COMPUTER RESEARCH AND MODELING
2023 VOL. 15 NO. 5 P. e1125–e1151
DOI: 10.20537/2076-7633-2023-15-5-1125-1151

Ки&М

MATHEMATICAL MODELING AND NUMERICAL SIMULATION

UDC: 519.8(075)

Using feedback functions to solve parametric programming problems

A. E. Umnov[a], E. A. Umnov

Moscow Institute of Physics and Technology (National Research University),
9 Institutskiy per., Dolgoprudny, Moscow region, 141700, Russia

E-mail: [a] mail@umnov.ru

Received 17.06.2023, after completion — 25.07.2023.
Accepted for publication 18.08.2023.

We consider a finite-dimensional optimization problem, the formulation of which in addition to the required variables contains parameters. The solution to this problem is a dependence of optimal values of variables on parameters. In general, these dependencies are not functions because they can have ambiguous meanings and in the functional case be non-differentiable. In addition, their domain of definition may be narrower than the domains of definition of functions in the condition of the original problem. All these properties make it difficult to solve both the original parametric problem and other tasks, the statement of which includes these dependencies. To overcome these difficulties, usually methods such as non-differentiable optimization are used.

This article proposes an alternative approach that makes it possible to obtain solutions to parametric problems in a form devoid of the specified properties. It is shown that such representations can be explored using standard algorithms, based on the Taylor formula. This form is a function smoothly approximating the solution of the original problem for any parameter values, specified in its statement. In this case, the value of the approximation error is controlled by a special parameter. Construction of proposed approximations is performed using special functions that establish feedback (within optimality conditions for the original problem) between variables and Lagrange multipliers. This method is described for linear problems with subsequent generalization to the nonlinear case.

From a computational point of view the construction of the approximation consists in finding the saddle point of the modified Lagrange function of the original problem. Moreover, this modification is performed in a special way using feedback functions. It is shown that the necessary conditions for the existence of such a saddle point are similar to the conditions of the Karush – Kuhn – Tucker theorem, but do not contain constraints such as inequalities and conditions of complementary slackness. Necessary conditions for the existence of a saddle point determine this approximation implicitly. Therefore, to calculate its differential characteristics, the implicit function theorem is used. The same theorem is used to reduce the approximation error to an acceptable level.

Features of the practical implementation feedback function method, including estimates of the rate of convergence to the exact solution are demonstrated for several specific classes of parametric optimization problems. Specifically, tasks searching for the global extremum of functions of many variables and the problem of multiple extremum (maximin-minimax) are considered. Optimization problems that arise when using multicriteria mathematical models are also considered. For each of these classes, there are demo examples.

Keywords: nonlinear programming problem with parameters, feedback function, modified Lagrange function, search for a global extremum, minimax, multicriteria model

Citation: *Computer Research and Modeling*, 2023, vol. 15, no. 5, pp. e1125–e1151.

КОМПЬЮТЕРНЫЕ ИССЛЕДОВАНИЯ И МОДЕЛИРОВАНИЕ 2023 Т. 15 № 5 С. e1125–e1151
DOI: 10.20537/2076-7633-2023-15-5-1125-1151

Ки&М

МАТЕМАТИЧЕСКИЕ ОСНОВЫ И ЧИСЛЕННЫЕ МЕТОДЫ МОДЕЛИРОВАНИЯ

УДК: 519.8(075)

Использование функций обратных связей для решения задач параметрического программирования

А. Е. Умнов[a], Е. А. Умнов

Московский физико-технический институт (Национальный исследовательский университет),
Россия, 141700, Московская область, г. Долгопрудный, Институтский пер., д. 9

E-mail: [a] mail@umnov.ru

Получено 17.06.2023, после доработки — 25.07.2023.
Принято к публикации 18.08.2023.

Рассматривается конечномерная оптимизационная задача, постановка которой, помимо искомых переменных, содержит параметры. Ее решение есть зависимость оптимальных значений переменных от параметров. В общем случае такие зависимости не являются функциями, поскольку могут быть неоднозначными, а в функциональном случае — быть недифференцируемыми. Кроме того, область их существования может оказаться уже области определения функций в условии задачи. Эти свойства затрудняют решение как исходной задачи, так и задач, в постановку которых входят данные зависимости. Для преодоления этих затруднений обычно применяются методы типа недифференцируемой оптимизации.

В статье предлагается альтернативный подход, позволяющий получать решения параметрических задач в форме, лишенной указанных свойств. Показывается, что такие представления могут исследоваться стандартными алгоритмами, основанными на формуле Тейлора. Данная форма есть функция, гладко аппроксимирующая решение исходной задачи. При этом величина погрешности аппроксимации регулируется специальным параметром. Предлагаемые аппроксимации строятся с помощью специальных функций, устанавливающих обратные связи между переменными и множителями Лагранжа. Приводится краткое описание этого метода для линейных задач с последующим обобщением на нелинейный случай.

Построение аппроксимации сводится к отысканию седловой точки модифицированной функции Лагранжа исходной задачи. Показывается, что необходимые условия существования такой седловой точки подобны условиям теоремы Каруша – Куна – Таккера, но не содержат в явном виде ограничений типа неравенств и условий дополняющей нежесткости. Эти необходимые условия аппроксимацию определяют неявным образом. Поэтому для вычисления ее дифференциальных характеристик используется теорема о неявных функциях. Эта же теорема применяется для уменьшения погрешности аппроксимации.

Особенности практической реализации метода функций обратных связей, включая оценки скорости сходимости к точному решению, демонстрируются для нескольких конкретных классов параметрических оптимизационных задач. Конкретно: рассматриваются задачи поиска глобального экстремума функций многих переменных и задачи на кратный экстремум (максимин-минимакс). Также рассмотрены оптимизационные задачи, возникающие при использовании многокритериальных математических моделей. Для каждого из этих классов приводятся демонстрационные примеры.

Ключевые слова: задача нелинейного программирования с параметрами, функция обратных связей, модифицированная функция Лагранжа, поиск глобального экстремума, минимакс, многокритериальная модель

1. Introduction

Consider the problem of parametric programming:

$$\begin{aligned}
&\textit{maximize } F(x, v) \textit{ with respect to } x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in E^n\\
&\textit{for a fixed parameter vector } v = (v_1, v_2, \ldots, v_K)^{\mathrm{T}} \in \Upsilon,\\
&\textit{where } \Upsilon \textit{ is a domain in } E^K,\\
&\textit{subject to}: x \in \Theta_v : \{x \mid f_i(x, v) \leqslant 0 \ \forall i = \overline{1, m}\}.
\end{aligned} \tag{1}$$

The solution of problem (1) is x_v^* — vector dependency $\operatorname*{argmax}_{x \in \Theta_v} F(x, v)$ on v.

Problems of the form (1), as well as those reduced to or related to them, were considered in a large number of both foreign and domestic studies, a detailed review of which can be found, for example, in [Измаилов, 2006].

In addition to the parameters, in the formulation of problem (1) the presence of constraints of type "inequality" is essential. For this reason,

– dependency scope x_v^* may *not match* the domain functions $F(x, v)$, $f_i(x, v)$,

– dependency x_v^* can be *non-functional* (*ambiguous*),

– dependency x_v^* can be *non-differentiable*.

These properties can complicate the procedure for solving both problem (1) and other problems that use x_v^*.

To date, a significant number of algorithms for solving parametric problems have been developed, for example, methods of nondifferentiable optimization and sensitivity theory [Danskin, 1967; Rockafellar, 1970; Демьянов, Васильев, 1981; Демьянов, Малоземов, 1972; Гольштейн, Третьяков, 1989; Нурминский, 1991; Измаилов, 2006]. These algorithms make it possible to overcome computational difficulties generated by marked features of x_v^* dependency.

However, of practical interest are also traditional methods for solving problems of the form (1), which are based on Taylor expansions. To date, such algorithms have been proposed, for example, in [Fiacco, McCormick, 1968; Гермейер, 1969; Умнов, 1974; Федоров, 1979; Fiacco, 1983; Скарин, 2010; Умнов, Умнов, 2014; Умнов, Умнов, 2018].

This article discusses an approach related to this direction. A method for constructing smooth function $\overline{x}(\tau, v)$, approximating dependence x_v^* is proposed. That is, a function for which the limit equality $\lim\limits_{\tau\to+0} F(\overline{x}(\tau, v), v) = F(x_v^*, v)$, is valid $\forall v \in \Upsilon$. In the case of unique x_v^* this equality is strengthened to $\lim\limits_{\tau\to+0} \overline{x}(\tau, v) = x_v^*$. In addition, the proposed approximation allows one to overcome the other above-mentioned computational difficulties arising in solving problems of the form (1).

Specifically, as $\overline{x}(\tau, v)$ it is suggested to use dependency on v the saddle point is modified in a special way, the Lagrange function for problem (1). This specificity is such that $\overline{x}(\tau, v)$ existence, functionality and smoothness are guaranteed: $\forall v \in \Upsilon$.

In the proposed approach the function $\overline{x}(\tau, v)$ is defined implicitly. However, the use of the classical theorem *on a system of implicit functions* allows one to overcome this difficulty and the build for $\overline{x}(\tau, v)$ Taylor approximations of the desired orders.

The procedure for modifying the Lagrange function and searching for its saddle point, called *the feedback function method*, was proposed and substantiated in [Умнов, Умнов, 2019] for the linear problem (1), the nonlinear case is considered in [Umnov, Umnov, 2022].

In this article, the application of the proposed approach is considered for problems reduced to the form (1):

– searching for the global extremum,

– finding multiple extremum and/or minimax,

– optimizations that occur when using multicriteria models.

In conclusion, we note that, in the proposed method, the smoothness of the approximation is combined with the ability to regulate its errors by selecting the value of the instrumental parameter τ.

This allows us to apply the considered approach in combination with other methods. That is, approximate estimates of solutions to the original problem, obtained using feedback functions, are used as initial approximations for algorithms of alternative types.

2. Feedback function method

Let us first give a brief description of the method of feedback functions. This method can be used for solutions both linear and non-linear problems. Therefore, it is advisable to start the description with a simpler, linear case.

Let the functions $F(x)$, $f_i(x)\ \forall i = \overline{1, n}$ be linear, and E^n_+ and E^m_+ be the non-negative orthants of the Euclidean spaces E^n and E^m. Then problem (1) can be written in the form

$$F(x) = \sum_{j=1}^{n} \sigma_j x_j \to \max,\ x \in E^n_+, \quad \textit{subject to } f_i(x) = -\beta_i + \sum_{j=1}^{n} \alpha_{ij} x_j \leqslant 0\ \forall i = \overline{1, m}. \tag{2}$$

Let us also formulate the problem dual to (2),

$$G(\lambda) = \sum_{i=1}^{m} \sigma_i \lambda_i \to \min \quad \textit{subject to } g_j(\lambda) = -\sigma_j + \sum_{i=1}^{m} \alpha_{ij} \lambda_i \geqslant 0\ \forall j = \overline{1, n}, \tag{3}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)^{\mathrm{T}} \in E^m_+$.

Let us apply, to solve problems (2) and (3), a variant of the method of *smooth penalty functions* [Умнов, 1974], which for problem (2) consists in sequential (in $\tau \to +0$) maximization over x auxiliary function $A_P(\tau, x) = F(x) - \sum\limits_{i=1}^{m} P(\tau, f_i(x)) - \sum\limits_{j=1}^{n} P(\tau, (-x_j))$, where the function $P(\tau, s)$, which determines the value of the "penalty" for violating the constraint $s \leqslant 0$, satisfies the following conditions:

2-1°. $\forall s$ and $\forall \tau > 0$: $\lim\limits_{\tau \to +0} P(\tau, s) = \begin{cases} +\infty, & s > 0, \\ 0, & s < 0. \end{cases}$

2-2°. The function $P(\tau, s)$ has continuous partial derivatives with respect to all its arguments up to and including the second order.

2-3°. For all $\tau > 0$ and $\forall s$ the inequalities $\frac{\partial P}{\partial s} > 0$ and $\frac{\partial^2 P}{\partial s^2} > 0$[1] are satisfied.

Note that from 2-3° it follows uniform convergence in $s\ \forall \varepsilon > 0$ on sets $s \leqslant -\varepsilon$ and $s \geqslant \varepsilon$ for limit transitions in 2-1°.

[1] Note that, for example, the standard quadratic penalty function of the form $P(\tau, s) = \begin{cases} \dfrac{s^2}{2\tau} \text{ if } s \geqslant 0, \\ 0 \text{ if } s < 0 \end{cases}$ does not meet these conditions.

For problem (3), the auxiliary function subject to successive minimization will look like:

$$A_D(\tau, \lambda) = G(\lambda) + \sum_{j=1}^{n} P(\tau, -g_j(\lambda)) + \sum_{i=1}^{m} P(\tau, (-\lambda_i)).$$

Let $\{\widetilde{x}(\tau), \widetilde{\lambda}(\tau)\}$ and $\{\breve{x}(\tau), \breve{\lambda}(\tau)\}$ be, respectively, stationary points of the functions $A_P(\tau, x)$ and $A_D(\tau, \lambda)$, defined for a fixed $\tau > 0$ by the equations

$$\operatorname*{grad}_x A_P(\tau, \widetilde{x}(\tau)) = o \quad \text{and} \quad \operatorname*{grad}_\lambda A_D(\tau, \breve{\lambda}(\tau)) = o.$$

If we additionally assume that problems (2) and (3) have uniquely defined solutions $x^* = \left(x_1^*, x_2^*, \ldots, x_n^*\right)^{\mathrm{T}}$ and $\lambda^* = \left(\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*\right)^{\mathrm{T}}$, then the following equalities are valid:

$$x_j^* = \lim_{\tau\to+0} \widetilde{x}_j(\tau) \qquad \forall j = \overline{1, n}, \qquad \lambda_i^* = \lim_{\tau\to+0} \breve{\lambda}_i(\tau) \qquad \forall i = \overline{1, m},$$
$$\lambda_i^* = \lim_{\tau\to+0} \frac{\partial P}{\partial s}(\tau, f_i(\widetilde{x}(\tau))) \quad \forall i = \overline{1, m}, \qquad x_j^* = \lim_{\tau\to+0} \frac{\partial P}{\partial s}(\tau, -g_j(\breve{\lambda}(\tau))) \quad \forall j = \overline{1, n},$$

since for problems (2)–(3) the components of the vector x^* are the Lagrange multipliers for problem (3), and the components of the vector λ^* are the Lagrange multipliers in problem (2).

In this case, $\forall\tau > 0$, in general,

$$\frac{\partial P}{\partial s}(\tau, f_i(\widetilde{x}(\tau))) \neq \breve{\lambda}_i(\tau)\ \forall i = \overline{1, m} \quad \text{or} \quad \frac{\partial P}{\partial s}(\tau, -g_j(\breve{\lambda}(\tau))) \neq \widetilde{x}_j(\tau)\ \forall j = \overline{1, n}. \tag{4}$$

Relations (4) will be valid equalities only in the limit, as $\tau \to +0$. However, formally after being written as *equalities* $\forall\tau > 0$, they can be combined into one system of equations. In [Умнов, Умнов, 2019] it is shown that that for any (i. e., without assumptions of compatibility or unique resolvability) linear problems (2)–(3) there are vector functions $\overline{x}(\tau)$ and $\overline{\lambda}(\tau)$ which are solutions of a similar, unified system of equations

$$\begin{cases} \overline{\lambda}_i(\tau) = \dfrac{\partial P}{\partial s}(\tau, f_i(\overline{x}(\tau))) & \forall i = \overline{1, m}, \\ \overline{x}_j(\tau) = \dfrac{\partial P}{\partial s}(\tau, -g_j(\overline{\lambda}(\tau))) & \forall j = \overline{1, n}, \end{cases} \tag{5}$$

for which (in the case of compatibility of problems (2)–(3)) it holds that: $\lim\limits_{\tau\to+0} F(\overline{x}(\tau)) = \lim\limits_{\tau\to+0} G(\overline{\lambda}(\tau)) = F^*$, and if x^* and λ^* are unique, then the following equalities hold:

$$\lambda_i^* = \lim_{\tau\to+0} \frac{\partial P}{\partial s}(\tau, f_i(\overline{x}(\tau)))\ \forall i = \overline{1, m}, \quad x_j^* = \lim_{\tau\to+0} \frac{\partial P}{\partial s}(\tau, -g_j(\overline{\lambda}(\tau)))\ \forall j = \overline{1, n}.$$

Note also that system (5) does not explicitly contain conditions for the non-negativity of the components of the vectors $\overline{x}(\tau)$ and $\overline{\lambda}(\tau)$, because $\frac{\partial P}{\partial s} > 0$ due to conditions 2-3°.

Now we use the fact that, under the assumptions made, the function $\frac{\partial P}{\partial s}(\tau, s)$, as a function continuously differentiable and strictly monotonically increasing by argument s $\forall s \in R$, has an *inverse one*, which is also continuously differentiable and strictly monotonically increasing by $(0, +\infty)$. By virtue of which system (5) can be written as

$$\begin{cases} f_i(\tau, \overline{x}(\tau)) = Q(\tau, \overline{\lambda}_i(\tau)) & \forall i = \overline{1, m}, \\ -g_j(\tau, \overline{\lambda}(\tau)) = Q(\tau, \overline{x}_j(\tau)) & \forall j = \overline{1, n} \end{cases} \quad \text{or} \quad \begin{cases} -\beta_i + \sum_{j=1}^{n} \alpha_{ij}\overline{x}_j = Q(\tau, \overline{\lambda}_i) & \forall i = \overline{1, m}, \\ \sigma_j - \sum_{i=1}^{m} \alpha_{ij}\overline{\lambda}_i = Q(\tau, \overline{x}_j) & \forall j = \overline{1, n}, \end{cases} \tag{6}$$

where the function $Q(\tau, s) = \operatorname*{inv}_s\left(\frac{\partial P}{\partial s}(\tau, s)\right)$ is an inverse function in s for $\frac{\partial P}{\partial s}(\tau, s)$.

It follows from (6) that that the function $Q(\tau, s)$ implements *feedback* between direct and dual variables under optimality conditions for problems (2) and (3). In other words, modulo small values functions $Q(\tau, s)$ on the right-hand sides of equations (6) are an indicator of activity of the corresponding constraints of problems (2) and (3) at the points x^* and λ^*. This justifies the use of term *feedback function* for $Q(\tau, s)$.

Finally, the transition to the nonlinear case is performed by introducing an auxiliary function of the form

$$U(\tau, x, \lambda) = \sum_{j=1}^{n}(\sigma_j x_j - R(\tau, x_j)) + \sum_{i=1}^{m}(\beta_i\lambda_i + R(\tau, \lambda_i)) - \sum_{j=1}^{n}\sum_{i=1}^{m}\alpha_{ij}x_j\lambda_i, \tag{7}$$

where $R(\tau, s) = \int\limits_{\alpha(\tau)}^{s} Q(\tau, u)\,du$, and the value of $\alpha(\tau)$ is found from the equation $Q(\tau, \alpha(\tau)) = 0$. This equation $\forall\tau > 0$ has a (unique) solution, since the function $Q(\tau, s)$ is strictly monotonically increasing in s and unbounded both from below and from above $\forall s \in (0, +\infty)$.

Due to (7), the solutions of system (6), that is, the vectors $\overline{x}(\tau)$ and $\overline{\lambda}(\tau)$ are the *stationary points* of the function $U(\tau, x, \lambda)$ in total $\{x; \lambda\}$, and the function (7) itself can be represented as some *modification of the Lagrange function* [Гольштейн, Третьяков, 1989; Жадан, 2015]

$$U(\tau, x, \lambda) = L(x, \lambda) - \sum_{j=1}^{n} R(\tau, x_j) + \sum_{i=1}^{m} R(\tau, \lambda_i), \tag{8}$$

where $L(x, \lambda)$ is a regular Lagrange function of problem (2) of the form

$$L(x, \lambda) = \sum_{j=1}^{n}\sigma_j x_j - \sum_{i=1}^{m}\lambda_i\left(-\beta_i + \sum_{j=1}^{n}\alpha_{ij}x_j\right) = F(x) - \sum_{i=1}^{m}\lambda_i f_i(x).$$

This form of writing the Lagrange function for problem (2) does not depend on whether the functions $F(x)$, $f_i(x)$ $\forall i = \overline{1, m}$ are linear or not. Therefore, equality (8) can be used as a *definition* of an auxiliary function $U(\tau, x, \lambda, \nu)$ for the nonlinear problem (1).

3. Smoothing property of feedback functions

Consider the applicability conditions for feedback functions to solve the *parametric* problem (1) with a constrained value F_ν^* and with maybe ambiguous dot x_ν^*. We will assume that all the conditions formulated below are satisfied: $\forall\nu \in \Upsilon$.

Suppose that in the problem under consideration the Lagrange function is regular. Also, let there be compact sets $\Omega_x \subset E^n$ and $\Omega_\lambda \subset E^m$ with non-empty interior, for which there is at least one pair of vectors $x_\nu^* \in \Omega_x$ and $\lambda_\nu^* \in \Omega_\lambda$, such that $L(x_\nu^*, \lambda_\nu^*, \nu) = F_\nu^*$.

Let the function *feedback* $Q(\tau, s)$ be defined $\forall\tau > 0$ and $\forall s \in (0, +\infty)$. By construction, it has the following properties:

3-1°. $Q(\tau, s)$ is strictly monotonically increasing in s and has for any fixed $\tau > 0$ $\lim\limits_{s\to+0} Q(\tau, s) = -\infty$, $\lim\limits_{s\to+\infty} Q(\tau, s) = +\infty$.

3-2°. $\forall s > 0$ and $\forall \tau > 0$: $\lim\limits_{\tau\to+0} Q(\tau, s) = \begin{cases} +\infty, & s \leqslant 0, \\ 0, & s > 0, \end{cases}$ and this limit transitions is *uniform* in s for $(-\infty, -\varepsilon_0]$ and $[\varepsilon_0, +\infty)$ $\forall \varepsilon_0 > 0$.

3-3°. In the domain of definition the function $Q(\tau, s)$ is continuously differentiable with respect to all its arguments.

As feedback functions, one can use, for example, $Q(\tau, s) = \tau \ln s$, $Q(\tau, s) = \frac{\tau}{2}\left(s - \frac{1}{s}\right)$.

Let's introduce $\forall s \in (0, +\infty)$ a function $R(\tau, s)$ such that $R(\tau, s) = \int\limits_{\alpha(\tau)}^{s} Q(\tau, u)\,du$, where $\alpha(\tau)$ is the solution to the equation $Q(\tau, \alpha(\tau)) = 0$ (under the assumptions made, $\alpha(\tau)$ exists and uniquely $\forall \tau > 0$). From the definition of $R(\tau, s)$ it also follows that $\frac{\partial R}{\partial s} = Q(\tau, s)$.

Taking formula (8) as the *definition*, we construct for the nonlinear problem (1), supplemented by the condition $x \in E^n_+$, the auxiliary function

$$U(\tau, x, \lambda, v) = L(x, \lambda, v) + W(\tau, x, \lambda), \quad \text{where} \quad W(\tau, x, \lambda) = -\sum_{j=1}^{n} R(\tau, x_j) + \sum_{i=1}^{m} R(\tau, \lambda_i). \tag{9}$$

Suppose that in problem (1) non-negativity conditions are not imposed on all components of the vector x, or among the restrictions there are equalities. Then into expression (9) for $W(\tau, x, \lambda)$ the corresponding terms are not included. For example, for problem (1) $W(\tau, x, \lambda) = \sum\limits_{i=1}^{m} R(\tau, \lambda_i)$.

We now describe the properties of the function $U(\tau, x, \lambda)$.

It is shown in [Umnov, Umnov, 2022] that under the above assumptions and a fixed vector of parameters the following statements hold.

Theorem 1. *$\forall \tau > 0$ function $U(\tau, x, \lambda, v)$ has $\{\overline{x}(\tau, v); \overline{\lambda}(\tau, v)\}$, a locally isolated saddle point inside $\Omega_x \times \Omega_\lambda$, where vectors $\overline{x}(\tau, v)$ and $\overline{\lambda}(\tau, v)$ are solutions of the system of equations*

$$\begin{cases} \operatorname*{grad}\limits_{x} U(\tau, \overline{x}, \overline{\lambda}, v) = o, \\ \operatorname*{grad}\limits_{\lambda} U(\tau, \overline{x}, \overline{\lambda}, v) = o. \end{cases} \tag{10}$$

For example, for problem (1), the system of equations (10) has the form

$$\begin{cases} \dfrac{\partial F}{\partial x_j} - \sum\limits_{i=1}^{m} \overline{\lambda}_i(\tau, v) \dfrac{\partial f_i}{\partial x_j} = 0 & \forall j = \overline{1, n}, \\ f_i(\overline{x}(\tau, v)) = -Q(\tau, \overline{\lambda}_i(\tau, v)) & \forall i = \overline{1, m}, \end{cases} \tag{11}$$

This form is similar in structure to the conditions of the Karush–Kuhn–Tucker theorem for problem (1), but does not contain explicit conditions for non-negativity of the Lagrange multipliers and conditions of complementary slackness.

Notice, that the vector functions $\overline{x}(\tau, v)$, $\overline{\lambda}(\tau, v)$ defined implicitly by system (10) describe in $\Omega_x \times \Omega_\lambda$ parametrically (by τ) a line which (by analogy with the extremal trajectory in the penalty function method) can be called the *saddle trajectory* of the problem (1).

Theorem 2. *On the saddle trajectory of the problem* (1)

$$\lim_{\tau\to+0} U(\tau,\, \overline{x}(\tau,\, v),\, \overline{\lambda}(\tau,\, v),\, v) = F_v^*, \tag{12}$$

In the case of local uniqueness solution of the problem (1) *the equalities are also valid*

$$\lim_{\tau\to+0} \overline{x}(\tau,\, v) = x_v^* \quad \textit{and} \quad \lim_{\tau\to+0} \overline{\lambda}(\tau,\, v) = \lambda_v^*. \tag{13}$$

Theorem 3. *On a saddle trajectory, the vector functions* $\{\overline{x}(\tau,\, v);\, \overline{\lambda}(\tau,\, v)\}$ *are continuously differentiable with respect to all its arguments* $\forall\tau > 0\ \forall v \in \Upsilon$.

We find from (10) for specific $\tau > 0$ and $v \in \Upsilon$ values of the vector functions $\overline{x}(\tau,\, v)$, $\overline{\lambda}(\tau,\, v)$. Then we apply to the system (10) the *implicit function theorem*, which gives $\forall r = \overline{1,\, K}$ system of linear equations

$$\begin{cases} \sum\limits_{j=1}^{n} \dfrac{\partial^2 U}{\partial x_p\, \partial x_j} \dfrac{\partial \overline{x}_j}{\partial v_r} + \sum\limits_{i=1}^{m} \dfrac{\partial^2 U}{\partial x_p\, \partial \lambda_i} \dfrac{\partial \overline{\lambda}_i}{\partial v_r} = -\dfrac{\partial^2 U}{\partial x_p\, \partial v_r} & \forall p = \overline{1,\, n}, \\ \sum\limits_{j=1}^{n} \dfrac{\partial^2 U}{\partial \lambda_q\, \partial x_j} \dfrac{\partial \overline{x}_j}{\partial v_r} + \sum\limits_{i=1}^{m} \dfrac{\partial^2 U}{\partial \lambda_q\, \partial \lambda_i} \dfrac{\partial \overline{\lambda}_i}{\partial v_r} = -\dfrac{\partial^2 U}{\partial \lambda_q\, \partial v_r} & \forall q = \overline{1,\, m}, \end{cases} \tag{14}$$

This system determines the values of the derivatives of the vector-function components $\overline{x}(\tau,\, v)$ and $\overline{\lambda}(\tau,\, v)$ over the components of the vector v.

If necessary, we strengthen the assumptions in an obvious way about the properties of feedback functions, as well as functions included in the condition of problem (1). As a result, we can find partial derivatives of a higher order in a similar way.

Theorems 1, 2 and 3 allow us to use the vector function $\overline{x}(\tau,\, v)$ as an *approximation* of the dependence x_v^*, having no properties preventing the use of computational procedures based on the Taylor formula.

It follows from Theorem 2 that the error of the feedback function method decreases as $\tau \to +0$. However, if for the value used τ it is unacceptably large, it is also possible to apply the implicit function theorem to reduce the approximation error.

Indeed, if the right-hand sides of the system (14) are replaced by $-\frac{\partial^2 U}{\partial x_p\, \partial \tau}$ and $-\frac{\partial^2 U}{\partial \lambda_i\, \partial \tau}$, then its solutions will be the values of the derivatives of $\{\overline{x}(\tau,\, v),\, \overline{\lambda}(\tau,\, v)\}$ by the τ parameter. These values of these derivatives for sufficiently small $\tau > 0$ improve the approximation accuracy, for example, according to the formulas

$$\widehat{x}_j = \overline{x}_j(\tau,\, v) - \tau \frac{\partial \overline{x}_j}{\partial \tau}\ \forall j = \overline{1,\, n} \quad \text{and} \quad \widehat{\lambda}_i = \overline{\lambda}_i(\tau,\, v) - \tau \frac{\partial \overline{\lambda}_i}{\partial \tau}\ \forall i = \overline{1,\, m}. \tag{15}$$

It is shown in [Umnov, Umnov, 2022] that although the point $\{\widehat{x},\, \widehat{\lambda}\}$ does not belong to the saddle trajectory, correction by formulas (15) can be performed iteratively in several steps. To do this, it suffices to replace the scalar parameter τ with the vector parameter by turning the saddle trajectory into a bundle of such trajectories.

We illustrate the application of the method of feedback functions by solving the following problem, with non-negative fixed p.

Problem 1.

$$\begin{aligned} &\textit{maximize } px \textit{ over } x \in \mathbb{R} \\ &\textit{subject to}\text{: } x \geqslant 0,\ x \leqslant 5,\ x \leqslant 5v \textit{ for fixed } v \in \mathbb{R}. \end{aligned} \tag{16}$$

Solutions to this problem in the case of $p > 0$ and, accordingly, $p = 0$ look like

$$x_v^* = \begin{cases} \textit{does not exist} & \text{for } -\infty < v < 0, \\ 5v & \text{for } 0 \leqslant v \leqslant 1, \\ 5 & \text{for } v > 1, \end{cases} \qquad x_v^* = \begin{cases} \textit{does not exist} & \text{for } -\infty < v < 0, \\ [0, 5v] & \text{for } 0 \leqslant v \leqslant 1, \\ [0, 5] & \text{for } v > 1. \end{cases}$$

In other words, for $v < 0$ x_v^* dependency is not defined, at the point $v = 1$ the derivative of x_v^* does not exist, and for $p = 0$ for $v > 0$ x_v^* dependency is ambiguous, and therefore is not a function.

Let us construct a smooth approximation x_v^* dependencies for 1 task, taking $Q(\tau, s) = \tau \ln s$ as a feedback function. The auxiliary function (9) in this case will be

$$U(\tau, x, \lambda_1, \lambda_2, \lambda_3, v) = px - \lambda_1(-x) - \lambda_2(-5 + x) - \lambda_3(-5v + x) + R(\tau, \lambda_1) + R(\tau, \lambda_1) + R(\tau, \lambda_1).$$

Conditions for its stationarity (system (10)) will be

$$\begin{cases} p + \overline{\lambda}_1 - \overline{\lambda}_2 - \overline{\lambda}_3 = 0, \\ \overline{x} + \tau \ln \overline{\lambda}_1 = 0, \\ -\overline{x} + 5 + \tau \ln \overline{\lambda}_2 = 0, \\ -\overline{x} + 5v + \tau \ln \overline{\lambda}_3 = 0. \end{cases}$$

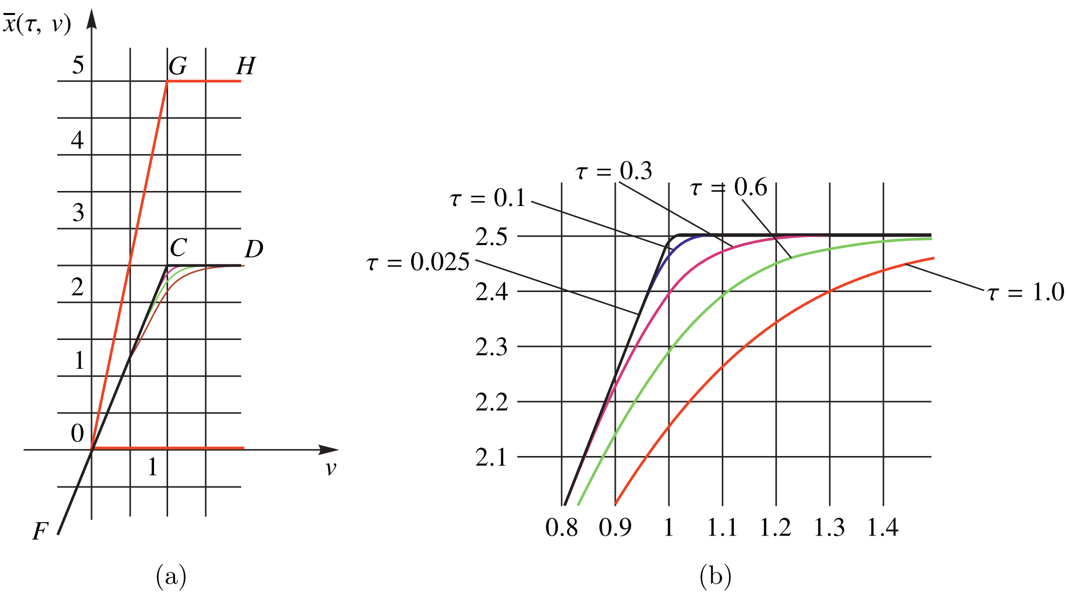

Figure 1. Graphical representation of functions $\overline{x}(\tau, v)$ for problem 1

Omitting the obvious transformations, we present the final form of the desired x_v^* dependency approximations:

$$\overline{x}(\tau, v) = -\tau \ln\left(\sqrt{\frac{p^2}{4} + e^{-\frac{5}{\tau}} + e^{-\frac{5v}{\tau}}} - \frac{p}{2}\right).$$

This function is defined $\forall \tau > 0$ and $\forall v \in \mathbb{R}$ and has a derivative of any order at each point. Disclosure of uncertainties in the case $p > 0$ gives $\lim\limits_{\tau \to +0} \overline{x}(\tau, v) = 5v$ at $0 \leqslant v \leqslant 1$ and $\lim\limits_{\tau \to +0} \overline{x}(\tau, v) = 5$ at $v > 1$.

Now we find the values of the limit $\lim\limits_{\tau\to+0} \overline{x}(\tau,\, v) = x_v^*$ for v for which x_v^* is defined ambiguously. With $p = 0$ and $0 < v < 1$ we have

$$\lim_{\tau\to+0} \overline{x}(\tau,\, v) = -\frac{1}{2}\lim_{\tau\to+0} \tau\ln\left(e^{-\frac{5}{\tau}} + e^{-\frac{5v}{\tau}}\right) = -\frac{1}{2}\lim_{\tau\to+0}\left(\tau\ln e^{-\frac{5v}{\tau}} + \tau\ln\left(1 + e^{\frac{5v-5}{\tau}}\right)\right) = \frac{5v}{2}.$$

For $v \geqslant 1$ the result is $\lim\limits_{\tau\to+0} \overline{x}(\tau,\, v) = \frac{5}{2}$ obtained similarly.

Next, we note that problem 1 is inconsistent $\forall v < 0$ due to inconsistency restrictions $x \geqslant 0$ and $x \leqslant 5v$. In this case we have $\lim\limits_{\tau\to+0} \overline{x}(\tau,\, v) = \frac{5v}{2}$. This value can be considered as some compromise for conflicting conditions.

Figure 1, *a* shows the graphs of the functions $\overline{x}(\tau,\, v)$ for $\tau = 1$, 0.6, 0.3, 0.1, 0.025. Figure 1, *b* shows on a larger scale the same graphs in the vicinity of the point $v = 1$, where the dependence x_v^* is non-differentiable.

4. Extremum of a function of several variables

Let us now consider the possibility of applying the method feedback functions in tasks which are reduced to problems of parametric programming.

Let us first show that approximations of solutions using feedback functions can be used to solve the problem of finding the maximum number in some finite set of numbers.

Let a set of numbers be given $v = \{v_1,\, v_2,\, \ldots,\, v_K\}$. The value of the maximum of them is the solution linear programming problems with scalar variable $f \in \mathbb{R}$ and with parameters $v_1,\, v_2,\, \ldots,\, v_K$:

$$\begin{aligned} &\textit{minimize } f\\ &\textit{subject to}\colon f \geqslant v_i \;\; \forall i = \overline{1,\, K}. \end{aligned} \tag{17}$$

Taking into account the problem statement format (1) and taking into account that f is not limited in sign, we build for this task auxiliary function (10)

$$U(\tau,\, f,\, \lambda,\, v) = -f - \sum_{i=1}^{K} \lambda_i(v_i - f) + \sum_{i=1}^{K} R(\tau,\, \lambda_i). \tag{18}$$

If we take as the feedback function $Q(\tau,\, s) = \tau\ln s$, then the stationarity conditions for (10) will look like

$$\begin{cases} -1 + \sum\limits_{i=1}^{K} \overline{\lambda}_i = 0,\\ v_i - \overline{f} = \tau\ln\overline{\lambda}_i \;\; \forall i = \overline{1,\, K} \end{cases} \quad\Longrightarrow\quad \begin{cases} \overline{\lambda}_i = e^{\frac{v_i - \overline{f}}{\tau}} \;\; \forall i = \overline{1,\, K},\\ \text{where } \overline{f} = \tau\ln\left(\sum\limits_{i=1}^{K} e^{\frac{v_i}{\tau}}\right). \end{cases} \tag{19}$$

Consequently, the solution of problem (17), that is, the value of the maximum (as well as the minimum) of the numbers in the set $\{v_1,\, v_2,\, \ldots,\, v_K\}$ will be equal to

$$f^*_{\max} = \lim_{\tau\to+0} \tau\ln\left(\sum_{i=1}^{K} e^{\frac{v_i}{\tau}}\right) \quad\Longrightarrow\quad f^*_{\min} = -\lim_{\tau\to+0} \tau\ln\left(\sum_{i=1}^{K} e^{-\frac{v_i}{\tau}}\right). \tag{20}$$

Let the passage to the limit in (20) not be performed, and let τ be a small positive number. Let us estimate the order of error of the obtained estimates. Without loss of generality, we can assume that the numbers in the set $\{v_1, v_2, \ldots, v_K\}$ sorted in descending order and the first M of them are equal to f^*, where $\{v_1, v_2, \ldots, v_K\}$. Then we have

$$\overline{f}(\tau) - f^* = \tau \ln\left(\sum_{i=1}^{K} e^{\frac{v_i}{\tau}}\right) - \tau \ln e^{\frac{f^*}{\tau}} = \tau \ln\left(\sum_{i=M+1}^{K} e^{\frac{v_i - f^*}{\tau}}\right) = \tau \ln\left(M + \sum_{i=1}^{K} e^{\frac{v_i - f^*}{\tau}}\right) \leqslant$$
$$\leqslant \tau \ln\left(M + (K-M)e^{\frac{A-f^*}{\tau}}\right) = \tau \ln M + \tau \ln\left(1 + \frac{K-M}{M} e^{\frac{A-f^*}{\tau}}\right) \leqslant \tau \ln M + \tau e^{\frac{A-f^*}{\tau}},$$

where $A = v_{M+1}$.

It follows from this estimate that the order of smallness of the error is determined by the term $\tau \ln M$ in the case when $M > 1$ (that is, in the set under study the maximum number is not unique). For $M = 1$, the order of error is determined by the term $\tau e^{\frac{A-f^*}{\tau}}$, which is much better at $\tau \to +0$.

The fact that the solution in the form (20) has been obtained for system (19) is, of course, an exception, not a rule. For example, if one uses the feedback function $Q(\tau, s) = \frac{\tau}{2}\left(s - \frac{1}{s}\right)$, then the system of equations (10) will have the form

$$\begin{cases} -1 + \sum_{i=1}^{K} \overline{\lambda}_i = 0, \\ v_i - \overline{f} = \dfrac{\tau}{2}\left(\overline{\lambda}_i - \dfrac{1}{\overline{\lambda}_i}\right) \ \forall i = \overline{1, K}, \end{cases} \tag{21}$$

for which only a numerical solution is possible.

For illustration Table 1 shows the results of solving system (19) for a set of numbers

$$\{v_1 = 5,\ v_2 = -2,\ v_3 = 4,\ v_4 = 7,\ v_5 = 0\}$$

with different values of the τ parameter.

Table 1. Solutions of system (19) for various values of the parameter τ

τ	$\overline{f}(\tau)$	$\overline{\lambda}_1(\tau)$	$\overline{\lambda}_2(\tau)$	$\overline{\lambda}_3(\tau)$	$\overline{\lambda}_4(\tau)$	$\overline{\lambda}_5(\tau)$
$10^{-0.00}$	7.170719212	0.114095529	$1.0404 \cdot 10^{-4}$	0.041973399	0.843058261	$7.6877 \cdot 10^{-4}$
$10^{-0.25}$	7.018454440	0.027615558	$1.0841 \cdot 10^{-7}$	$4.6651 \cdot 10^{-3}$	0.967715479	$3.7990 \cdot 10^{-6}$
$10^{-0.50}$	7.000590038	$1.7884 \cdot 10^{-3}$	$4.355 \cdot 10^{-13}$	$7.5703 \cdot 10^{-5}$	0.998135874	$2.430 \cdot 10^{-10}$
$10^{-0.75}$	7.000002329	$1.3048 \cdot 10^{-5}$	$4.355 \cdot 10^{-13}$	$4.7135 \cdot 10^{-8}$	0.999986904	0.000000000
$10^{-1.00}$	7.000000000	$2.0612 \cdot 10^{-9}$	0.000000000	$9.358 \cdot 10^{-14}$	0.999999998	0.000000000
$10^{-1.25}$	7.000000000	0.000000000	0.000000000	0.000000000	1.000000000	0.000000000

For comparison, Table 2 presents numerical solutions of system (21) for a set of numbers $\{v_1 = 5, v_2 = 5, v_3 = 4, v_4 = 5, v_5 = 0\}$ also with different values of the τ parameter.

Consider now the problem of finding extreme values for numerical sets of cardinality continuum.

Suppose we are given a function $f(x)$ continuous on a compact $\Omega \subset E^n$. Replacing the summation operation by integration in formula (20), we obtain an estimate for the value of global maximum of a function of several variables

$$f^*_{\max} = \lim_{\tau \to +0} \tau \ln \int_{\Omega} e^{\frac{f(x)}{\tau}}\, dx. \tag{22}$$

Table 2. Solutions of system (21) for various values of the parameter τ

τ	$\overline{f}(\tau)$	$\overline{\lambda}_1(\tau)$	$\overline{\lambda}_2(\tau)$	$\overline{\lambda}_3(\tau)$	$\overline{\lambda}_4(\tau)$	$\overline{\lambda}_5(\tau)$
$10^{-1.00}$	5.109862742	0.333328289	0.333328289	$1.5133 \cdot 10^{-5}$	0.333328289	0.000000000
$10^{-1.20}$	5.069317752	0.333333319	0.333333319	$4.3629 \cdot 10^{-8}$	0.333333319	0.000000000
$10^{-1.50}$	5.034741173	0.333333333	0.333333333	0.000000000	0.333333333	0.000000000
$10^{-2.00}$	5.010986124	0.333333333	0.333333333	0.000000000	0.333333333	0.000000000
$10^{-4.00}$	5.000109861	0.333333333	0.333333333	0.000000000	0.333333333	0.000000000
$10^{-7.00}$	5.000000110	0.333333333	0.333333333	0.000000000	0.333333333	0.000000000

The validity of formula (22) follows from conditions $f(x) \leqslant f^*$ and estimates:

$$\lim_{\tau\to+0} \tau \ln \int\limits_{\Omega} e^{\frac{f(x)}{\tau}}\, dx = \lim_{\tau\to+0} \tau \ln e^{\frac{f^*}{\tau}} \int\limits_{\Omega} e^{\frac{f(x)-f^*}{\tau}}\, dx = f^* + \lim_{\tau\to+0} \tau \ln \int\limits_{\Omega} e^{\frac{f(x)-f^*}{\tau}}\, dx,$$

$$0 \leqslant \lim_{\tau\to+0} \tau \ln \int\limits_{\Omega} e^{\frac{f(x)-f^*}{\tau}}\, dx \leqslant \lim_{\tau\to+0} \tau \ln \operatorname{mes} \Omega = 0.$$

Relationship of integration operations and extremum search was previously used to solve problems of different classes. For example, in the saddle-point method, described in [Федорюк, 1977], or when searching for a maximin in game problems [Федоров, 1979].

The following example illustrates the application of formula (22).

Problem 2. *Find global extremes by $x \in \Omega \subseteq E^2$ for the function $f(x) = |x_1| + |x_2|$, where* $\Omega = \begin{cases} -3 \leqslant x_1 \leqslant 4, \\ -2 \leqslant x_2 \leqslant 1. \end{cases}$

Solution. For the global maximum, we have:

$$f^*_{\max} = \lim_{\tau\to+0} \tau \ln \iint\limits_{\Omega} e^{\frac{|x_1|+|x_2|}{\tau}}\, dx_1\, dx_2 = \lim_{\tau\to+0} \tau \ln \left[\tau^2 \left(e^{\frac{4}{\tau}} + e^{\frac{3}{\tau}} - 2\right)\left(e^{\frac{2}{\tau}} + e^{\frac{1}{\tau}} - 2\right)\right] = 6.$$

It is achieved at the boundary of Ω region at $x_1 = 4$ and $x_2 = -2$.

The global minimum here is internal, at the point $x_1 = x_2 = 0$

$$f^*_{\min} = -\lim_{\tau\to+0} \tau \ln \iint\limits_{\Omega} e^{-\frac{|x_1|+|x_2|}{\tau}}\, dx_1\, dx_2 = -\lim_{\tau\to+0} \tau \ln \left[\tau^2 \left(e^{-\frac{4}{\tau}} + e^{-\frac{3}{\tau}} - 2\right)\left(e^{-\frac{2}{\tau}} + e^{-\frac{1}{\tau}} - 2\right)\right] = 0.$$

Note also that the optimal value of the objective function in the problem of mathematical programming

$$\begin{gathered} \textit{maximize } F(x),\ x \in E^n \\ \textit{subject to } x \in \Omega \subseteq E^n \end{gathered}$$

can be represented (under appropriate assumptions about the properties of $F(x)$ and Ω) as $F^* = \lim_{\tau\to+0} \tau \ln \int\limits_{\Omega} e^{\frac{F(x)}{\tau}}\, dx.$

5. Multiple extremum and minimax problems

Smoothed dependency estimates for solutions to parametric programming problems can also be useful for studying the properties of superpositions of extremum search operators. Note that a similar approach based on the use of external penalty functions was proposed in [Гермейер, 1969] and substantiated in [Федоров, 1979].

Consider the following minimax problem:

$$\begin{aligned}&\textit{minimize } F(x) \textit{ in } x\\&\textit{subject to } x\in\Omega,\ \textit{where } \Omega\subseteq E^n \textit{is compact and } F(x)=\max_{k\in[1,K]}\{f_k(x)\}.\end{aligned}\tag{23}$$

We also assume that the functions $f_k(x)$ $\forall k\in[1, K]$ are continuously differentiable on the set Ω.

In the case where the set Ω is given by a system of inequalities of the form $y_i(x)\leqslant 0$, $i=[1, m]$, problem (23) is equivalent to the problem of mathematical programming:

$$\begin{aligned}&\textit{maximize } -V \textit{ by } \{x, V\}\\&\textit{subject to } f_k(x)-V\leqslant 0\ \forall k=[1, K] \textit{ and } y_i(x)\leqslant 0\ \forall i=[1, m].\end{aligned}\tag{24}$$

Here we also assume that the functions $y_i(x)$ $\forall i\in[1, m]$ are continuously differentiable on the set Ω.

Dependence V_x^* under the above assumptions is continuous but non-differentiable throughout its domain of definition. To solve problem (24), we apply the method of feedback functions with an auxiliary function

$$U(\tau, x, \Lambda, V)=-V-\sum_{k=1}^{K}\lambda_k(f_k(x)-V)-\sum_{i=1}^{m}\mu_i y_i(x)+\sum_{k=1}^{K}R(\tau,\lambda_k)+\sum_{k=1}^{m}R(\tau,\mu_i),\tag{25}$$

where $\Lambda=\{\lambda_1,\ldots,\lambda_K,\mu_1,\ldots,\mu_m\}$.

Stationarity conditions for the function (25) can be written as a system of equations:

$$\begin{cases}\sum\limits_{k=1}^{K}\overline{\lambda}_k\cdot\operatorname*{grad}\limits_x f_k(\overline{x})+\sum\limits_{i=1}^{m}\overline{\mu}_i\cdot\operatorname*{grad}\limits_x y_i(\overline{x})=o,\\-f_k(\overline{x})+\overline{V}+Q(\tau,\overline{\lambda}_k)=0\quad\forall k=\overline{1, K},\\-y_i(\overline{x})+Q(\tau,\overline{\mu}_i)=0\quad\forall i=\overline{1, m},\\-1+\sum\limits_{k=1}^{K}\overline{\lambda}_k=0.\end{cases}\tag{26}$$

Let the problem of minimax search has no constraints, and the feedback function is defined as $Q(\tau, s)=\tau\ln s$. In this case, the stationarity conditions for the auxiliary function are simplified

$$\begin{cases}\sum\limits_{k=1}^{K}\overline{\lambda}_k\cdot\operatorname*{grad}\limits_x f_k(\overline{x})=o,\\\overline{\lambda}_k=e^{\frac{f_k(\overline{x})-\overline{V}}{\tau}}\quad\forall k=\overline{1, K},\\-1+\sum\limits_{k=1}^{K}\overline{\lambda}_k=0.\end{cases}\tag{27}$$

Note that the last two equalities give a smoothed approximation of the function maximum $\overline{V}(x)=$ $=\tau\ln\sum\limits_{k=1}^{K}e^{\frac{f_k(x)}{\tau}}$, whereas, the first two equalities in (27) are necessary conditions for stationarity for $\overline{V}(x)$ over x.

As illustrative examples, consider two problems.

Problem 3. *Find the minimum for* $V(x) = \max\left\{x^2;\ \sin 4x\right\}$.

Solution. This problem is reduced to the problem of mathematical programming

$$\begin{aligned}&\textit{maximize } -V \textit{ by } \{x,\ V\}\\ &\textit{subject to } x^2 - V \leqslant 0,\ \sin 4x - V \leqslant 0.\end{aligned}$$

Using (27) we find that the approximation of the maximum function has the form

$$\overline{V}(x) = \tau \ln\left(e^{\frac{x^2}{\tau}} + e^{\frac{\sin 4x}{\tau}}\right), \tag{28}$$

Its stationary points are the roots of the equation

$$\overline{x}e^{\frac{\overline{x}^2}{\tau}} + 2e^{\frac{\sin 4\overline{x}}{\tau}} \cos 4\overline{x} = 0. \tag{29}$$

Equation (29) has three roots, which are approximations to the points

$$x_1^* = 0, \quad x_2^* = \frac{\pi}{8} = 0.392699082 \quad \text{and} \quad x_3^* = 0.669283188,$$

The last one is the root of the equation $x^2 = \sin 4x$.

Solutions of equation (29) for different values of the τ parameter are given in Table 3, and the graphs of the function $\overline{V}(x)$ are shown in Fig. 2.

Table 3. Solutions of equation (29) for various values of the parameter τ

τ	$\overline{x}_1(\tau)$	$\overline{x}_2(\tau)$	$\overline{x}_3(\tau)$
0.50	−0.166596357	0.413732397	0.723794952
0.25	−0.124388280	0.398475953	0.704617207
0.10	−0.077449382	0.392845484	0.688202392
0.05	−0.046509302	0.392699415	0.679216477
Exact solution	0	0.392699082	0.669283188

As a second illustration, consider the minimax problem, arising in the process of searching using the method of dichotomy of the extremum of the function of n variables in a given direction to E^n.

Suppose we are given a continuous, unimodal function $f(x)$ (that is, having a single extremum on segment $[\alpha, \beta]$). It is required to find on this segment two points the values of $f(x)$ in which allow us to build a new segment minimum possible length containing the extremum point.

Let's denote the required points by x_1 and x_2. Then it follows from the properties of unimodality that these points are the solution to the following problem:

Problem 4. *Find minimum by* $\{x_1,\ x_2\}$ *for* $\max\{x_2 - \alpha,\ \beta - x_1\}$ *subject to* $x_1 \leqslant x_2$.

Solution. This minimax problem reduces to parametric programming problem of the form

$$\begin{aligned}&\textit{find maximum } -v \textit{ by } \{v,\ x_1,\ x_2\}\\ &\textit{subject to } x_2 - \alpha \leqslant v,\ \beta - x_1 \leqslant v,\ x_1 - x_2 \leqslant 0.\end{aligned}$$

The auxiliary function (10) for this problem look like

$$U(\tau,\ x,\ \lambda,\ v) = -v - \lambda_1(x_2 - \alpha - v) - \lambda_2(\beta - x_1 - v) - \lambda_3(x_1 - x_2) + R(\tau,\ \lambda_1) + R(\tau,\ \lambda_2) + R(\tau,\ \lambda_3).$$

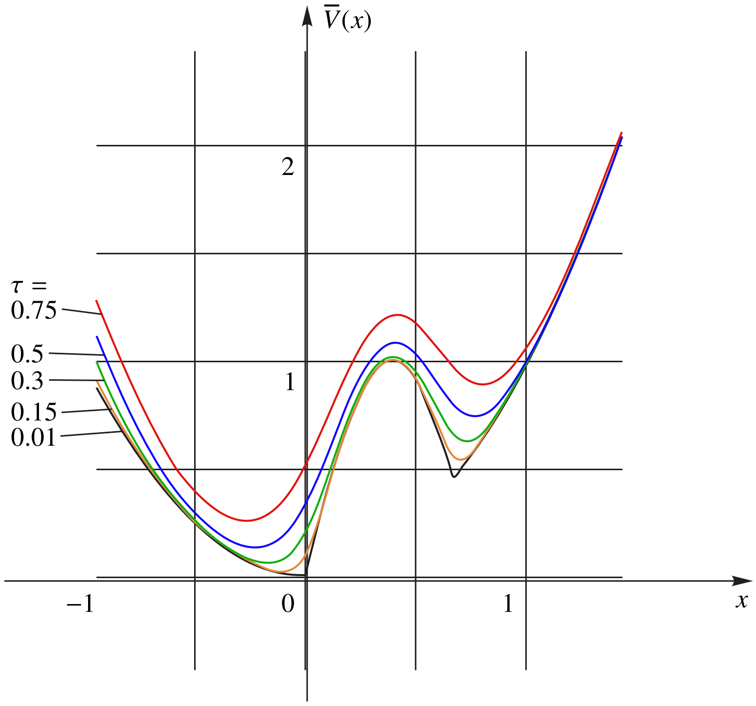

Figure 2. Graphical interpretation of problem 3

The stationarity conditions for v, x_1, x_2, λ_1, λ_1 and λ_3 with feedback function $Q(\tau, s) = \tau \ln s$ are

$$\begin{cases} -1 + \overline{\lambda}_1 + \overline{\lambda}_2 = 0, \\ \overline{\lambda}_2 - \overline{\lambda}_3 = 0, \\ -\overline{\lambda}_1 + \overline{\lambda}_3 = 0, \\ -\overline{x}_2 + \alpha + \overline{v} + \tau \ln(\overline{\lambda}_1) = 0, \\ -\beta + \overline{x}_1 + \overline{v} + \tau \ln(\overline{\lambda}_2) = 0, \\ -\overline{x}_1 + \overline{x}_2 + \tau \ln(\overline{\lambda}_3) = 0. \end{cases} \tag{30}$$

From (30) we get $\overline{v} = \tau \ln\left(e^{\frac{\overline{x}_2-\alpha}{\tau}} + e^{\frac{-\overline{x}_1+\beta}{\tau}}\right)$ and $\overline{x}_2 - \overline{x}_1 = \tau \ln 2 > 0$, which gives for v_{x_1} the approximation $\overline{v}(\tau, x_1) = \tau \ln\left(2e^{\frac{x_1-\alpha}{\tau}} + e^{\frac{-x_1+\beta}{\tau}}\right)$. The plot of this approximation is shown for the parameter values $\alpha = 1$, $\beta = 5$ and $\tau = 0.5, 0.2, 0.01$ in Fig. 3, *a*.

To take into account the limitation $x_1 \leqslant x_2$ we can use any sufficiently smooth penalty function. This allows us to obtain an approximation of two-dimensional dependence v_{x_1,x_2}. The type of isoline system for such an approximation with $\alpha = -5$, $\beta = 11$ and $\tau = 0.05$ is shown in Fig. 3, *b*.

6. Using feedback functions in problems for multiobjective models

From the above, it follows that a function approximating x_v^* dependency can be explicitly found only in exceptional cases. However, often it turns out that it is sufficient to construct only its Taylor polynomial. That is, it is enough to be able to calculate for $\forall v \in \Upsilon$ both values of the approximating function $\overline{x}(\tau, v)$ and its partial derivatives up to some order.

Let us show that the method of feedback functions is applicable in solving parametric optimization problems in multicriteria mathematical models.

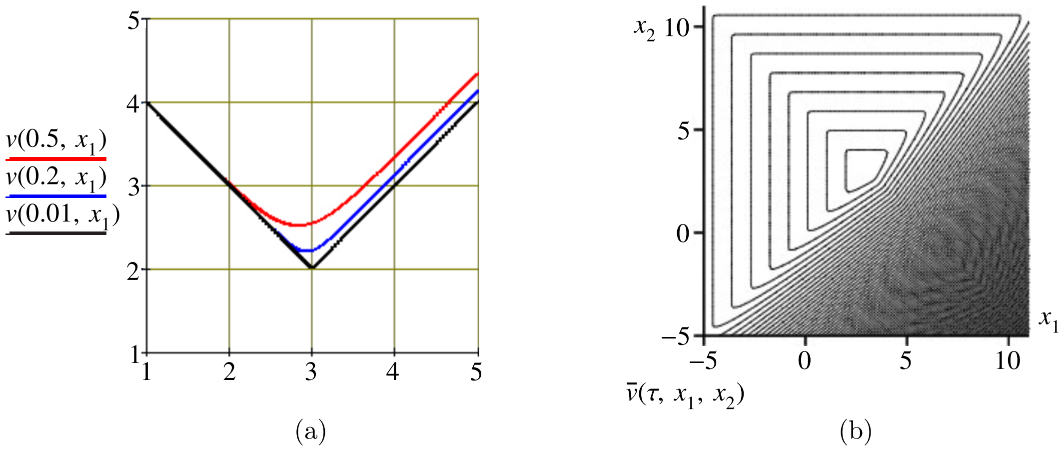

Figure 3. Graphical representation of smoothed approximations for the problem 4

As is well known, in mathematical modeling, in the formation of quality criteria for the states of the object being modeled, several objective functions independent of each other may arise. Let's assume that there are also restrictions on the arguments of these objective functions.

In this article the term *multiobjective model* means the set of functions to be maximized with respect to $x \in E^n$, also depending on the parameter vector $v \in \Upsilon \subseteq E^K$

$$F_k(x, v) \quad \forall k = \overline{1, N}, \tag{31}$$

subject to

$$f_i(x, v) \leqslant 0 \quad \forall i = \overline{1, m}. \tag{32}$$

We will also assume that all functions $F_k(x, v)$ and $f_i(x, v)$ are twice continuously differentiable.

Naturally, various optimization problems may arise, when multicriteria models are used in the process of making managerial decisions.

Indeed, the simultaneous achievement by all objective functions (31) of extrema at some point of the set defined by the system of inequalities (32), is generally impossible. Therefore, problems whose solutions are, in a way, a compromise for a set of criteria (31) are useful.

As a basis for constructing a compromise one can use solutions of single-criteria problems of the form

$$\begin{aligned} &\textit{maximize } F_k(x, v) \textit{ in } x \in E^n \\ &\textit{subject to } f_i(x, v) \leqslant 0 \ \forall i = \overline{1, m}. \end{aligned} \tag{33}$$

The solution to each of these problems $\left(\text{vector } x^*_{(k)v} \text{ and number } F^*_{(k)v} = F_k\left(x^*_{(k)v}, v\right)\right)$ $\forall k = \overline{1, N}$ is the best state of the model (31)–(32) from the point of view of the kth criterion. Consider the case where the problem of determining the compromise state is to find a valid point x^{**}_v which is the extremum of some value of the "compromise quality".

This value can be *minimum* by x, satisfying the constraints (32), *maximum* from differences $F^*_{(k)v} - F_k(x^{**}_v, v)$ $\forall k = \overline{1, N}$, that is, the solution to the problem

$$\begin{aligned} &\textit{maximize } -\rho \textit{ in } \{x \in E^n;\ \rho \geqslant 0\} \\ &\textit{subject to } f_i(x, v) \leqslant 0 \ \forall i = \overline{1, m},\ F^*_{(k)v} - F_k(x, v) - \rho \leqslant 0 \ \forall k = \overline{1, N}. \end{aligned} \tag{34}$$

We denote the solutions of the parametric programming problem (34) as ρ_v^{**} and x_v^{**}. Note that the dependence value ρ_v^{**} can be interpreted as a measure *inconsistency* system of objective functions in the multicriteria model (31)–(32).

For brevity, in what follows we will call problems (33) problems *of the first level*, and problem (34), that of searching for a compromise state multicriteria model x_v^{**}, will be called the problem *of the second level*.

The well-known methods [Лотов, Поспелова, 2008] in the practice of mathematical modeling are: *objective function convolution methods* (31), *methods of searching for Pareto equilibrium*, and *ideal point* methods.

Note that the minimax value of the quantity ρ, as a solution to problem (34), is some dependence on the parameter vector v. This dependence has the above-mentioned features that complicate the solution of optimization problems, in which it is included.

Finally, it also seems natural to formulate, for the model (31)–(32), an optimization problem *of the third level*, for example, the following:

$$\textit{find extremum } \rho_v^{**} \textit{ in } v \in \Upsilon \subseteq E^K. \tag{35}$$

The solution to this problem is the vector $v^{***} \in \Upsilon$ and number $\rho^{***} = \rho_{v^{***}}^{**}$, that is, the minimax extremum in problem (34).

Implementation of the algorithm for solving problems of optimization problems for multicriteria models

One of the reasons for the computational difficulties in solving problem (35) is the use of ρ_v^{**} dependencies in its statement. To overcome them we apply the method of feedback functions, which allows us to build the function $\overline{\rho}(\tau, v)$, a smooth approximation of dependence ρ_v^{**}.

Suppose that a method of the second order is used to find a local solution of problem (35). Then it is enough have values $\overline{\rho}(\tau, v)$ and all its partial derivatives up to and including the second order.

According to Section 3, the values $\overline{\rho}(\tau, v)$ are found from the stationarity conditions for the auxiliary function (10). The values of derivatives can be calculated using the implicit function theorem applied to these stationarity conditions.

Let's consider this approach in more detail.

We first note that the dependency ρ_v^{**} is a solution to problem (34), whose statement contains non-smooth dependencies $F_{(k)v}^{*}$ $\forall k = \overline{1, N}$ — solutions of problems (33). Therefore, we will use the smoothing property of the method of feedback functions not only when solving problems of the second level (34), but also for problems (33).

The auxiliary function for the kth problem of the first level (33) looks like

$$U_k(\tau, x, \Lambda, v) = F_k(x, v) - \sum_{i=1}^{m} \lambda_i f_i(x, v) + \sum_{i=1}^{m} R(\tau, \lambda_i). \tag{36}$$

Stationarity conditions for the function (36) have the form

$$\begin{cases} \dfrac{\partial U_k}{\partial x_p} = 0 \quad \forall p = \overline{1, n}, \\ \dfrac{\partial U_k}{\partial \lambda_i} = 0 \quad \forall i = \overline{1, m} \end{cases} \quad \text{or} \quad \begin{cases} \operatorname*{grad}_x F_k\left(\overline{x}_{(k)}, v\right) - \sum\limits_{i=1}^{m} \overline{\lambda}_{(k)i} \operatorname*{grad}_x f_i\left(\overline{x}_{(k)}, v\right) = o, \\ -f_i\left(\overline{x}_{(k)}, v\right) + Q\left(\tau, \overline{\lambda}_{(k)i}\right) = 0 \quad \forall i = \overline{1, m}. \end{cases} \tag{37}$$

The solution of system (37) allows us to construct the function

$$\overline{U}_{(k)}(\tau,\ v) = U_{(k)}\left(\tau,\ \overline{x}_{(k)}(\tau,\ v),\ \overline{\lambda}_{(k)}(\tau,\ v),\ v\right). \tag{38}$$

This smooth function can be used as an approximation for dependencies $F^*_{(k)v}$ in the condition if the problem of the second level (34).

Applying the theorem on a system of functions given implicitly to the stationarity conditions for the auxiliary function (36) gives a system of linear equations similar to system (**??**) for derivatives of the functions $\overline{x}(\tau,\ v)$ and $\overline{\lambda}(\tau,\ v)$ by v_t $\forall t = \overline{1,\ k}$

$$\begin{cases} \sum\limits_{q=1}^{n} \dfrac{\partial^2 U_k}{\partial x_p\, \partial x_q} \dfrac{\partial \overline{x}_q}{\partial v_t} + \sum\limits_{q=1}^{m} \dfrac{\partial^2 U_k}{\partial x_p\, \partial \lambda_q} \dfrac{\partial \overline{\lambda}_q}{\partial v_t} = -\dfrac{\partial^2 U_k}{\partial x_p\, \partial v_t} & \forall p = \overline{1,\ n}, \\ \sum\limits_{q=1}^{n} \dfrac{\partial^2 U_k}{\partial \lambda_i\, \partial x_q} \dfrac{\partial \overline{x}_q}{\partial v_t} + \sum\limits_{q=1}^{m} \dfrac{\partial^2 U_k}{\partial \lambda_i\, \partial \lambda_q} \dfrac{\partial \overline{\lambda}_q}{\partial v_t} = -\dfrac{\partial^2 U_k}{\partial \lambda_i\, \partial v_t} & \forall i = \overline{1,\ m}. \end{cases} \tag{39}$$

The main matrix of the system of linear equations (38) is the Hessian matrix for the auxiliary function (36), whose elements, as well as the components of the column of the right-hand sides, are calculated at a stationary point $\left\{\overline{x}_{(k)}(\tau,\ v),\ \overline{\lambda}_{(k)}(\tau,\ v)\right\}$.

We now apply the method of feedback functions to solve the problem *of the second level* (34).

Let us write the condition of this problem in a form that is more convenient for constructing the auxiliary function by introducing the functions $Y_k(\rho,\ x,\ v) = \overline{U}_{(k)}(\tau,\ v) - F_k(x,\ v) - \rho\ \forall k = \overline{1,\ N}$. Then the condition of the problem of the second level takes the form

$$\begin{aligned} &\textit{maximize}\ \ -\rho\ \textit{in}\ \{x \in E^n;\ \rho \geqslant 0\} \\ &\textit{subject to}\ f_i(x,\ v) \leqslant 0\ \forall i = \overline{1,\ m},\ Y_k(\rho,\ x,\ v) \leqslant 0\ \forall k = \overline{1,\ N}. \end{aligned}$$

The auxiliary function (10) for this problem of the second level looks like

$$U(\tau,\ \rho,\ x,\ \lambda,\ \mu,\ v) = -\rho - \sum_{i=1}^{m} \lambda_i f_i(x,\ v) - \sum_{k=1}^{N} \mu_k Y_k(\rho,\ x,\ v) - R(\tau,\ \rho) + \sum_{i=1}^{m} R(\tau,\ \lambda_i) + \sum_{k=1}^{N} R(\tau,\ \mu_k). \tag{40}$$

The stationarity conditions for this function have the form

$$\begin{cases} \dfrac{\partial U}{\partial \rho} = 0, \\ \dfrac{\partial U}{\partial x_j} = 0 & \forall j = \overline{1,\ n}, \\ \dfrac{\partial U}{\partial \lambda_i} = 0 & \forall i = \overline{1,\ m}, \\ \dfrac{\partial U}{\partial \mu_k} = 0 & \forall k = \overline{1,\ N} \end{cases} \quad \text{or} \quad \begin{cases} -1 + \sum\limits_{k=1}^{N} \overline{\overline{\mu}}_k - Q(\tau,\ \overline{\overline{\rho}}) = 0, \\ -\sum\limits_{i=1}^{m} \overline{\overline{\lambda}}_i \dfrac{\partial f_i}{\partial x_j}(\overline{\overline{x}},\ v) - \sum\limits_{k=1}^{N} \overline{\overline{\mu}}_k \dfrac{\partial Y_k}{\partial x_j}(\overline{\overline{\rho}},\ \overline{\overline{x}},\ v) = 0 & \forall j = \overline{1,\ n}, \\ -f_i(\overline{\overline{x}},\ v) + Q(\tau,\ \overline{\overline{\lambda}}_i) = 0 & \forall i = \overline{1,\ m}, \\ -Y_k(\overline{\overline{\rho}},\ \overline{\overline{x}},\ v) + Q(\tau,\ \overline{\overline{\mu}}_k) = 0 & \forall k = \overline{1,\ N}, \end{cases} \tag{41}$$

where $\{\overline{\overline{\rho}},\ \overline{\overline{x}},\ \overline{\overline{\lambda}},\ \overline{\overline{\mu}}\}$ is a stationary point of the auxiliary function (40).

Applying the implicit function theorem to the stationarity conditions for the auxiliary function (40) gives a system of linear equations similar to system (14):

$$\begin{cases} \dfrac{\partial^2 U}{\partial \rho^2}\dfrac{\partial \overline{\overline{\rho}}}{\partial v_t} + \sum\limits_{j=1}^{n} \dfrac{\partial^2 U}{\partial \rho\, \partial x_j}\dfrac{\partial \overline{\overline{x}}_j}{\partial v_t} + \sum\limits_{q=1}^{m} \dfrac{\partial^2 U}{\partial \rho\, \partial \lambda_q}\dfrac{\partial \overline{\overline{\lambda}}_q}{\partial v_t} + \sum\limits_{k=1}^{N} \dfrac{\partial^2 U}{\partial \rho\, \partial \mu_k}\dfrac{\partial \overline{\overline{\mu}}_k}{\partial v_t} = -\dfrac{\partial^2 U}{\partial \rho\, \partial v_t}, \\ \dfrac{\partial^2 U}{\partial x_p\, \partial \rho}\dfrac{\partial \overline{\overline{\rho}}}{\partial v_t} + \sum\limits_{j=1}^{n} \dfrac{\partial^2 U}{\partial x_p\, \partial x_j}\dfrac{\partial \overline{\overline{x}}_j}{\partial v_t} + \sum\limits_{q=1}^{m} \dfrac{\partial^2 U}{\partial x_p\, \partial \lambda_q}\dfrac{\partial \overline{\overline{\lambda}}_q}{\partial v_t} + \sum\limits_{k=1}^{N} \dfrac{\partial^2 U}{\partial x_p\, \partial \mu_k}\dfrac{\partial \overline{\overline{\mu}}_k}{\partial v_t} = -\dfrac{\partial^2 U}{\partial x_p\, \partial v_t} \quad \forall p = \overline{1, n}, \\ \dfrac{\partial^2 U}{\partial \lambda_i\, \partial \rho}\dfrac{\partial \overline{\overline{\rho}}}{\partial v_t} + \sum\limits_{j=1}^{n} \dfrac{\partial^2 U}{\partial \lambda_i\, \partial x_j}\dfrac{\partial \overline{\overline{x}}_j}{\partial v_t} + \sum\limits_{q=1}^{m} \dfrac{\partial^2 U}{\partial \lambda_i\, \partial \lambda_q}\dfrac{\partial \overline{\overline{\lambda}}_q}{\partial v_t} + \sum\limits_{k=1}^{N} \dfrac{\partial^2 U}{\partial \lambda_i\, \partial \mu_k}\dfrac{\partial \overline{\overline{\mu}}_k}{\partial v_t} = -\dfrac{\partial^2 U}{\partial \lambda_i\, \partial v_t} \quad \forall i = \overline{1, m}, \\ \dfrac{\partial^2 U}{\partial \mu_s\, \partial \rho}\dfrac{\partial \overline{\overline{\rho}}}{\partial v_t} + \sum\limits_{j=1}^{n} \dfrac{\partial^2 U}{\partial \mu_s\, \partial x_j}\dfrac{\partial \overline{\overline{x}}_j}{\partial v_t} + \sum\limits_{q=1}^{m} \dfrac{\partial^2 U}{\partial \mu_s\, \partial \lambda_q}\dfrac{\partial \overline{\overline{\lambda}}_q}{\partial v_t} + \sum\limits_{k=1}^{N} \dfrac{\partial^2 U}{\partial \mu_s\, \partial \mu_k}\dfrac{\partial \overline{\overline{\mu}}_k}{\partial v_t} = -\dfrac{\partial^2 U}{\partial \mu_s\, \partial v_t} \quad \forall s = \overline{1, N}. \end{cases} \tag{42}$$

Its solutions are the partial derivatives of the functions $\{\overline{\overline{\rho}}, \overline{\overline{x}}, \overline{\overline{\lambda}}, \overline{\overline{\mu}}\}$ with respect to the parameter v_t $\forall t = \overline{1, k}$.

Here we note that, the implicit function theorem is also applicable to calculate the derivatives of $\{\overline{\overline{\rho}}, \overline{\overline{x}}, \overline{\overline{\lambda}}, \overline{\overline{\mu}}\}$ with respect to the τ parameter. The values of these derivatives can be used in extrapolation procedures (see, for example, [Umnov, Umnov, 2022]) to reduce the error of the smoothed approximation by the formulas

$$\widetilde{x}_j = \overline{x}_j(\tau, v) - \tau\frac{\partial \overline{x}_j}{\partial \tau} \ \forall j = \overline{1, n} \quad \text{and} \quad \widetilde{\lambda}_i = \overline{\lambda}_i(\tau, v) - \tau\frac{\partial \overline{\lambda}_i}{\partial \tau} \ \forall i = \overline{1, m}.$$

Let us now consider the *problem of the third level*. As a smooth approximation of the dependence ρ_v^{**} choose a function

$$\overline{\overline{U}}(\tau, v) = U(\tau, \overline{\overline{\rho}}(\tau, v), \overline{\overline{x}}(\tau, v), \overline{\overline{\lambda}}(\tau, v), \overline{\overline{\mu}}(\tau, v), v). \tag{43}$$

According to the rule of differentiation of a complex function, the partial derivative of $\overline{\overline{U}}$ with respect to v_t has the form

$$\frac{\partial \overline{\overline{U}}}{\partial v_t} = \frac{\partial U}{\partial v_t} + \frac{\partial U}{\partial \rho}\frac{\partial \overline{\overline{\rho}}}{\partial v_t} + \sum_{j=1}^{n}\frac{\partial U}{\partial x_j}\frac{\partial \overline{\overline{x}}_j}{\partial v_t} + \sum_{i=1}^{m}\frac{\partial U}{\partial \lambda_i}\frac{\partial \overline{\overline{\lambda}}_i}{\partial v_t} + \sum_{j=1}^{n}\frac{\partial U}{\partial \mu_k}\frac{\partial \overline{\overline{\mu}}_k}{\partial v_t}.$$

All partial derivatives of the function $\overline{\overline{U}}$ in this formula are calculated at the point $\{\tau, \overline{\overline{\rho}}, \overline{\overline{x}}, \overline{\overline{\lambda}}, \overline{\overline{\mu}}, v\}$. Therefore, by virtue of equalities (41), the expression for the derivative simplifies to

$$\frac{\partial \overline{\overline{U}}}{\partial v_t} = \frac{\partial U}{\partial v_t}(\tau, \overline{\overline{\rho}}(\tau, v), \overline{\overline{x}}(\tau, v), \overline{\overline{\lambda}}(\tau, v), \overline{\overline{\mu}}(\tau, v), v) \quad \forall t = \overline{1, K}. \tag{44}$$

Thus, when solving a problem of the third level by any first-order method, it is sufficient to be able to find solutions to systems (37) and (41) $\forall v \in \Upsilon$. The final error is determined by the value of the τ parameter.

In the case of using second-order methods, we will also need the values of the second derivatives of the approximating function (43). Direct differentiation of formula (44) in view of (41) gives

$$\frac{\partial^2 \overline{\overline{U}}}{\partial v_s\, \partial v_t} = \frac{\partial^2 U}{\partial v_s\, \partial v_t} + \frac{\partial^2 U}{\partial v_s\, \partial \rho}\frac{\partial \overline{\overline{\rho}}}{\partial v_t} + \sum_{j=1}^{n}\frac{\partial^2 U}{\partial v_s\, \partial x_j}\frac{\partial \overline{\overline{x}}_j}{\partial v_t} + \sum_{i=1}^{m}\frac{\partial^2 U}{\partial v_s\, \partial \lambda_i}\frac{\partial \overline{\overline{\lambda}}_i}{\partial v_t} + \sum_{j=1}^{n}\frac{\partial^2 U}{\partial v_s\, \partial \mu_k}\frac{\partial \overline{\overline{\mu}}_k}{\partial v_t} \quad \forall s, t = \overline{1, K}. \tag{45}$$

Whence it follows that in this case the linear system (42) needs to be solved as well.

Illustrative example

The use of feedback functions to solve a three-level optimization parametric problem is illustrated by the following example.

Although this example is relatively simple, it demonstrates quite well the features of parametric optimization problems for multicriteria mathematical models.

Suppose that the objective functions of the model that are to be maximized with respect to $x \in E^3$ with $v \in E^2$ look like

$$F_1(x, v) = x_1, \quad F_2(x, v) = x_2, \quad F_3(x, v) = x_3.$$

The set of feasible states of the model is given by the system of inequalities

$$x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0, \quad \frac{x_1}{v_1} + \frac{x_2}{v_2} + \frac{x_3}{A - v_1 - v_2} \leqslant 1.$$

Finally, the set Υ is defined as

$$1 \leqslant v_1 \leqslant B, \quad 1 \leqslant v_2 \leqslant B, \quad 2v_1 + v_2 \geqslant C,$$

where $A = 11$, $B = 5$ and $C = 7$.

For the considered model, the number of objective functions is equal to $K = 3$. The problems of the first level look like

$$\forall k = 1, 2, 3 \ \textit{maximize} \ F_{(k)} = x_{(k)k} \ \textit{in} \ x \in E^3$$
$$\textit{subject to} \ x_{(k)1} \geqslant 0, \ x_{(k)2} \geqslant 0, \ x_{(k)3} \geqslant 0, \ \frac{x_{(k)1}}{v_1} + \frac{x_{(k)2}}{v_2} + \frac{x_{(k)3}}{A - v_1 - v_2} \leqslant 1.$$

To solve these problems, we choose the feedback function $Q(\tau, s) = \frac{\tau}{2}\left(s - \frac{1}{s}\right)$ and, by virtue of condition (9), $R(\tau, s) = \frac{\tau}{2}\left(\frac{s^2}{2} - \ln s - \frac{1}{2}\right)$. Then the auxiliary function (36) for these problems can be written as

$$U_{(k)}(x, v) = x_{(k)k} - \lambda_{(k)}\left(\frac{x_{(k)1}}{v_1} + \frac{x_{(k)2}}{v_2} + \frac{x_{(k)3}}{A - v_1 - v_2} - 1\right) - \\ - R\left(\tau, x_{(k)1}\right) - R\left(\tau, x_{(k)2}\right) - R\left(\tau, x_{(k)3}\right) + R\left(\tau, \lambda_{(k)}\right). \quad (46)$$

The conditions for its stationarity are

$$\begin{cases} \delta_{k1} - \dfrac{\overline{\lambda}_{(k)}}{v_1} - Q\left(\tau, \overline{x}_{(k)1}\right) = 0, \\ \delta_{k2} - \dfrac{\overline{\lambda}_{(k)}}{v_2} - Q\left(\tau, \overline{x}_{(k)2}\right) = 0, \\ \delta_{k3} - \dfrac{\overline{\lambda}_{(k)}}{A - v_1 - v_2} - Q\left(\tau, \overline{x}_{(k)3}\right) = 0, \\ \dfrac{\overline{x}_{(k)1}}{v_1} + \dfrac{\overline{x}_{(k)2}}{v_2} + \dfrac{\overline{x}_{(k)3}}{A - v_1 - v_2} - 1 - Q\left(\tau, \overline{\lambda}_{(k)}\right) = 0. \end{cases} \quad (47)$$

In turn, the solutions of system (47) allow us to find the values of the functions (38), which in the example under consideration are determined by the formulas following from (46) and (47)

$$\overline{U}_{(k)}(\tau, v) = x_{(k)k}(\tau, v) - \\ - R\left(\tau, \overline{x}_{(k)1}(\tau, v)\right) - R\left(\tau, \overline{x}_{(k)2}(\tau, v)\right) - R\left(\tau, \overline{x}_{(k)3}(\tau, v)\right) + R\left(\tau, \overline{\lambda}_{(k)}\right) - \overline{\lambda}_{(k)} Q\left(\tau, \overline{\lambda}_{(k)}\right). \quad (48)$$

The components of their gradients over v are:

$$\frac{\partial \overline{U}_{(k)}}{\partial v_1} = \frac{\overline{\lambda}_{(k)}\overline{x}_{(k)1}}{v_1^2} - \frac{\overline{\lambda}_{(k)}\overline{x}_{(k)3}}{(A - v_1 - v_2)^2}, \quad \frac{\partial \overline{U}_{(k)}}{\partial v_2} = \frac{\overline{\lambda}_{(k)}\overline{x}_{(k)2}}{v_2^2} - \frac{\overline{\lambda}_{(k)}\overline{x}_{(k)3}}{(A - v_1 - v_2)^2} \quad \forall k = \overline{1, K}. \tag{49}$$

Let us now apply the method of feedback functions to solve the problem of the second level.

In the formulation of this problem, instead of dependencies $F^*_{(k)v}$ we use their smoothed approximations $\overline{U}_{(k)}(\tau, v_1, v_2)$ $\forall k = 1, 2, 3$. Then the statement of the problem of the second level for the model under consideration takes the form

$$\begin{aligned} & \textit{maximize } -\rho \\ & \textit{subject to } \rho \geqslant 0,\ x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_3 \geqslant 0, \\ & \qquad \frac{x_1}{v_1} + \frac{x_2}{v_2} + \frac{x_3}{A - v_1 - v_2} \leqslant 1, \\ & \qquad \overline{U}_{(1)}(\tau, v_1, v_2) - x_1 - \rho \leqslant 0, \\ & \qquad \overline{U}_{(2)}(\tau, v_1, v_2) - x_2 - \rho \leqslant 0, \\ & \qquad \overline{U}_{(3)}(\tau, v_1, v_2) - x_3 - \rho \leqslant 0. \end{aligned} \tag{50}$$

The auxiliary function for the second level problem is

$$\begin{aligned} U(\tau, \rho, x, \lambda, \mu) = -\rho - \lambda\left(\frac{x_1}{v_1} + \frac{x_2}{v_2} + \frac{x_3}{A - v_1 - v_2} - 1\right) - \\ - \mu_1\left(\overline{U}_{(1)}(\tau, v_1, v_2) - x_1 - \rho\right) - \mu_2\left(\overline{U}_{(2)}(\tau, v_1, v_2) - x_2 - \rho\right) - \mu_3\left(\overline{U}_{(3)}(\tau, v_1, v_2) - x_3 - \rho\right) - \\ - R(\tau, \rho) - R(\tau, x_1) - R(\tau, x_2) - R(\tau, x_3) + R(\tau, \lambda) + R(\tau, \mu_1) + R(\tau, \mu_2) + R(\tau, \mu_3), \end{aligned}$$

where λ and $\mu = \{\mu_1, \mu_2, \mu_3\}$ are the Lagrange multipliers of the last four restrictions in (50).

Whence the conditions that determine its stationary points are

$$\begin{cases} \dfrac{\partial U}{\partial \rho} = 0, \\ \dfrac{\partial U}{\partial x_1} = 0, \\ \dfrac{\partial U}{\partial x_2} = 0, \\ \dfrac{\partial U}{\partial x_3} = 0, \\ \dfrac{\partial U}{\partial \lambda} = 0, \\ \dfrac{\partial U}{\partial \mu_1} = 0, \\ \dfrac{\partial U}{\partial \mu_2} = 0, \\ \dfrac{\partial U}{\partial \mu_3} = 0 \end{cases} \quad \text{or} \quad \begin{cases} -1 + \overline{\overline{\mu}}_1 + \overline{\overline{\mu}}_2 + \overline{\overline{\mu}}_3 - Q(\tau, \overline{\overline{\rho}}) = 0, \\ -\dfrac{\overline{\overline{\lambda}}}{v_1} + \overline{\overline{\mu}}_1 - Q(\tau, \overline{\overline{x}}_1) = 0, \\ -\dfrac{\overline{\overline{\lambda}}}{v_2} + \overline{\overline{\mu}}_2 - Q(\tau, \overline{\overline{x}}_2) = 0, \\ -\dfrac{\overline{\overline{\lambda}}}{A - v_1 - v_2} + \overline{\overline{\mu}}_3 - Q(\tau, \overline{\overline{x}}_3) = 0, \\ \dfrac{\overline{\overline{x}}_1}{v_1} + \dfrac{\overline{\overline{x}}_2}{v_2} + \dfrac{\overline{\overline{x}}_3}{A - v_1 - v_2} - 1 - Q(\tau, \overline{\overline{\lambda}}) = 0, \\ \overline{U}_{(1)}(\tau, v_1, v_2) - \overline{\overline{x}}_1 - \overline{\overline{\rho}} - Q(\tau, \overline{\overline{\mu}}_1) = 0, \\ \overline{U}_{(2)}(\tau, v_1, v_2) - \overline{\overline{x}}_2 - \overline{\overline{\rho}} - Q(\tau, \overline{\overline{\mu}}_2) = 0, \\ \overline{U}_{(3)}(\tau, v_1, v_2) - \overline{\overline{x}}_3 - \overline{\overline{\rho}} - Q(\tau, \overline{\overline{\mu}}_3) = 0. \end{cases} \tag{51}$$

Having obtained solutions of system (51), we can construct a smoothing approximation for the dependence ρ_v^{**}, which is used in the statement for the problem of the third level. As such an approximation, we use the function

$$\begin{aligned}
&\overline{\overline{U}}(\tau, v) = U(\tau, \overline{\overline{\rho}}(\tau, v), \overline{\overline{x}}(\tau, v), \overline{\overline{\lambda}}(\tau, v), \overline{\overline{\mu}}(\tau, v)) = \\
&= -\overline{\overline{\rho}} - \overline{\overline{\lambda}}Q(\tau, \overline{\overline{\lambda}}) - \overline{\overline{\mu}}_1 Q(\tau, \overline{\overline{\mu}}_1) - \overline{\overline{\mu}}_2 Q(\tau, \overline{\overline{\mu}}_2) - \overline{\overline{\mu}}_3 Q(\tau, \overline{\overline{\mu}}_3) - R(\tau, \overline{\overline{\rho}}) - R(\tau, \overline{\overline{x}}_1) - R(\tau, \overline{\overline{x}}_2) - R(\tau, \overline{\overline{x}}_3) + \\
&\qquad + R(\tau, \overline{\overline{\lambda}}) + R(\tau, \overline{\overline{\mu}}_1) + R(\tau, \overline{\overline{\mu}}_2) + R(\tau, \overline{\overline{\mu}}_3),
\end{aligned}$$

the partial derivatives of which, according to (44), are

$$\begin{aligned}
\frac{\partial \overline{\overline{U}}}{\partial v_1} &= \frac{\overline{\overline{\lambda}}\,\overline{\overline{x}}_1}{v_1^2} - \frac{\overline{\overline{\lambda}}\,\overline{\overline{x}}_3}{(A - v_1 - v_2)^2} - \overline{\overline{\mu}}_1 \frac{\partial \overline{U}_{(1)}}{\partial v_1} - \overline{\overline{\mu}}_2 \frac{\partial \overline{U}_{(2)}}{\partial v_1} - \overline{\overline{\mu}}_3 \frac{\partial \overline{U}_{(3)}}{\partial v_1}, \\
\frac{\partial \overline{\overline{U}}}{\partial v_2} &= \frac{\overline{\overline{\lambda}}\,\overline{\overline{x}}_2}{v_2^2} - \frac{\overline{\overline{\lambda}}\,\overline{\overline{x}}_3}{(A - v_1 - v_2)^2} - \overline{\overline{\mu}}_1 \frac{\partial \overline{U}_{(1)}}{\partial v_2} - \overline{\overline{\mu}}_2 \frac{\partial \overline{U}_{(2)}}{\partial v_2} - \overline{\overline{\mu}}_3 \frac{\partial \overline{U}_{(3)}}{\partial v_2},
\end{aligned} \tag{52}$$

where the values of the partial derivatives on the right-hand sides are determined from (49).

In conclusion, we consider a variant of the third level problem:

find local extreme values of dependence ρ_v^{**} *in the case where the set* $\Upsilon \subset E^2$ *is determined by the system of inequalities* $\begin{cases} 1 \leqslant v_1 \leqslant 5, \\ 1 \leqslant v_2 \leqslant 5, \\ 2v_1 + v_2 \geqslant 7. \end{cases}$

Geometrically, this problem admits the following interpretation: it is required to find parameter values that optimize (according to the size of the criteria mismatch) the form of the Pareto set of the considered multicriteria model.

The contour system and the spatial graph function $-\overline{U}(v_1, v_2)$, which is an approximation of the dependence ρ_v^{**}, are shown in Fig. 4.

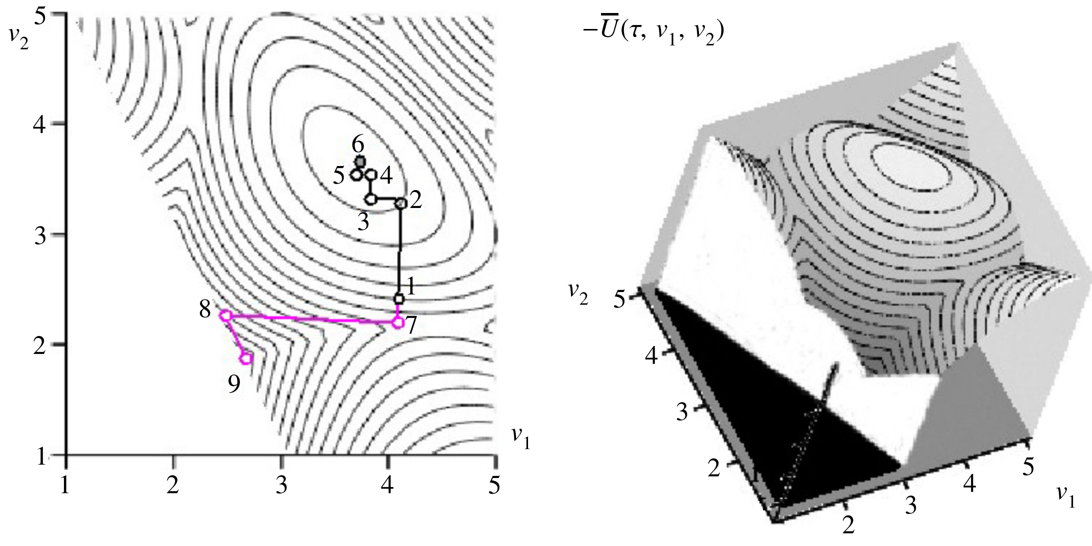

Figure 4. Contour system and 3d function graph for $-\overline{\overline{U}}(\tau, v_1, v_2)$

To search for the maximum in the problem of the third level the *steepest ascent* method has been applied. As a direction vector w, the *normalized gradient* of the function $-\overline{U}$ was used. The step value in this direction was estimated from the condition of reaching a maximum along w.

The main quantitative characteristics for several initial iterations of the solution process with $\tau =$ $= 0.01$ are given in Table 4. The exact solution of the third level problem (local maximum search) in this example (point 6 in Fig. 4) has the form

$$v_1^{**} = \frac{11}{3}, \quad v_2^{**} = \frac{11}{3}, \quad \rho_{v^{**}}^{**} = \frac{22}{9}.$$

Table 4. Solving the problem of the third level to the maximum

Point in Fig. 4	v_1	v_2	$-\overline{\overline{U}}$	Grad. norm	w_1	w_2	Step
1	4.000000000	2.500000000	2.292876919	0.292812466	0.117759800	0.993042109	0.925000000
2	4.108927815	3.418563950	2.427442369	0.070259924	−0.993862751	0.110620212	0.360000000
3	3.751137224	3.458387226	2.440214361	0.041937320	0.104185521	0.994557880	0.151750000
4	3.766947377	3.609311384	2.443407623	0.017053825	−0.994280382	0.106801322	0.080651500
5	3.686757173	3.617925071	2.444096417	0.009512746	. . .	. . .	. . .
Exact solution	3.666666667	3.666666667	2.444444444				

The local minimum in the problem of the third level was determined by the method of *antigradient projections*. Quantitative characteristics of the first three steps of the corresponding iterative process at $\tau = 0.01$ are given in Table 5. The exact solution of the problem in this example (point 9 in Fig. 4) looks like

$$v_1^{**} = \frac{1 + \sqrt{141}}{5}, \quad v_2^{**} = \frac{33 - 2\sqrt{141}}{5}, \quad \rho_{v^{**}}^{**} = \frac{33 - 2\sqrt{141}}{5}.$$

Table 5. Solving the problem of the third level to the minimum

Point in Fig. 4	v_1	v_2	$-\overline{\overline{U}}$	Grad. norm	w_1	w_2	Step
1	4.000000000	2.500000000	2.292876919	0.292812466	−0.117759800	−0.993042109	0.320363750
7	3.962274029	2.181865306	2.181828016	0.085751139	−0.993052354	0.117673374	1.662578285
8	2.311246750	2.377506503	1.976736712	0.433886194	0.447213595	−0.894427191	0.589481987
9	2.574871109	1.850257785	1.850234349	. . .	. . .	. . .	. . .
Exact solution	2.574868417	1.850263165	1.850263165				

To demonstrate the smoothing properties of the feedback function method, we show in Fig. 5 the graphs of the function $\overline{\Phi}(\tau, v_1) = -\overline{\overline{U}}(\tau, v_1, 7 - 2v_1)$ for the parameter values $\tau = 0.05, 0.025, 0.01$, as well as the exact solution.

Conclusion

This article describes a scheme for solving problems of parametric programming, which allows, in its implementation, the use of the representation of functions according to the Taylor formula.

The basis of the proposed scheme is the construction of a smooth approximation of the dependence of the optimal values of variables on parameters. Building an approximation reduces to finding saddle points for Lagrange functions modified in a special way.

The specifics of the applied modification is to use functions that establish feedback between direct variables and Lagrange multipliers under optimality conditions.

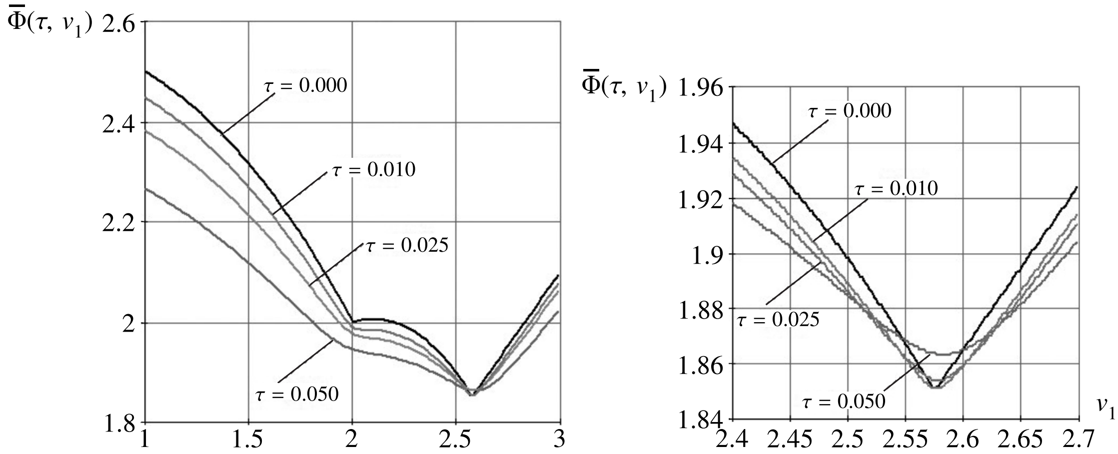

Figure 5. Function graphs for $\overline{\Phi}(\tau, v_1)$

In this case, the search for saddle points consists in solving a system of equations similar in structure to the optimality conditions in the Karush – Kuhn – Tucker theorem. This system, however, does not contain restrictions of nonnegativity and complementary nonrigidity.

Descriptions of the properties of feedback functions are given which provide the required smoothness approximation and its error is also estimated.

Options for using the proposed approach are illustrated by solutions of various classes of problems which contain parameters or reduce to them.

Comparing the proposed approach with other algorithms for solving problems (1) using Taylor expansions, the following can be pointed out.

Methodologically, this approach is similar to both the penalty function method and the Lagrange function modification methods.

The main similarity is that the original problem of mathematical programming is reduced to solving systems of nonlinear equations with some instrumental parameter successively tending to the limit value.

In this case, individual types of feedback functions can be obtained by some transformation directly from penalty functions of a special class. It is also true that from the feedback functions one can get some types of penalty functions.

Finally, the system of nonlinear equations (10) which is the basis of the method of feedback functions (and realizes the feedback between the estimates of direct variables and Lagrange multipliers) turns out to be at the same time a necessary condition for stationarity for the modified Lagrange function of problem (1).

There is also a significant difference between the above-mentioned methods and the method of feedback functions: for basic standard types penalty or modifying functions (such as slice functions, piecewise smooth or barrier functions) the construction of feedback functions is impossible.

The experience gained so far allows us to give the following assessment of the feasibility of the practical use of the method of feedback functions.

1. In the approach under consideration, the smoothness of the approximation is combined with the possibility of regulating its errors (by selecting the value of the instrumental parameter τ).

This implies that the approach considered here is appropriate for a preliminary approximate evaluation of solutions, which can then be used as initial approximations in other methods.

2. The proposed algorithm is a special case of primal-dual methods for solving problem (1), the effectiveness of which is known [Жадан, 2015]. This is especially evident in the procedure for solving

system (10) in the case where one of the problem in the dual pair is ill-conditioned, while the second one is overdetermined [Умнов, Умнов, 2018, § 4.3].

3. On the other hand, it is obvious that the reduction (described in § 4) of the procedure for solving problem (1) to integration, followed by passage to the limit, is computationally not the most efficient. However, using feedback functions in integral form may be useful in theoretical research.

We now indicate some possible directions for further development of the proposed approach.

The first of them is assessment of the effectiveness of the practical use of the method of feedback functions and similar computational schemes. Although, according to the authors, such an assessment falls beyond the scope of this article and seems to be the subject of a separate study, some observation are in order here.

For example, one can avoid the need for special control over the condition $s > 0$ when calculating the values of the feedback function at trial points of the process of solving the system (10) and reduce its dependence on the choice of initial approximations. To do this, it is enough to replace non-negative unknowns in (10) with their absolute values. In other words, one can solve, instead of the system (10), the system

$$\begin{cases} \operatorname*{grad}_x U(\tau, |\overline{x}_1|, |\overline{x}_2|, \ldots, |\overline{x}_n|, |\overline{\lambda}_1|, |\overline{\lambda}_2|, \ldots, |\overline{\lambda}_m|, v) = o, \\ \operatorname*{grad}_\lambda U(\tau, |\overline{x}_1|, |\overline{x}_2|, \ldots, |\overline{x}_n|, |\overline{\lambda}_1|, |\overline{\lambda}_2|, \ldots, |\overline{\lambda}_m|, v) = o, \end{cases} \tag{53}$$

in which the left-hand sides of the equations are even functions of the components of the vectors $\overline{x}$ and $\overline{\lambda}$.

In this case, the solution process (53) can terminate in any orthant $E^n \otimes E^m$, however, the solution of system (10) is obviously obtained from the solution (53) by replacing the found values of the components $\overline{x}$ and $\overline{\lambda}$ with their absolute values.

The second possible direction for further research is an extension of the class of feedback functions. We can talk about, say, functions $Q(\tau, s)$ defined $\forall s \in R$. Indeed, by direct verification one can, for example, verify that $Q(\tau, s) = \tau s - \exp\left(-\frac{s}{\tau}\right)$, defined $\forall s \in R$, has all the necessary properties feedback functions.

In the authors's opinion, the study of the convergence of the procedure of solving the system (10) in the case where one of the problems in the dual pair is ill-conditioned, and the second is overdetermed, holds much promise as well.

Finally, of interest is also, the analysis of new classes of problems which reduce to the problem statement (1) and which can efficiently be solved by the method of feedback functions.

For example, the method of feedback functions can be used to solve optimization problems for *a complex* of mathematical models or in the case where the transformation of a certain subset of variables into parameters makes it possible to reduce the solution of a nonlinear problem to a series of linear ones. Similar approaches using the method of smooth penalty functions, namely, the *distributed simulation method* and the *parametric linearization scheme*, are considered in [Умнов, Умнов, 2018].

References

Гермейер Ю. Б. Приближенное сведение с помощью штрафных функций задачи определения максимина к задаче определения максимума // ЖВМ-МФ. — 1969. — Т. 9, № 3. — С. 730–731.
Germeyyer Yu. B. Approximate reduction using penalty functions of the problem of determining the maximin to the problem of determining the maximum // USSR Computational Mathematics and Mathematical Physics. — 1969. — Vol. 9, Issue 3. — P. 325–328. — DOI: https://doi.org/10.1016/0041-5553(69)90086-X (Original Russian paper: *Germeyyer Yu. B.* Priblizhennoye svedeniye s pomoshch'yu shtrafnykh funktsiy zadachi opredeleniya maksimina k zadache opredeleniya maksimuma // ZHVM-MF. — 1969. — Vol. 9, No. 3. — P. 730–731.)

Гольштейн Е. Г., Третьяков Н. В. Модифицированные функции Лагранжа. — М.: Наука, 1989. — С. 400.

Gol'shteyn E. G., Tret'yakov N. V. Modifitsirovannyye funktsii Lagranzha [Modified Lagrange functions]. — Moscow: Nauka, 1989. — P. 400 (in Russian).

Демьянов В. Ф., Васильев Л. В. Недифференцируемая оптимизация. — М.: Наука, 1981. — С. 384.

Dem'yanov V. F., Vasil'yev L. V. Nedifferentsiruyemaya optimizatsiya [Non-differentiable optimization]. — Moscow: Nauka, 1981. — P. 384 (in Russian).

Демьянов В. Ф., Малоземов В. Н. Введение в минимакс. — М.: Наука, 1972. — С. 368.

Dem'yanov V. F., Malozemov V. N. Vvedeniye v minimaks [Introduction to minimax]. — Moscow: Nauka, 1972. — P. 368 (in Russian).

Жадан В. Г. Методы оптимизации. Ч. 2. Численные алгоритмы. § 7.2. — М.: МФТИ, 2015. — 320 с.

Zhadan V. G. Metody optimizatsii. Ch. 2. Chislennyye algoritmy. § 7.2 [Optimization methods. Part 2. Numerical algorithms]. — Moscow: MFTI, 2015. — 320 p. (in Russian).

Измаилов А. Ф. Чувствительность в оптимизации. — М.: Физматлит, 2006. — 245 с.

Izmailov A. F. Chuvstvitel'nost' v optimizatsii [Sensitivity in optimization]. — Moscow: Fizmatlit, 2006. — 245 p. (in Russian).

Лотов А. В., Поспелова И. И. Многокритериальные задачи принятия решений. — М.: МАКС Пресс, 2008. — 197 с.

Lotov A. V., Pospelova I. I. Mnogokriterial'nyye zadachi prinyatiya resheniy [Multicriteria decision making problems]. — Moscow: MAKS Press, 2008. — 197 p. (in Russian).

Нурминский Е. А. Численные методы выпуклой оптимизации. — М.: Наука, 1991. — 168 с.

Nurminskiy E. A. Chislennyye metody vypukloy optimizatsii [Numerical methods for convex optimization]. — Moscow: Nauka, 1991. — 168 p. (in Russian).

Скарин В. Д. Об одном общем подходе в оптимальной коррекции несобственных задач выпуклого программирования // Труды Института математики и механики УрО РАН. — 2010. — Т. 16, № 3. — С. 265–275.

Skarin V. D. Ob odnom obshchem podkhode v optimal'noy korrektsii nesobstvennykh zadach vypuklogo programmirovaniya [On a general approach to optimal correction of improper convex programming problems] // Trudy Instituta matematiki i mekhaniki UrO RAN. — 2010. — Vol. 16, No. 3. — P. 265–275 (in Russian).

Умнов А. Е. Многошаговая линейная экстраполяция в методе штрафных функций // ЖВМ-МФ. — 1974. — Т. 14, № 6. — С. 1451–1463.

Umnov A. E. Multi-step linear extrapolation in the penalty function method // USSR Computational Mathematics and Mathematical Physics. — 1974. — Vol. 14, Issue 6. — P. 77–89. — DOI: https://doi.org/10.1016/0041-5553(74)90171-2 (Original Russian Paper: *Umnov A. E.* Mnogoshagovaya lineynaya ekstrapolyatsiya v metode shtrafnykh funktsiy // ZHVM-MF. — 1974. — Vol. 14, No. 6. — P. 1451–1463.)

Умнов А. Е., Умнов Е. А. Использование функций обратных связей в задачах линейного программирования // Ж. вычисл. матем. и матем. физ. — 2019. — Т. 59, № 10. — С. 1681–1694.

Umnov A. E. Using feedback functions in linear programming problems // Computational Mathematics and Mathematical Physics. — 2019. — Vol. 59, Issue 10. — P. 1626–1638. — DOI: https://doi.org/10.1134/S0965542519100142 (Original Russian paper: *Umnov A. E., Umnov E. A.* Ispol'zovaniye funktsiy obratnykh svyazey v zadachakh lineynogo programmirovaniya // Zh. vychisl. matem. i matem. fiz. — 2019. — Vol. 59, No. 10. — P. 1681–1694.)

Умнов Е. А., Умнов А. Е. Параметрические задачи в математическом программировании. — М.: МФТИ, 2018. — 297 с. — www.umnov.ru

Umnov E. A., Umnov A. E. Parametricheskiye zadachi v matematicheskom programmirovanii [Parametric problems in mathematical programming]. — Moscow: MFTI, 2018. — 297 p. — www.umnov.ru (in Russian).

Умнов Е. А., Умнов А. Е. Параметрический анализ в задачах математического программирования // Труды МФТИ. — 2014. — Т. 6, № 3. — С. 73–83.

Umnov E. A., Umnov A. E. Parametricheskiy analiz v zadachakh matematicheskogo programmirovaniya [Parametric analysis in problems of mathematical programming] // Trudy MFTI. — 2014. — Vol. 6, No. 3. — P. 73–83 (in Russian).

Федоров В. В. Численные методы максимина. — М.: Наука, 1979. — 279 с.

Fedorov V. V. Chislennyye metody maksimina [Numerical methods of maximin]. — Moscow: Nauka, 1979. — 279 p. (in Russian).

Федорюк М. В. Метод перевала. — М.: Наука, 1977. — 368 с.

Fedoryuk M. V. Metod perevala [Pass method]. — Moscow: Nauka, 1977. — 368 p. (in Russian).

Danskin J. The theory of max-min and its application to weapons allocation problems. — NY: Springer, 1967.

Fiacco A. V. Introduction to sensitivity and stability analysis in nonlinear programming. — NY: Academic Press, 1983.

Fiacco A. V., McCormick G. P. Nonlinear programming: sequential unconstrained minimization techniques. — NY: John Wiley and Sons, 1968. — 210 p.

Rockafellar T. Convex analysis. — Prinseton University Press, 1970. — Vol. 3.

Umnov A. E., Umnov E. A. Feedback functions in problems of nonlinear programming // SCIREA Journal of Mathematics. — 2022. — Vol. 7, Issue 5. — P. 67–82. — DOI: 10.54647/mathematics11348