is a smooth embedding, therefore the plaques are connected k -dimensional submanifolds of M . This shows that each leaf L_α is union of plaques and there exists differentiable structure σ_α on L_α such that $(L_\alpha, \sigma_\alpha)$ is a k -dimensional connected manifold. Note that the canonical injection $i: (L_\alpha, \sigma_\alpha) \rightarrow M$ is an immersion, but it is not necessarily an embedding [20].

An example of a foliation is given by a smooth submersion $f: M \rightarrow B$, where B — $(n - k)$ -dimensional manifold. The connected components of the inverse images of the points of $y \in B$ define a k -dimensional foliation of M .

We will denote by (M, F) manifold M with k -dimensional foliation F on M . Let $L(p)$ be a leaf of the foliation F passing through point the p , $T_p F$ is the tangent space to the the leaf $L(p)$ at p .

We get subbundle (smooth distribution) $TF = \{T_p F: p \in M\}$ of the tangent bundle TM of the manifold M .

Let us denote by $V(M), V(F)$ the set of smooth sections of bundles TM, TF respectively. The set $V(F)$ consists of vector fields tangent to the foliation F . The set $V(M)$ is a Lie algebra with respect to the Lie bracket of vector fields, the set $V(F)$ is a Lie subalgebra of $V(M)$.

Definition 2. If for the some C^r - diffeomorphism $\varphi: M \rightarrow M$ the image $\varphi(L_\alpha)$ of any leaf L_α of foliation F is a leaf of foliation F , we say that the φ is C^r - diffeomorphism of foliated manifold and write as $\varphi: (M, F) \rightarrow (M, F)$.

Let's denote as $Diff_F(M)$ the set of all C^r - diffeomorphisms of foliated manifold (M, F) , where $r \geq 0$. The group $Diff_F(M)$ is subgroup of $Diff(M)$ and therefore it is topological group in compact open topology.

Example 1. Let $M = \mathbb{R}^2(x, y)$ is a Euclidean plane with the Cartesian coordinates (x, y) . Leaves L_α of foliation F are given by the equations $x^2 - y = \alpha = \text{const}$. Then the plan diffeomorphism $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ determined by the formula

$$\varphi(x, y) = (x, y + \lambda f(x, y))$$

is diffeomorphism of foliated plane (\mathbb{R}^2, F) , for every $\lambda \in \mathbb{R}$, such that $\lambda \neq 1$, where $f(x, y) = x^2 - y$. It sends a leaf L_α to leaf $L_{(1-\lambda)\alpha}$. It is easy to check that in fact $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is an isometry of foliated plane (\mathbb{R}^2, F) .

Let (M, F) be a foliated manifold, where F is k -dimensional smooth foliation where $0 < k < n$. Recall a vector field X is called a foliated field if for every vector field Y , tangent to F , Lie brocket $[X, Y]$ also is tangent to F . It is known that flow of every foliated field consists of diffeomorphisms of foliated manifold (M, F) [20]. The set $L(M, F)$ of foliated vector fields is a Lie subalgebra of Lie algebra $V(M)$ [8, p. 35]. It follows from here that the group $Diff_F(M)$ contains the Lie group for which the Lie algebra is an algebra $L(M, F)$.

For foliated plane from example-1 vector field $X = (x^2 - y)\frac{\partial}{\partial y}$ is foliated field and its flow consists of diffeomorphisms $\varphi^t: (x, y) \in \mathbb{R}^2 \rightarrow (x, x^2 - e^{-t}(x^2 - y)) \in \mathbb{R}^2$ of foliated plane (\mathbb{R}^2, F) . Every diffeomorphism $\varphi^t: (x, y) \in \mathbb{R}^2 \rightarrow (x, x^2 - e^{-t}(x^2 - y)) \in \mathbb{R}^2$ sends a leaf L_α to leaf $L_{e^{-t}\alpha}$.

As shown in the following simple example, from differentiability of mapping on each leaf does not follow it's differentiability on all manifold M .

Example 2. Let $M = \mathbb{R}^2(x, y)$ is a Euclidean plane with the Cartesian coordinates (x, y) . Leaves L_α of foliation F are given by the equations $y = \alpha = \text{const}$. Then the plan homeomorphism $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ determined by the formula $\varphi(x, y) = (x + y, y^{\frac{1}{3}})$ is a diffeomorphism on every leaf of the foliation F , but is not a diffeomorphism of plane.

Theorem 1. Let (M, F) be a foliated manifold where M is a smooth connected finite-dimensional manifold. Then the group $\text{Diff}_F(M)$ is a closed subgroup of $\text{Diff}(M)$ in compact open topology.

P r o o f. Since a manifold M is a separable metric space for $r \geq 0$ the set $C^r(M)$ of all C^r -maps of manifold M onto itself with compact open topology has complete metric and a countable base [5, p. 35], [4, p. 202]. Consequently topological space $\text{Diff}(M)$ with induced compact open topology has a countable base. Thus we can use sequences to prove the theorem.

Let us assume that sequence $f_i \in \text{Diff}_F(M)$ convergences to $f \in \text{Diff}(M)$ at $i \rightarrow \infty$ in compact-open topology and $K \subset L$ is a connected compact subset of some leaf L of foliation of F .

We can assume that $f(K)$ is contained in a distinguished chart (U, x, y) . Then sets $f_i(K)$ are contained in local plaques which are determined by equations $y_1 = c_1^i, y_2 = c_2^i, \dots, y_{n-k} = c_{n-k}^i$. It follows from here the set $f(K)$ is contained in the plaque which is determined by equations $y_1 = \text{const}, y_2 = \text{const}, \dots, y_{n-k} = \text{const}$. Hence the set $f(K)$ is subset of some leaf of foliation F .

In the case when $f(K)$ is not contained in a distinguished chart we can cover $f(K)$ with finite number of distinguished charts and by using the local coordinate system of a distinguished chart we get the set $f(K)$ is contained in some leaf of foliation F .

Now we consider case when $K \subset L$ is not connected compact, let K_1 and K_2 be two of its connected components. Assume the $f(K_1)$ is contained in a leaf L_1 , the set $f(K_2)$ is contained in a leaf L_2 . Since every leaf of foliation is arcwise connected by definition, there exists continuous path $\varphi: [0, 1] \rightarrow L$ from K_1 to K_2 , say $\varphi(0) \in K_1$ and $\varphi(1) \in K_2$. We cover compact $\varphi([0, 1])$ with finite number of distinguished charts and by using the local coordinate system of a distinguished chart we get the set $\varphi([0, 1])$ is contained in some leaf of foliation F . It follows from here that $L_1 = L_2$, i. e., $f(K)$ is contained in some leaf of foliation F .

Every leaf of foliation as manifold is paracompact topological space and hence it can be written as $L = \bigcup_1^\infty K_j$, where K_j — are compact subsets of L and $K_1 \subset K_2 \subset \dots$.

We can repeat this process for compacts K_j and get all the sets $f(K_j)$ are contained in the same leaf of foliation F . It follows the set $f(L)$ is a subset of some leaf L' .

Since L is k -dimensional manifold [19, p. 86], canonical injection $i: (L_\alpha, \sigma_\alpha) \rightarrow M$ is an immersion and f is a homeomorphism, the image $f(L)$ is a k -dimensional topological manifold, i. e., every point of $f(L)$ has a neighborhood in $f(L)$ homeomorphic to an open ball in R^k . Hence the set $f(L)$ is a open subset of L' .

We will show it is closed subset of L' . Let sequence $q_i \in f(L)$ convergences to a point q in topology of L' , U_q be a neighborhood of the point of q with compact closure in L' . As $\text{Diff}(M)$ is topological group in compact open topology, it follows from $f_i \rightarrow f$ at $i \rightarrow \infty$ that $f_i^{-1} \rightarrow f^{-1}$

at $i \rightarrow \infty$. By using arguments above we get $f^{-1}(U_q)$ is contained in the some leaf. But the set U_q contains points q_i for large i . It follows from here that the set $f^{-1}(U_q)$ is contained in the leaf L . Then sequence $p_i = f^{-1}(q_i) \in L$ convergences to a point $p = f^{-1}(q)$ in topology of L . Therefore $q = f(p) \in f(L)$. As L' is connected manifold we have $f(L) = L'$. From here we have $f \in Diff_F(M)$. \square

The closedness of the set $Diff_F(M)$ allows us to state the following corollary [2, p. 30].

Corollary 1. *Factor space $Diff(M)/Diff_F(M)$ is regular homogeneous topological space.*

Let M be a smooth connected finite-dimensional Riemannian manifold.

Definition 3. An isometry $\varphi: M \rightarrow M$ is called an isometry of foliated manifold (M, F) if it is diffeomorphism of foliated manifold (M, F) .

We will denote by $Iso_F(M)$ the set of all C^r -isometries of foliated manifold (M, F) , where $r \geq 0$. We have that $Iso_F(M) = Diff_F(M) \cap Iso(M)$.

Example 3. Let $M = \mathbb{R}^2(x, y)$ is a Euclidean plane with the Cartesian coordinates (x, y) , foliation F are given by the equations $x^2 - y = \alpha = \text{const}$. Then the plan isometry $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ determined by the formula

$$\varphi_\lambda(x, y) = (x, y + \lambda)$$

is isometry of foliated plane (\mathbb{R}^2, F) for every $\lambda \in \mathbb{R}$. It sends a leaf L_α to $L_{\alpha-\lambda}$. Family of isometries φ_λ is the flow of Killing vector field $X = \frac{\partial}{\partial y}$.

Let us recall that vector field X on riemannian manifold (M, g) is called Killing field if its flow consists of isometries of Riemannian manifold (M, g) , that is $L_X g = 0$, where g is riemannian metric, $L_X g$ denotes Lie derivative of the metric g with respect to X . Geometry of orbits of Killing vector fields is studied in [10–12, 14]. If X is foliated Killing vector field, it's flow consists of isometries of foliated manifold (M, F) . The set $K(M, F)$ of foliated Killing vector fields is a Lie subalgebra of Lie algebra $L(M, F)$. It follows from here that the group $Iso_F(M)$ contains the Lie group for which the Lie algebra is an algebra $K(M, F)$.

It is known that limit of isometries is an isometry. By using this fact as direct consequence of Theorem 1 and the definition of the induced topology we have following theorem.

Theorem 2. *Let (M, F) be a foliated manifold where M is a smooth connected finite-dimensional Riemannian manifold. Then the group $Iso_F(M)$ is closed subset of $Iso(M)$ in compact open topology.*

From the Theorem 2 and Cartan theorem [17, p. 242] it follows very important fact that the group $Iso_F(M)$ is Lie group. Really Cartan's theorem states that on a closed subgroup of a Lie group there exists a differential structure with respect to which the closed subgroup is a Lie subgroup of a given Lie group. This fact we formulate as theorem in following form.

Theorem 3. *Let (M, F) be a foliated manifold where M is a smooth connected finite-dimensional Riemannian manifold. Then the group $Iso_F(M)$ is Lie subgroup of Lie group $Iso(M)$.*

We will introduce some topology on the group $Diff_F(M)$, which depends on foliation F and coincides with compact open topology when F is n -dimensional foliation.

Let $\{K_\lambda\}$ be a family of all compact sets where each K_λ is a subset of some leaf L_λ of foliation F and let $\{U_\beta\}$ is the family of all open sets on M . We consider for each pair K_λ

and U_β set of all mappings $f \in Diff_F(M)$ for which $f(K_\lambda) \subset U_\beta$. This set of mappings we denote through $[K_\lambda, U_\beta] = \{f: M \rightarrow M | f(K_\lambda) \subset U_\beta\}$.

It isn't difficult to show that every possible finite intersections of sets of the form $[K_\lambda, U_\beta]$ forms a base for some topology. This topology we call foliated compact open topology or in brief F -compact open topology. The space $Diff_F(M)$ with F -compact open topology is Hausdorff topological space [13]. Since K runs only over all compact subsets of leaves, F -compact open topology on $Diff_F(M)$ is weaker than induced from $Diff(M)$ usual compact open topology.

Theorem 4. *Let (M, F) be a foliated manifold, where M is a smooth connected finite-dimensional Riemannian manifold. Then the group $Iso_F(M)$ is a topological group with F -compact open topology.*

P r o o f. We denote through $d(x, y)$ the distance between points x and y , determined by Riemannian metric g .

First of all we will show that mapping $(g, h) \rightarrow g \circ h$ is continuous mapping for $g, h \in Iso_F(M)$.

Let $g \circ h \in [K, G]$, where K is a compact on some leaf of foliation F , G is a open subset of M . Since $K' = g(h(K))$ is a compact set, there is $\varepsilon > 0$ such that neighborhood $V_\varepsilon = \{y \in M: d(y, K') < \varepsilon\}$ of K' is contained in G , where $d(y, K') = \inf\{d(y, z): z \in K'\}$.

Now we put

$$U_{\varepsilon_1}(h) = \{h' \in Iso_F(M): d(h'(x), h(x)) < \varepsilon_1, \quad \forall x \in K\}$$

and

$$U_{\varepsilon_2}(g) = \{g' \in Iso_F(M): d(g'(y), g(y)) < \varepsilon_2, \quad \forall y \in h(K)\}$$

where $\varepsilon_1 > 0, \varepsilon_2 > 0$, and $\varepsilon_1 + \varepsilon_2 < \varepsilon$.

The set $U_{\varepsilon_1}(h)$ is a neighborhood of h in F -compact open topology as $U_{\varepsilon_1}(h) = [K, V_{\varepsilon_1}]$, the set $U_{\varepsilon_2}(g)$ is a neighborhood of g in F -compact-open topology, as $U_{\varepsilon_2}(g) = [h(K), V'_{\varepsilon_2}]$, where $V_{\varepsilon_1} = \{x \in M: d(x, h(K)) < \varepsilon_1\}$, $V'_{\varepsilon_2} = \{y \in M: d(y, K') < \varepsilon_2\}$.

If $h' \in U_{\varepsilon_1}(h)$, $g' \in U_{\varepsilon_2}(g)$, $x \in K$, $y = h(x)$ and $y' = h'(x)$ then

$$d(g'(h'(x)), g(h(x))) = d(g'(y'), g(y)) \leq d(g'(y'), g'(y)) + d(g'(y), g(y)).$$

Since g, g' are isometries we have

$$d(g'(y'), g'(y)) + d(g'(y), g(y)) \leq d(y', y) + \varepsilon_2 \leq \varepsilon_1 + \varepsilon_2 < \varepsilon.$$

It follows from here that if $(g', h') \in U_{\varepsilon_2}(g) \times U_{\varepsilon_1}(h)$, then $g' \circ h' \in [K, G]$.

Now we shall prove, that mapping $\chi: f \rightarrow f^{-1}$ is continuous.

For this purpose we shall prove, that the full inverse image $\chi^{-1}(A)$ of the open set $A \subset Iso_F(M)$ is a open set. Really it is enough to show this fact when A is an element of prebase, i. e., $A = \{f \in Iso_F(M): f(K) \subset V\}$ where K is compact subset of the leaf L of foliation F , V is open set. In this case $\chi^{-1}(A) = \{f \in Iso_F(M): f^{-1}(K) \subset V\}$.

We shall show, that $\chi^{-1}(A)$ is a open set in F -compact open topology.

Let $g \in \chi^{-1}(A)$, U — a neighborhood of $g^{-1}(K)$ in M with compact closure such, that $\bar{U} \subset V$. We put $K_1 = g^{-1}(K)$, $U_g = \{h \in Iso_F(M): d(g(x), h(x)) < \frac{\varepsilon}{2}, \quad \forall x \in K_1\}$, where $\varepsilon = \text{dist}(K_1, M \setminus U) = \inf\{d(x, y): x \in K_1, y \in (M \setminus U)\}$.

We need to show that if $h \in U_g$ then $h^{-1}(K) \subset U$. In other words, we must show that $h^{-1}(y) \in U$ for all $y \in K$. Indeed, since $g^{-1}(y) \in K_1$,

$$d(y, h(g^{-1}(y))) = d(g(g^{-1}(y)), h(g^{-1}(y))) < \frac{\varepsilon}{2}.$$

Since h is an isometry we have $d(y, h(g^{-1}(y))) = d(h^{-1}(y), h^{-1} \cdot h \cdot g^{-1}(y)) = d(h^{-1}(y), g^{-1}(y))$. Thus $d(h^{-1}(y), g^{-1}(y)) < \frac{\varepsilon}{2}$ and $h^{-1}(y)$ belongs to a $\frac{\varepsilon}{2}$ -neighborhood of $g^{-1}(y)$ and therefore of $\frac{\varepsilon}{2}$ -neighborhood of K_1 . Hence it is contained in U , that is $h^{-1}(y) \in U$. \square

Let's denote as $Diff_F^0(M)$ set of all C^r diffeomorphisms $g \in Diff_F(M)$ of foliated manifold (M, F) , such that $g(L_\alpha) = L_\alpha$ for every L_α leaf of foliation F . Flow of every tangent vector field consists of diffeomorphisms of foliated manifold (M, F) , which belong to the group $Diff_F^0(M)$. It follows from here that the group $Diff_F^0(M)$ contains the Lie group for which the Lie algebra is an algebra $V(F)$.

It can be proven following theorem.

Theorem 5. *Let (M, F) be a foliated manifold, where M is a smooth, connected and finite-dimensional manifold. Assume every leaf L_α of F is a proper, i. e., canonic injection $i: L_\alpha \rightarrow M$ is an embedding. Then the group $Diff_F^0(M)$ is a topological group with F -compact open topology.*

P r o o f. First of all we will show that mapping $(g, h) \rightarrow g \circ h$ is continuous mapping for $g, h \in Diff_F^0(M)$.

Let $g \circ h \in [K, G]$, where K is a compact subset of some leaf L of foliation F , G is a open subset of M . Then $h(K) \subset g^{-1}(G)$ and since $h(K)$ is a compact set, there is open neighborhood A of $h(K)$ in the leaf $h(L)$ with compact closure, i. e., \bar{A} compact in $h(L)$. Since the leaf $h(L)$ is a proper there exists open subset A' of M such that $A = h(L) \cap A'$. Since h is leaf-preserving map we have $A = L \cap A'$. If $h' \in [K, A']$, then $h'(K) \subset A' \cap h(L)$ and $h(K) \subset A$. It follows from here that if $(g', h') \in [\bar{A}, G] \times [K, A']$, then $g' \circ h' \in [K, G]$. Thus, the inverse image of the open set under the mapping $(g, h) \rightarrow g \circ h$ is open; therefore, the map $(g, h) \rightarrow g \circ h$ is continuous.

We shall prove, that mapping $\chi: f \rightarrow f^{-1}$ is continuous.

Let g be complete Riemannian metric on M . It is known, smooth manifold M possess complete Riemannian metric [3, p. 186], [4].

The restriction of the Riemannian metric g to $T_p F$ for all $p \in M$ induces complete Riemannian metric on the leaves [11].

Let L_α be a some leaf of foliation F , d_α is the distance on L_α , determined by induced Riemannian metric g_α .

Now using complete Riemannian metric g_α we shall prove, that mapping $\chi: f \rightarrow f^{-1}$ is continuous. For this purpose we shall prove, that the full inverse image $\chi^{-1}(A)$ of the open set $A \subset Diff_F^0(M)$ is a open set.

Really it is enough to show this fact when A is an element of prebase, i. e., $A = \{f \in Diff_F^0(M): f(K) \subset V\}$ where K is compact subset of a leaf L_α , V is open subset of M . In this case $\chi^{-1}(A) = \{f \in Diff_F^0(M): f^{-1}(K) \subset V\}$.

We shall show, that $\chi^{-1}(A)$ is a open set in F -compact open topology.

Let $g \in \chi^{-1}(A)$, U a neighborhood of $g^{-1}(K)$ in L_α with compact closure \bar{U} in topology of L_α such, that $\bar{U} \subset V$.

We put $U(g) = \{g \in Diff_F^0(M): d_\alpha(g(x), h(x)) < \frac{\varepsilon}{2}, \forall x \in \bar{U}\}$, where

$$\varepsilon = d_\alpha(K, L_\alpha \setminus g(U)) = \inf\{d_\alpha(x, y): x \in K, y \in (L_\alpha \setminus g(U))\}.$$

Let's show, that if $h \in U(g)$ takes place $h^{-1}(K) \subset V$, i. e., $U(g) \subset \chi^{-1}(A)$. We shall show, that $h^{-1}(K) \subset U$.

Let's assume, it is not true. Let for the some $h \in U(g)$ there exists a point $y \in K$ such that $h^{-1}(y) \in L_\alpha \setminus U$, i. e., $y \in L_\alpha \setminus h(U)$. Then by virtue of that $g^{-1}(y) \in U$, takes place $d_\alpha(y = g(g^{-1}(y)), h(g^{-1}(y))) < \frac{\varepsilon}{2}$.

Let γ — the shortest geodesic in L_α (by virtue of completeness (L_α, g_α)) there exists a shortest geodesic between any two points) going from the point y to the point $h(g^{-1}(y))$, and

$z \in \gamma \cap \partial(h(U))$. Then $h^{-1}(z) \in \bar{U}$ and besides $d_\alpha(g(h^{-1}(z)), h(h^{-1}(z))) < \frac{\varepsilon}{2}$. In addition $d_\alpha(y, z) < \frac{\varepsilon}{2}$. Hence $d_\alpha(y, g(h^{-1}(z))) \leq d_\alpha(y, z) + d_\alpha(z, g(h^{-1}(z))) < \varepsilon$.

But on the other hand, by virtue of that $z \notin h(U)$, we have $g(h^{-1}(z)) \in M \setminus g(U)$. Since $y \in K \subset g(U)$, $g(h^{-1}(z)) \notin g(U)$ and $\varepsilon = d_\alpha(K, L \setminus g(U))$ we have $d_\alpha(y, g(h^{-1}(z))) \geq \varepsilon$. This contradiction shows that $h^{-1}(K) \subset U$. Hence, $U(g) \subset \chi^{-1}(A)$. \square

Proposition 1. *Let (M, F) be a foliated manifold, where M is a smooth, connected and finite-dimensional manifold. Assume all leaves of foliated manifold (M, F) are closed subsets of M . Then the group $Diff_F^0(M)$ is closed subset of $Diff_F(M)$ in F -compact open topology.*

P r o o f. Let us consider the set $A = Diff_F(M) \setminus Diff_F^0(M)$ and show that the set A is open subset of $Diff_F(M)$ in F -compact open topology. If $f \in A$ then there exists a leaf L of foliation F such that $f(L) = L'$, and $L \neq L'$, L' – another leaf of foliation F .

Let K be a compact subset of the leaf L , ε_i – real numbers, $i = 1, 2, \dots$, and $\varepsilon_i \rightarrow 0$ at $i \rightarrow \infty$.

We put

$$U_i(f) = \{g \in Diff_F(M) : d(g(x), f(x)) < \varepsilon_i, \forall x \in K\}, \quad V_{\varepsilon_i} = \{x \in M : d(x, f(K)) < \varepsilon_i\},$$

where d – the distance on M , determined by Riemannian metric g . The set $U_i(f)$ is a open subset of $Diff_F(M)$ in F -compact open topology since $U_i(f) = [K, V_{\varepsilon_i}]$.

If for some i the set $U_i(f)$ does not contain elements of $Diff_F^0(M)$ then the set $U_i(f) \subset A$ and f is a interior point of A .

Assume that for all i the set $U_i(f)$ contains element f_i of $Diff_F^0(M)$. Then sequence $f_i(x)$ convergences to $f(x)$ at $i \rightarrow \infty$ for every $x \in K$. It follows from here $f(x) \in \bar{L}$, where \bar{L} – closure of L in M . Since $\bar{L} = L$, we have $f(x) \in L$. On the other hand since $f(L) = L'$ and $L \cap L' = \emptyset$ it follows from here $f(K) \subset L'$. This contradiction shows that for large i the set $U_i(f)$ does not contain elements of $Diff_F^0(M)$ and $U_i(f) \subset A$, i. e., element f is a interior point of A . \square

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О группе диффеоморфизмов слоеных многообразий

Ключевые слова: многообразие, слоение, группа диффеоморфизмов, компактно открытая топология.

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В настоящее время теория слоений является интенсивно развивающимся разделом современной дифференциальной геометрии, что показывают многочисленные исследования по теории слоений. Целью нашей работы является изучение структуры группы диффеоморфизмов $Diff_F(M)$ и группы изометрий $ISO_F(M)$ слоеного многообразия (M, F) . Показано, что группа $Diff_F(M)$ является замкнутой подгруппой группы $Diff(M)$ диффеоморфизмов многообразия M в компактно-открытой топологии, а также доказано, что группа изометрий $ISO_F(M)$ слоеного многообразия является группой Ли. Введена новая топология на $Diff_F(M)$, которая зависит от слоения F и называется F -компактно открытой топологией. Доказано, что некоторые подгруппы группы $Diff_F(M)$ являются топологическими группами с F -компактно открытой топологией.

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