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Raising convergence order of grid-characteristic schemes for 2D linear elasticity problems using operator splitting

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The grid-characteristic method is successfully used for solving hyperbolic systems of partial differential equations (for example, transport / acoustic / elastic equations). It allows to construct correctly algorithms on contact boundaries and boundaries of the integration domain, to a certain extent to take into account the physics of the problem (propagation of discontinuities along characteristic curves), and has the property of monotonicity, which is important for considered problems. In the cases of two-dimensional and three-dimensional problems the method makes use of a coordinate splitting technique, which enables us to solve the original equations by solving several one-dimensional ones consecutively. It is common to use up to 3-rd order one-dimensional schemes with simple splitting techniques which do not allow for the convergence order to be higher than two (with respect to time). Significant achievements in the operator splitting theory were done, the existence of higher-order schemes was proved. Its peculiarity is the need to perform a step in the opposite direction in time, which gives rise to difficulties, for example, for parabolic problems.

In this work coordinate splitting of the 3-rd and 4-th order were used for the two-dimensional hyperbolic problem of the linear elasticity. This made it possible to increase the final convergence order of the computational algorithm. The paper empirically estimates the convergence in L_1 and L_∞ norms using analytical solutions of the system with the sufficient degree of smoothness. To obtain objective results, we considered the cases of longitudinal and transverse plane waves propagating both along the diagonal of the computational cell and not along it. Numerical experiments demonstrated the improved accuracy and convergence order of constructed schemes. These improvements are achieved with the cost of three- or fourfold increase of the computational time (for the 3-rd and 4-th order respectively) and no additional memory requirements. The proposed improvement of the computational algorithm preserves the simplicity of its parallel implementation based on the spatial decomposition of the computational grid.

Keywords: computer modeling, numerical methods, hyperbolic system, grid-characteristic method, operator splitting, convergence order

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Повышение порядка точности сеточно-характеристического метода для задач двумерной линейной упругости с помощью схем операторного расщепления

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Сеточно-характеристический метод успешно применяется для решения различных гиперболических систем уравнений в частных производных (например, уравнения переноса, акустики, линейной упругости). Он позволяет корректно строить алгоритмы на контактных границах и границах области интегрирования, в определенной степени учитывать физику задачи (распространение разрывов вдоль характеристических поверхностей), обладает важным для рассматриваемых задач свойством монотонности. В случае двумерных и трехмерных задач используется процедура расщепления по пространственным направлениям, позволяющая решить исходную систему путем последовательного решения нескольких одномерных систем. На настоящий момент во множестве работ используются схемы до третьего порядка точности при решении одномерных задач и простейшие схемы расщепления, которые в общем случае не позволяют получить порядок точности по времени выше второго. Значительное развитие получило направление операторного расщепления, доказана возможность повышения порядка сходимости многомерных схем. Его особенностью является необходимость выполнения шага в обратном направлении по времени, что порождает сложности, например, для параболических задач.

В настоящей работе схемы расщепления 3-го и 4-го порядка были применены непосредственно к решению двумерной гиперболической системы уравнений в частных производных линейной теории упругости. Это позволило повысить итоговый порядок сходимости расчетного алгоритма. В работе эмпирически оценена сходимость по нормам L_1 и L_∞ с использованием аналитических решений определяющей системы достаточной степени гладкости. Для получения объективных результатов рассмотрены случаи продольных и поперечных плоских волн, распространяющихся как вдоль диагонали расчетной ячейки, так и не вдоль нее. Проведенные численные эксперименты подтверждают повышение точности метода и демонстрируют теоретически ожидаемый порядок сходимости. При этом увеличивается в 3 и в 4 раза время моделирования (для схем 3-го и 4-го порядка соответственно), но не возрастает потребление оперативной памяти. Предложенное усовершенствование вычислительного алгоритма сохраняет простоту его параллельной реализации на основе пространственной декомпозиции расчетной сетки.

Ключевые слова: компьютерное моделирование, численные методы, гиперболические системы, сеточно-характеристический численный метод, операторное расщепление, порядок сходимости

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Introduction

Hyperbolic systems of partial differential equations describe wave propagation. A special case of linear elasticity equations describes seismic waves in elastic bodies and is widely used, for example, for modeling seismic waves in a geological medium [Petrov et al., 2021; Golubev, Khokhlov, 2018], the process of dynamic loading of composites [Beklemysheva et al., 2021], ultrasound studies, problems of seismic resistance of structures, problems of global seismics [Bagaev, Golubev, Golubeva, 2019]. The large size of the computational domain determines the computational complexity of the numerical solution of direct seismic problems; at the same time, many methods of full-wave inversion require solving a large number of direct problems in a reasonable time. In combination, these factors ensure the relevance of the problem of fast and accurate numerical simulation of the propagation of seismic waves accounting for a specified medium model.

In world practice, various methods are used to solve direct problems of seismics: finite-difference [Koene, Robertsson, Andersson, 2021; Wang et al., 2021], finite-volume [LeVeque, 1986], finite-element [Moczo et al., 2007], spectral [Liu et al., 2014; Chaljub et al., 2007], grid-characteristic [Kholodov, 1980; Golubev, Shevchenko, Petrov, 2020], mesh-free [Benito et al., 2018], discontinuous Galerkin method [Dumbser, Kaser, 2006; Lisitsa, Tcheverda, Botter, 2016]. An overview of various methods is provided in [Virieux et al., 2016; Moczo et al., 2021]. In this article we propose a combination of the grid-characteristic method with high-order accuracy splitting schemes (3rd and 4th), which allows us to obtain a new high-order accuracy numerical scheme for the system of linear elasticity equations in the two-dimensional case. The grid-characteristic method used in this work has already been successfully applied to modeling elastic media [Favorskaya et al., 2018], however other methods with desired properties can also be used to solve one-dimensional hyperbolic problems.

In computational mathematics, operator splitting methods feature prominently. They are used in various fields, for example, in methods for solving ordinary differential and partial differential equations, in optimization methods [MacNamara, Strang, 2016]. Often, the use of splitting for the numerical solution of mathematical physics problems is largely limited to the simplest splitting schemes (the Lee scheme, the Marchuk – Yanenko scheme, the Strang scheme, and others), providing an order of accuracy not higher than the second [Marchuk, 1990]. However, advances in the theoretical construction of high-order splitting schemes make it possible to obtain new interesting results when combined with known numerical schemes for systems of partial differential equations.

The section Numerical method discusses: the grid-characteristic method for the one-dimensional case, the operator splitting schemes used and their main properties, the system of linear elasticity equations. The section Numerical investigation of the order of convergence presents a numerical study of the convergence of the method on test homogeneous problems in order, firstly, to verify theoretical reasoning and software implementation, and, secondly, to choose the most appropriate splitting scheme in the sense of the minimum required computer time to achieve the desired accuracy. In the Conclusion, findings obtained as a result of this study are formulated.

Numerical Method

Grid-characteristic method for one-dimensional problems.

Let us consider the following homogeneous system of hyperbolic partial differential equations in one-dimensional case:

$$\vec{q}_t + A\vec{q}_x = \vec{0}, \tag{1}$$

where the vector $\vec{q} = \vec{q}(x, t)$ includes all unknown functions of time t and the spatial variable $x \in \mathbb{R}^1$. The matrix $A(x)$ is determined by the parameters of the medium (for example, for the transport equation — the transfer rate, for the acoustics equations — the wave propagation velocity and density).

Let us examine the case of a homogeneous medium when the matrix A is constant throughout the spatial domain.

The hyperbolicity of the system means that the matrix A has a complete set of eigenvectors, so it can be represented in the following form:

$$A = \Omega^{-1}\Lambda\Omega, \tag{2}$$

where the columns of the matrix Ω^{-1} are the eigenvectors of the matrix A, and the matrix Λ is diagonal, and its diagonal contains the eigenvalues of the matrix A corresponding to the eigenvectors in the columns of Ω^{-1}. Substituting (2) into equation (1) and introducing the change $\vec{\omega} = \Omega\vec{q}$ (the components of the vector $\vec{\omega}$ will be called Riemann invariants), we obtain the system

$$\vec{\omega}_t + \Lambda\vec{\omega}_x = \vec{0}. \tag{3}$$

Since the matrix Λ is diagonal, this system is a set of independent transfer equations of the form $w_t + cw_x = 0$, where $w(x, t)$ is the ith component of the vector $\vec{\omega}$, $c = \lambda_i$ is the ith eigenvalue (it is also the diagonal element of Λ). Knowing the values of the invariants at the current time step, we can find the values at the next time step using the following formula based on the characteristic property of the transport equation:

$$w(x, t + \tau) = w(x - c\tau, t). \tag{4}$$

To find the right part of the last equality, the polynomial interpolation procedure is used. Interpolation by 2 points (linear), by 3 points (quadratic), by 4 points, and so on is possible. The first scheme, known as the Courant – Isakson – Reese (hereafter CIR) scheme, has first-order convergence. The 4-point interpolation, known as the Rusanov scheme, exhibits 3rd order convergence (on a homogeneous one-dimensional problem with constant material). Rusanov's scheme is often used in practice due to its high accuracy and proximity to monotonic schemes. These schemes are stable if the Courant condition $\tau < \frac{h}{c}$ is satisfied, where h is the grid step, c is the maximum eigenvalue modulus.

The final algorithm can be written as follows: at each time step (n) at each point

1) Go to the Riemann invariants $\vec{\omega}^{(n)}$ using formula $\vec{\omega}^{(n)} = \Omega\vec{q}^{(n)}$;
2) Calculate the values of the Riemann invariants at the next time step $\vec{\omega}^{(n+1)}$ by interpolating the desired order of values at the current time step (n);
3) Go from the Riemann invariants $\vec{\omega}^{(n+1)}$ to the initial unknowns $\vec{q}^{(n+1)}$.

REMARK 1. For the case of heterogeneous equation (in the right-hand side non-zero vector $\vec{f}(x, t)$) item 4 is added to the algorithm: perform a time step for the equation $\vec{q}_t = \vec{f}$.

Directional splitting for two-dimensional problems

Let us consider the following homogeneous hyperbolic system of equations in the two-dimensional case:

$$\vec{q}_t + A_1\vec{q}_x + A_2\vec{q}_y = \vec{0}. \tag{5}$$

Similarly to the previous subparagraph, the vector $\vec{q}(x, y, t)$ contains all the unknown functions, and matrices A_1, A_2 are the same throughout the computational domain.

To solve this system by the grid-characteristic method, a coordinate splitting procedure, also known as splitting along spatial directions, is applied. This approach makes it possible to reduce the solution of a two-dimensional problem to the sequential solution of several one-dimensional problems. The method described in the previous subparagraph can be used to solve these subproblems.

REMARK 2. Due to the hyperbolicity of the system, both matrices A_1, A_2 have a complete set of eigenvectors.

The simplest widely used splitting scheme is as follows (we will refer to it below as S1):

1) Make a time step for the problem $\vec{q}_t + A_1\vec{q}_x = \vec{0}$, find $\vec{q}^{(n+1/2)}$;

2) Make a time step for the problem $\vec{q}_t + A_2\vec{q}_y = \vec{0}$ with initial values after step 1, find $\vec{q}^{(n+1)}$.

Although this scheme has 1st order time accuracy, it is very popular due to its ease of implementation, speed of operation, and fairly accurate results in practice. Nevertheless, for a more accurate solution of the system of equations (5), numerical methods of higher order accuracy based on more accurate splitting schemes may be useful.

To describe several splitting schemes, let us present their algorithm in a more general way: at each time step

FOR i FROM 1 TO s EXECUTE:
STEP on X with $\tau = \alpha_i^X\, dt$
STEP on Y with $\tau = \alpha_i^Y\, dt$

Here dt is the time step, τ — the used («fractional») time step; the coefficients α_i^j, $i \in \{1, 2, \ldots, s\}$, $j \in \{X, Y\}$, which specify the fractional time steps, uniquely define the splitting scheme. The coefficients for several of the schemes used in this paper are given in the tables below. The X step is defined as performing one step τ to solve the equation $\vec{q}_t + A_1\vec{q}_x = \vec{0}$, the Y step is $\vec{q}_t + A_2\vec{q}_y = \vec{0}$.

REMARK 3. The following relations are satisfied to ensure the complete execution of one step in time dt when all fractional steps are executed:

$$\sum_{i=1}^{s} \alpha_i^X = 1, \quad \sum_{i=1}^{s} \alpha_i^Y = 1. \tag{6}$$

Table 1. Splitting scheme of the 1st order (hereinafter Scheme S1). $s = 1$

i	α^x	α^y
1	1	1

Table 2. Splitting scheme of the 3rd order (hereinafter scheme R3). $s = 3$

i	α^x	α^y
1	$\frac{7}{24}$	$\frac{2}{3}$
2	$\frac{3}{4}$	$-\frac{2}{3}$
3	$-\frac{1}{24}$	1

Table 3. Splitting scheme of the 4th order (hereinafter scheme Y4). $s = 4$. $\theta = \frac{1}{2-\sqrt[3]{2}}$

1	$\frac{\theta}{2}$	θ
2	$\frac{1-\theta}{2}$	$1 - 2\theta$
3	$\frac{1-\theta}{2}$	θ
4	$\frac{\theta}{2}$	0

Remark 4. Scheme R3 requires 3 times more computation time than scheme S1, scheme Y4 — 4 times more. There is no additional memory cost.

In the splitting schemes R3, Y4 there are negative coefficients α_i, indicating the need to perform negative steps in time. In general, it can be shown that if the splitting of the specified type has an order of accuracy higher than the second, then at least one of the coefficients α_i is strictly negative. For hyperbolic equations, the negative time step does not present fundamental difficulties. In the case of using the grid-characteristic method, both the general approach and the formula (4) remain true even at $\tau < 0$.

When using the grid-characteristic method for one-dimensional (split) equations, the condition for their stability will be $\tau < \frac{h}{c}$ (with the correct choice of the template for interpolation).

Linear elasticity equations

The approach described above was successfully applied to the hyperbolic system of linear elasticity equations. In this model, the vector of unknowns $\vec{q}$ includes the components of the velocity vector of the medium's particles and the components of the stress tensor in series: $\vec{q} = (v_x\ v_y\ s_{xx}\ s_{yy}\ s_{xy})^T$. The matrices A_1, A_2 are defined by the Lamé parameters λ, μ and the density ρ as follows:

$$A_1 = -\begin{pmatrix} 0 & 0 & \rho^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho^{-1} \\ \lambda + 2\mu & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = -\begin{pmatrix} 0 & 0 & 0 & 0 & \rho^{-1} \\ 0 & 0 & 0 & \rho^{-1} & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & \lambda + 2\mu & 0 & 0 & 0 \\ \mu & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of the matrices are the same and correspond to the propagation velocities of the longitudinal $\left(c_p = \sqrt{\frac{\lambda+2\mu}{\rho}}\right)$ and transverse $\left(c_s = \sqrt{\frac{\mu}{\rho}}\right)$ waves: $\Lambda = \mathrm{diag}(0, -c_p, c_p, -c_s, c_s)$.

Numerical investigation of the order of convergence

To verify the increase in the order of accuracy of the method, a series of calculations of the wave propagation problems in the medium described by the linear elasticity model was carried out. Several problem statements were considered. For the numerical solution, various splitting schemes and various grid-characteristic schemes for solving one-dimensional problems were used. In all cases, a homogeneous system of equations with constant parameters was solved, the initial conditions were set by a plane wave. A more detailed description is provided below. The numerical solution was compared with the analytical one according to the norms L_1, L_∞:

$$\|f\|_{L_1} = \sum_i |f_i| \cdot h^2, \quad \|f\|_{L_\infty} = \max_i |f_i|.$$

The calculations were repeated with halving the spatial grid steps h and the time steps dt, so that the Courant number $\frac{c \cdot dt}{h}$ remained constant. Based on a series of calculations, the numerical value of the convergence order was calculated using the formula $p = \log_2\left(\frac{E_{2h}}{E_h}\right)$, where E_h means the error — the norm of the difference between the analytical solution and the numerical one at the grid step h.

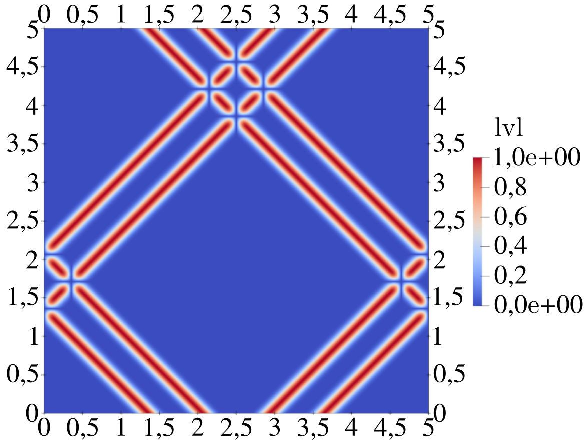

Figure 1. Wavefield at the initial moment of time for the statement of the problem No. 1, the velocity modulus is displayed

Setting No. 1 — 45°

A square area of 5 m × 5 m was considered. Periodic boundary conditions were set on the boundary. The initial values set two plane longitudinal waves propagating in the 45° direction and two plane transverse waves propagating in the 135° direction. The plane waveform was set in the form $f(\xi) = \sin^6(\xi)$, where the variable ξ is directed along the wave propagation direction perpendicular to the wave front. This momentum ensures a sufficient order of smoothness of the initial conditions over the entire computational domain. Figure 1 shows the distribution of the velocity modulus at the initial moment of time. With this configuration of initial and boundary conditions, the waves propagate in the medium and restore the configuration of the initial time moment without distortion from the boundary. The medium parameters were as follows: $c_p = 20\sqrt{2}$ m/s, $c_s = 10\sqrt{2}$ m/s, $\rho = 15$ kg/m^3. One complete period was calculated, i. e., wave propagation until the initial wave field was restored. The results of the calculations are presented in the tables below.

Setting No. 2 — 30°

A 17 m × 17 m square area was considered. The initial values set one plane wave (longitudinal or transverse) propagating in the 30° direction. The shape of the plane wave was set in the form $f(\xi) = \sin^6(\xi)$, where the variable ξ is directed along the wave propagation direction perpendicular to the wave front. This momentum ensures a sufficient order of smoothness of the initial conditions over the entire computational domain. Figure 2 shows the distribution of the velocity modulus at the initial and final moments of time. The time in which the wave passes 3 m is considered. To avoid errors near the boundary (since periodic boundary conditions do not match the physical nature of the process), we considered a 7 m × 7 m subarea in the center in order to calculate the error and then estimate the order of accuracy. The results of the calculations are given in the tables below.

Calculation Results

The results of the calculations are presented in the tables, where the step of the computational grid, the absolute and relative error by L_1 and L_∞ results of the calculations are shown. Then the order of convergence by L_1 and L_∞, and the computation time are reflected. An increase in the accuracy of the method at the cost of a three- and fourfold increase in the computation time when using the 3rd- and 4th-order splitting schemes, respectively, is confirmed. The following can also be noted:

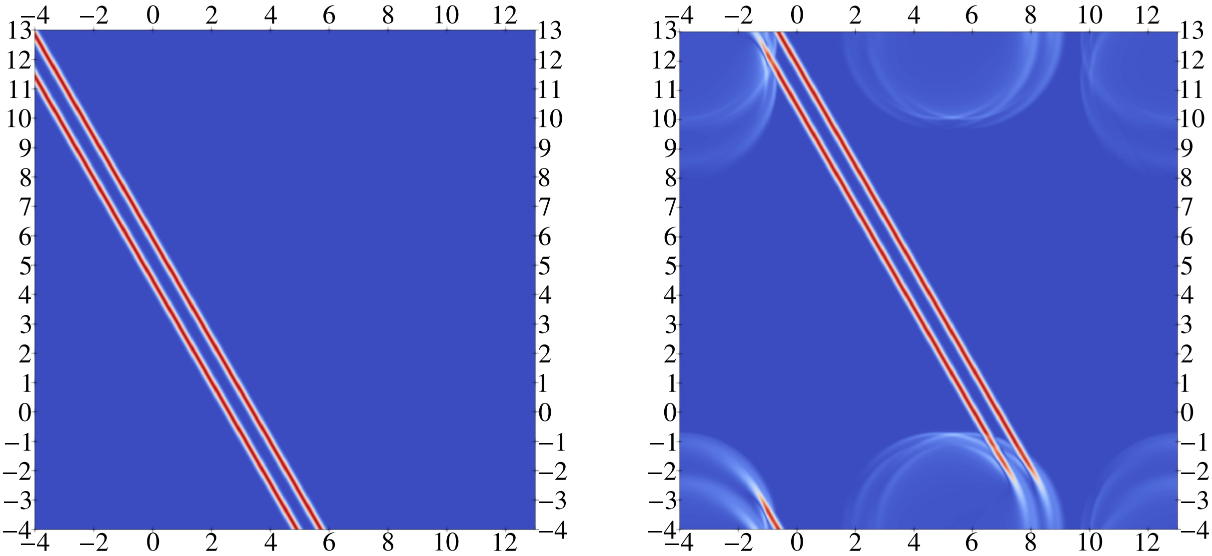

Figure 2. Wavefield at the initial and finite time moments for the statement of problem No. 2, the velocity modulus is represented by a linear color scale from 0 to 1

- For the two-dimensional problem, it is not possible to obtain an order of convergence higher than the order of solution of the one-dimensional scheme for the split system of equations.

- The use of a one-dimensional scheme of higher order of convergence than the splitting scheme makes it possible to increase the accuracy of the calculation in practice in spite of the theoretical limitation of the finite order of the scheme to the order of the splitting scheme.

Table 4. First-order S1 splitting and one-dimensional 3rd-order Rusanov scheme for split equations. Setting No. 1

h	Abs. L_1	Abs. L_∞	Rel. L_1, %	Rel. L_∞, %	p, L_1	p, L_∞	Time, s
0,0500000	1,7131e+00	4,9140e–01	0,6091	0,6955	—	—	0,1
0,0250000	7,7484e–01	2,4442e–01	0,2769	0,3459	1,145	1,008	0,5
0,0125000	1,7294e–01	6,2142e–02	0,0620	0,0879	2,164	1,976	4,1
0,0062500	2,5343e–02	9,5585e–03	0,0091	0,0135	2,771	2,701	41,1
0,0031250	3,4883e–03	1,3511e–03	0,0013	0,0019	2,861	2,823	322,7
0,0015625	5,4836e–04	2,1659e–04	0,0002	0,0003	2,669	2,641	2667,0

Table 5. 3rd-order R3 splitting and one-dimensional 3rd-order Rusanov scheme for split equations. Setting No. 1

h	Abs. L_1	Abs. L_∞	Rel. L_1, %	Rel. L_∞, %	p, L_1	p, L_∞	Time, s
0,0500000	2,2272e+00	5,7499e–01	0,7919	0,8138	—	—	0,2
0,0250000	1,0799e+00	3,2743e–01	0,3860	0,4634	1,044	0,812	1,2
0,0125000	3,1769e–01	1,0782e–01	0,1139	0,1525	1,765	1,603	8,7
0,0062500	5,0677e–02	1,8472e–02	0,0182	0,0261	2,648	2,545	101,0
0,0031250	6,6015e–03	2,4319e–03	0,0024	0,0034	2,940	2,925	815,2
0,0015625	8,2922e–04	3,0603e–04	0,0003	0,0004	2,993	2,990	7456,8

Table 6. 4th-order Y4 splitting and one-dimensional 3rd-order Rusanov scheme for split equations. Setting No. 1

h	Abs. L_1	Abs. L_∞	Rel. L_1, %	Rel. L_∞, %	p, L_1	p, L_∞	Time, s
0,0500000	2,2904e+00	5,8187e–01	0,8143	0,8235	—	—	0,3
0,0250000	1,1172e+00	3,3834e–01	0,3993	0,4788	1,036	0,782	1,6
0,0125000	3,4231e–01	1,1545e–01	0,1227	0,1633	1,706	1,551	12,0
0,0062500	5,5570e–02	2,0236e–02	0,0199	0,0286	2,623	2,512	162,0
0,0031250	7,2676e–03	2,6780e–03	0,0026	0,0038	2,935	2,918	1240,7
0,0015625	9,1344e–04	3,3721e–04	0,0003	0,0005	2,992	2,989	10 055,7

Table 7. 4th-order Y4 splitting and 4th-order one-dimensional scheme for split equations. Setting No. 1

h	Abs. L_1	Abs. L_∞	Rel. L_1, %	Rel. L_∞, %	p, L_1	p, L_∞	Time, s
0,0500000	1,4162e+00	3,9852e–01	0,5035	0,5640	—	—	0,4
0,0250000	3,6368e–01	1,2270e–01	0,1300	0,1737	1,961	1,699	2,4
0,0125000	2,9529e–02	1,0622e–02	0,0106	0,0150	3,622	3,530	18,3
0,0062500	1,8500e–03	6,6471e–04	0,0007	0,0009	3,997	3,998	162,0
0,0031250	1,1521e–04	4,1480e–05	0,0000	0,0001	4,005	4,002	1297,6
0,0015625	7,1910e–06	2,5908e–06	0,0000	0,0000	4,002	4,001	11 372,8

Table 8. First-order S1 splitting and one-dimensional 3rd-order Rusanov scheme for split equations. Setting No. 2, longitudinal wave

h	Abs. L_1	Abs. L_∞	Rel. L_1, %	Rel. L_∞, %	p, L_1	p, L_∞	Time, s
0,100000	2,5239e+00	3,8957e–01	64,1675	44,9841	—	—	0,1
0,050000	1,1635e+00	1,8297e–01	29,4147	21,1280	1,117	1,090	0,4
0,025000	2,7136e–01	4,6735e–02	6,8414	5,3964	2,100	1,969	4,0
0,012500	4,3434e–02	7,0858e–03	1,0935	0,8182	2,643	2,721	32,6
0,006250	7,9699e–03	1,0503e–03	0,2005	0,1213	2,446	2,754	296,6
0,003125	2,8097e–03	2,9422e–04	0,0707	0,0340	1,504	1,836	3462,5

Table 9. 3rd-order R3 splitting and one-dimensional 3rd-order Rusanov scheme for split equations. Setting No. 2, longitudinal wave

h	Abs. L_1	Abs. L_∞	Rel. L_1, %	Rel. L_∞, %	p, L_1	p, L_∞	Time, s
0,100000	2,7357e+00	4,1324e–01	69,5516	47,7164	—	—	0,1
0,050000	1,2758e+00	2,0196e–01	32,2541	23,3206	1,101	1,033	1,0
0,025000	3,1363e–01	5,5247e–02	7,9072	6,3794	2,024	1,870	11,1
0,012500	4,6847e–02	8,5965e–03	1,1794	0,9926	2,743	2,684	101,1
0,006250	6,0309e–03	1,1122e–03	0,1517	0,1284	2,958	2,950	886,5
0,003125	7,5710e–04	1,3971e–04	0,0190	0,0161	2,994	2,993	10 358,2

Conclusion

This paper proposes a higher-order grid-characteristic method for two-dimensional problems using special splitting techniques along the spatial directions of the 3rd and 4th orders. The implementation for linear elasticity equations has confirmed an increase in the order of accuracy and error reduction achieved at the cost of increasing the computation time by a factor of 3–4 (without additional memory costs). Using the simplest 1st order splitting together with one-dimensional 3rd or 4th order schemes for split problems showed high practical efficiency in spite of the theoretically first order accuracy in terms of time, so this combination can be recommended for most cases. To obtain more accurate results (with relative error less than 1 %) the calculation by a higher-order splitting

Table 10. 4th-order Y4 splitting and 4th-order one-dimensional scheme for split equations. Setting No. 2, longitudinal wave

h	Abs. L_1	Abs. L_∞	Rel. L_1, %	Rel. L_∞, %	p, L_1	p, L_∞	Time, s
0,100000	2,6052e+00	3,0910e–01	66,2340	35,6916	—	—	0,3
0,050000	7,7195e–01	1,1590e–01	19,5164	13,3834	1,755	1,415	2,1
0,025000	6,3839e–02	1,0311e–02	1,6095	1,1906	3,596	3,491	17,7
0,012500	4,0959e–03	6,5655e–04	0,1031	0,0758	3,962	3,973	199,9
0,006250	2,5711e–04	4,1282e–05	0,0065	0,0048	3,994	3,991	2045,3
0,003125	1,6089e–05	2,5835e–06	0,0004	0,0003	3,998	3,998	17 568,7

Table 11. First-order S1 splitting and one-dimensional 3rd-order Rusanov scheme for split equations. Setting No. 2, transverse wave

h	Abs. L_1	Abs. L_∞	Rel. L_1, %	Rel. L_∞, %	p, L_1	p, L_∞	Time, s
0,100000	1,5569e+00	2,3655e–01	68,5582	47,3101	—	—	0,1
0,050000	7,1143e–01	1,1374e–01	31,1533	22,7486	1,130	1,056	0,4
0,025000	1,7328e–01	3,1015e–02	7,5667	6,2030	2,038	1,875	4,1
0,012500	3,1947e–02	6,3870e–03	1,3931	1,2774	2,439	2,280	33,0
0,006250	1,1867e–02	2,0103e–03	0,5171	0,4021	1,429	1,668	284,2
0,003125	5,6625e–03	8,6626e–04	0,2467	0,1733	1,067	1,215	3343,3

Table 12. 3rd-order R3 splitting and one-dimensional 3rd-order Rusanov scheme for split equations. Setting No. 2, transverse wave

h	Abs. L_1	Abs. L_∞	Rel. L_1, %	Rel. L_∞, %	p, L_1	p, L_∞	Time, s
0,100000	1,6855e+00	2,5144e–01	74,2225	50,2876	—	—	0,1
0,050000	7,9109e–01	1,2582e–01	34,6418	25,1631	1,091	0,999	1,0
0,025000	2,0868e–01	3,6629e–02	9,1128	7,3258	1,923	1,780	12,0
0,012500	3,2085e–02	5,9016e–03	1,3991	1,1803	2,701	2,634	92,9
0,006250	4,1558e–03	7,6839e–04	0,1811	0,1537	2,949	2,941	852,5
0,003125	5,2226e–04	9,6600e–05	0,0228	0,0193	2,992	2,992	9991,4

Table 13. 4th-order Y4 splitting and 4th-order one-dimensional scheme for split equations. Setting No. 2, transverse wave

h	Abs. L_1	Abs. L_∞	Rel. L_1, %	Rel. L_∞, %	p, L_1	p, L_∞	Time, s
0,100000	1,4442e+00	1,7829e–01	63,5949	35,6586	—	—	0,3
0,050000	4,2732e–01	6,4872e–02	18,7121	12,9744	1,757	1,459	2,0
0,025000	3,6720e–02	5,9308e–03	1,6035	1,1862	3,541	3,451	17,5
0,012500	2,3562e–03	3,7702e–04	0,1027	0,0754	3,962	3,976	180,6
0,006250	1,4779e–04	2,3654e–05	0,0064	0,0047	3,995	3,994	1996,4
0,003125	9,2457e–06	1,4798e–06	0,0004	0,0003	3,999	3,999	17 429,7

scheme may be preferable due to the use of a coarser computational grid and shorter computational time.

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References

Bagaev R. A., Golubev V. I., Golubeva Yu. A. Full-wave 3D earthquake simulation using the double-couple model and the grid-characteristic method // Computer Research and Modeling. — 2019. — Vol. 11, No. 6. — P. 1061–1067.

Beklemysheva K., Golubev V., Petrov I., Vasyukov A. Determining effects of impact loading on residual strength of fiber-metal laminates with grid-characteristic numerical method // Chinese Journal of Aeronautics. — 2021. — Vol. 34, No. 7. — P. 1–12.

Benito J., Prieto F. U., Urena M., Casino E. S., Gavete L. A new meshless approach to deal with interfaces in seismic problems // Applied Mathematical Modelling. — 2018. — Vol. 58. — P. 447–458.

Chaljub E., Komatitsch D., Vilotte J.-P., Capdeville Y., Valette B. Spectral-element analysis in seismology // Advances in Geophysics. — 2007. — Vol. 48. — P. 365–419.

Dumbser M., Kaser M. An arbitrary high-order discontinuous Galerkin method for elastic waves on unstructured meshes II. The three-dimensional isotropic case // Geophysical Journal International. — 2006. — Vol. 167, No. 1. — P. 319–336.

Favorskaya A. V., Zhdanov M. S., Khokhlov N. I., Petrov I. B. Modelling the wave phenomena in acoustic and elastic media with sharp variations of physical properties using the grid-characteristic method // Geophysical Prospecting. — 2018. — Vol. 66. — P. 1485–1502.

Golubev V. I., Khokhlov N. I. Estimation of anisotropy of seismic response from fractured geological objects // Computer Research and Modeling. — 2018. — Vol. 10, No. 2. — P. 231–240.

Golubev V., Shevchenko A., Petrov I. Simulation of Seismic Wave Propagation in a Multicomponent Oil Deposit Model // International Journal of Applied Mechanics. — 2020. — Vol. 12, No. 8. — No. 2050084.

Kholodov A. S. The construction of difference schemes of increased order of accuracy for equations of hyperbolic type // USSR Computational Mathematics and Mathematical Physics. — 1980. — Vol. 20, No. 6. — P. 234–253.

Koene E. F. M., Robertsson J. O. A., Andersson F. Anisotropic elastic finite-difference modeling of sources and receivers on Lebedev grids // Geophysics. — 2021. — Vol. 86, No. 2. — P. A21–A25.

LeVeque R. J. Intermediate boundary conditions for time-split methods applied to hyperbolic partial differential equations // Mathematical Computations. — 1986. — Vol. 47. — P. 37–54.

Lisitsa V., Tcheverda V., Botter C. Combination of the discontinuous Galerkin method with finite differences for simulation of seismic wave propagation // Journal of Computational Physics. — 2016. — Vol. 311. — P. 142–157.

Liu Y., Teng J., Lan H., Si X., Ma X. A comparative study of finite element and spectral element methods in seismic wavefield modeling // Geophysics. — 2014. — Vol. 79, No. 2. — P. T91–T104.

MacNamara S., Strang G. Operator splitting. In Splitting methods in communication, imaging, science, and engineering. — Springer, Cham: Scientific Computation (formerly: Sprin. Ser. Comp. Sciences), 2016. — P. 95–114.

Marchuk G. I. Splitting and alternating direction methods // Handbook of numerical analysis. — 1985. — Vol. 1. — P. 197–462.

Moczo P., Kristek J., Gabriel A.-A., Chaljub E., Ampuero J.-P., Sanchez-Sesma F. J., Galis M., Gregor D., Kristekova M. Numerical wave propagation simulation // The 6th IASPEI / IAEE International Symposium: Effects of Surface Geology on Seismic Motion, August 2021.

Moczo P., Kristek J., Galis M., Pazak P., Balazovjech M. The finite-difference and finite-element modeling of seismic wave propagation and earthquake motion // Acta Physica Slovaca. — 2007. — Vol. 57, No. 2. — P. 177–406.

Petrov I. B., Golubev V. I., Petrukhin V. Y., Nikitin I. S. Simulation of seismic waves in anisotropic media // Doklady Mathematics. — 2021. — Vol. 103, No. 3. — P. 146–150.

Virieux J., Etienne V., Cruz-Atienza V., Brossier R., Chaljub E., Coutant O. Modelling seismic wave propagation for geophysical imaging, seismic waves–research and analysis / ed. Dr. Masaki Kanao. — InTech, 2012. — P. 304.

Wang W., Wen X., Tang C., Li B., Li L., Wang W. Variable-order optimal implicit finite-difference schemes for explicit time-marching solutions to wave equations // Geophysics. — 2021. — Vol. 86, No. 2. — P. T91–T106.