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MODELS OF ECONOMIC AND SOCIAL SYSTEMS

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Game-theoretic and reflexive combat models

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Combat operations modeling is an urgent scientific and practical task aimed at providing commanders and headquarters with quantitative basis for making decisions. The authors proposed the function of victory in combat and military operations, based on the function of the conflict by G. Tullock and taking into account the scale of combat (military) operations. With a sufficient volume of military statistics, the scale parameter was assessed and its values were found for the tactical, operational and strategic levels. The game-theoretic models «offensive – defense», in which the sides solve the immediate and subsequent tasks, having the formation of troops in one or several echelons, have been investigated. At the first stage of modeling, the solution of the immediate task is found — the breakthrough (holding) of defense points, at the second — the solution of the subsequent task — the defeat of the enemy in the depth of the defense (counterattack and restoration of defense). For the tactical level, using the Nash equilibrium, solutions were found for the closest problem (distribution of the sides forces at points of defense) in an antagonistic game according to three criteria: a) breakthrough of the weakest point, b) breakthrough of at least one point, and c) weighted average probability. It is shown that it is advisable for the attacking side to use the criterion of «breaking through at least one point», in which, all other things being equal, the maximum probability of breaking through the points of defense is ensured. At the second stage of modeling for a particular case (the sides are guided by the criterion of breaking through the weakest point when breaking through and holding defense points), the problem of distributing forces and facilities between tactical tasks (echelons) was solved according to two criteria: a) maximizing the probability of breaking through the defense point and the probability of defeating the enemy in depth defense, b) maximizing the minimum value of the named probabilities (the criterion of the guaranteed result). Awareness is an important aspect of combat operations. Several examples of reflexive games (games characterized by complex mutual awareness) and information management are considered. It is shown under what conditions information control increases the player's payoff, and the optimal information control is found.

Keywords: mathematical model, battle, offensive, defense, victory function, game-theoretic model, reflexive and information control

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МОДЕЛИ ЭКОНОМИЧЕСКИХ И СОЦИАЛЬНЫХ СИСТЕМ

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Теоретико-игровые и рефлексивные модели боевых действий

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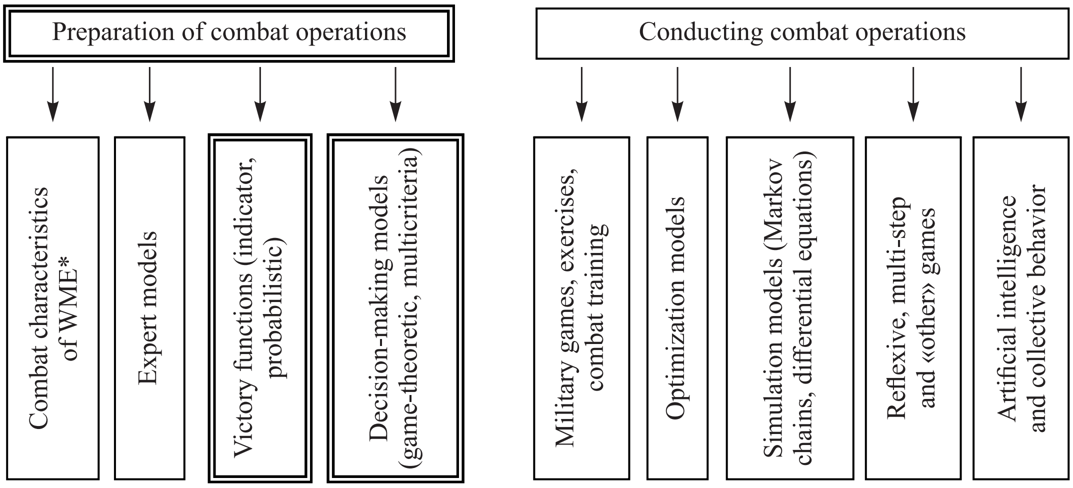
Моделирование боевых действий является актуальной научной и практической задачей, направленной на предоставление командирам и штабам количественных оснований для принятия решений. Авторами предложена функция победы в боевых и военных действиях, основанная на функции конфликта Г. Таллока и учитывающая масштаб боевых (военных) действий. На достаточном объеме данных военной статистики выполнена оценка параметра масштаба и найдены его значения для тактического, оперативного и стратегического уровней. Исследованы теоретико-игровые модели «наступление – оборона», в которых стороны решают ближайшую и последующую задачи, имея построение войск в один или несколько эшелонов. На первом этапе моделирования находится решение ближайшей задачи — прорыв (удержание) пунктов обороны, на втором — решение последующей задачи — разгром противника в глубине обороны (контратака и восстановление обороны). Для тактического уровня с использованием равновесия Нэша найдены решения ближайшей задачи (распределение сил сторон по пунктам обороны) в антагонистической игре по трем критериям: а) прорыв слабейшего пункта; б) прорыв хотя бы одного пункта; в) средневзвешенная вероятность. Показано, что наступающей стороне целесообразно использовать критерий «прорыв хотя бы одного пункта», при котором, при прочих равных условиях, обеспечивается максимальная вероятность прорыва пунктов обороны. На втором этапе моделирования для частного случая (стороны при прорыве и удержании пунктов обороны руководствуются критерием прорыва слабейшего пункта) решена задача распределения сил и средств между тактическими задачами (эшелонами) по двум критериям: а) максимизация вероятности прорыва пункта обороны и вероятности разгрома противника в глубине обороны; б) максимизация минимального значения из названных вероятностей (критерий гарантированного результата). Важным аспектом боевых действий является информированность. Рассмотрены несколько примеров рефлексивных игр (игр, характеризующихся сложной взаимной информированностью) и осуществления информационного управления. Показано, при каких условиях информационное управление увеличивает выигрыш игрока, и найдено оптимальное информационное управление.

Ключевые слова: математическая модель, бой, наступление, оборона, функция победы, теоретико-игровая модель, рефлексивное и информационное управление

1. Introduction

Control of troops is a purposeful activity of commanders, headquarters and other command and control bodies to maintain the combat readiness and fighting capacity of troops, prepare them for battle and guide them in the performance of assigned tasks [Taktika, 1987]. The two main phases of troop command and control (preparation and conduct of combat operations) can be associated with a set of models, the classification of which is shown in fig. 1.



*WME — Weapons and Military Equipments

Fig. 1. Classification of combat operations' models

At the stage of preparation, in the general case the modeling of combat operations comes down to finding the deployment of troops (deployment of forces and facilities to the area, their allocation), in which the maximum possible damage is inflicted on the enemy.

The sequence of modeling may be as follows.

At the first stage, the tactical characteristics of combat assets are analyzed and calculate the parameter of the combat superiority of the formation (unit, small/large unit) over the enemy formation expected in the fight, battle, operation in moral and technological sense are computed (possibly with the involvement of experts) [Buravlev, Tsyrendorzhiev, Brezgin, 2009; Dorokhov and Ishchuk, 2017].

At the second stage, the type of the formation victory function is selected [Shumov, 2020]: indicator (Colonel Blotto's game [Application, 1961]) or probabilistic type, and in the second case, based on the ratio (Yu. B. Germeier [Germeier, 1971], the function of G. Tullock [Tullock, 1980]), or on the force difference (the model of D. McFadden and D. Hirschleifen [Jia, Skaperdas, Vaidya, 2013]).

The third stage of modeling usually consists in setting the game-theoretic task «offensive – defense» and finding optimal solutions for the distribution of troops between tasks and points (regions, positions). If it is possible, planning (prediction) of offensives during the breakthrough of the defense and in its depth is also carried out, as well as proof for activities to mislead the enemy.

At the final stage, the model is verified, computation results are checked for compliance with the principles of military art and combat experience (the evidence of the «correctness» of the models is the compliance of the simulation results with the principles of military art [Osipov, 1915]).

The conduct of combat operations (the second phase of control) is explored during military games, exercises and combat trainings conducted using simulation and other models and decision support systems with different extents of automation during the process of combat actions [Novikov, 2012; Aggregated, 2000] and is not the subject of this paper.

The Russian General Mikhail Pavlovich Osipov is considered to be the founder of the simulation of combat operations. In his work «The influence of the number of combatants on their losses» [Osipov, 1915], published in 1915 in the «Military collection» (now the journal «Military Thought»), he elaborated a model of battle dynamics based on an analysis of the results of 38 battles of standing troops from the 19th and 20th centuries, the solution was found, and the model parameters were estimated.

Formal justification of the Osipov – Lanchester models [Osipov, 1915; Lanchester, 1916] (mean dynamics method) can be found in [Wentzel, 1964]. The basic concepts of the theory of antagonistic games were developed by E. Borel [Borel, 1921]. In Russia and abroad, models of combat operations were developed within the scientific discipline framework «Operations research» (see, for example, the works [Germeier, 1971; Krasnoshchekov, Petrov, 1983; Vasin, Morozov, 2003; Vasin, 2005; Vasin, Krasnoshchekov, Morozov, 2008; Morse and Kimball, 1951; Karlin, 1964; Wagner, 1972]). The classical game-theoretic problem «offensive – defense» was formulated and solved by Yu. B. Germeier [Germeier, 1971] (as the O. Gross model modification), where two sides allocate limited resources among defense points.

An overview of papers on modeling combat operations can be found in the article by D. A. Novikov «Hierarchical models of warfare» [Novikov, 2012], where the Lanchester models, the game of Colonel Blotto (a game where two sides simultaneously and independently distribute their resources between objects — battlefields, simultaneous competitions / auctions, groups of voters, etc.), as well as conflict functions of indicator and probabilistic types are considered.

Subjects make decisions based on a hierarchy of ideas about essential parameters, and inevitably, for one reason or another, there is a discrepancy between ideas (reflexive reality) and objective reality. A systematic study of reflexion in Control Science began in the 60s of the XX century [Lefevre, 1973], in the last decade a set of mathematical models of information and strategic reflexion has been developed [Novikov, Chkhartishvili, 2012].

The purpose of this paper is to generalize and elaborate the results of combat operations simulations [Korepanov, Novikov, 2011; Shumov, 2019; Shumov, Korepanov, 2020; Shumov, Korepanov, 2021] in the following directions:

- firstly, the statistical justification of the function of victory in a fight, battle, operation, taking into account the moral and technological characteristics of combat units, the scale of combat operations;
- secondly, the development of game-theoretic «offensive – defense» models, in which the offensive side solves two tasks: the first-phase one (breaking through the enemy's defense) and the subsequent one (destroying the enemy's reserves, capturing an object in the depths of the defense). Note that in the existing game-theoretic models only the first task is formalized, i. e. such models can be called, for example, counter-fight models, but not offensive (defense) models;
- thirdly, considering the awareness of the sides in the battle models.

Thus, a feature of the combat models considered below is the use of functions of victory in them, expressing the antagonistic nature of the conflict: rise in the efforts of the first side increases its chances of success, as well as a fall in the efforts of the second side [Hirshleifer, 2000]. The main emphasis in this work is on the analysis of the victory functions and the solution of game-theoretic problems of the optimal distribution of resources over tasks and directions (objects, points).

2. The «offensive – defense» game-theoretic model

The game-theoretic «offensive – defense» model is a development of the Gross – Germeier – Vasin «attack – defense» models [Karlin, 1964; Germeier, 1971; Vasin, Morozov, 2003], where the task of breaking through defense points is studied. The function of victory in a fight (battle, operation) is used as an aggregated function of combat technology.

2.1. The function of victory in fight, battle, operation

In the general case, the functions of the conflict (competition) are divided into (classification basis — a justification method of the model): probabilistic or stochastic models; models built on the basis of axioms (assumptions); competitive and auction models based on the economic mechanism design, models based on aggregation of microeconomic indicators (submodels).

Assume that two sides are involved in a conflict (competition, auction). Their efforts (resources) will be denoted by $x > 0$ and $y > 0$, respectively. Any combination of the efforts of the sides is assigned with the probability of success (victory) — $p_x(x, y)$ and $p_y(x, y)$. The following functions of victory class is well studied:

$$p_x(x, y) = \frac{f_x(x)}{f_x(x) + f_y(y)}, \tag{2.1}$$

where $f_x(\cdot)$ and $f_y(\cdot)$ — non-negative, strictly increasing functions. We note the most common functional forms of model (2.1). Model of G. Tullock

$$p_x(x, y) = \frac{x^\mu}{x^\mu + y^\mu} = \frac{(x/y)^\mu}{(x/y)^\mu + 1},$$

where $\mu > 0$ — parameter of the decisiveness of the sides, belongs to the class of models based on the ratio of efforts (the result depends on the ratio of the efforts of the sides). The model by D. McFadden and D. Hirschleifer

$$p_x(x, y) = \frac{\exp(\mu x)}{\exp(\mu x) + \exp(\mu y)} = \frac{1}{1 + \exp(\mu(y - x))},$$

belongs to the class of models based on efforts difference. The probit model belongs to the same class: $p_x(x, y) = \Phi(x - y)$, where Φ — the Laplace function.

Historically, the first victory function used in combat simulation is the Gross – Germeyer victory function. In [Germeyer, 1971], the following goal function of offense is considered (the total number of offensive facilities that have broken through all defense points):

$$f(x, y) = \sum_{i=1}^{n} \max[x_i - \mu_i y_i, 0], \tag{2.2}$$

where μ_i — the number of offensive facilities that can be destroyed by one unit of defense facilities at point i, $x_i(y_i)$ is the number of offensive (defense) facilities, n is the number of defense points. Let's restore the function of victory from the Yu. B. Germeyer's model. Note that the last expression uses the following probability of breaking through the defense point i

$$\pi_x(x_i, y_i) = \begin{cases} \dfrac{\beta_i x_i - y_i}{\beta_i x_i}, & \beta_i x_i - y_i \ge 0, \\ 0, & \beta_i x_i - y_i < 0, \end{cases} \quad \beta_i = \frac{1}{\mu_i},$$

and the number of units that have broken through is defined as the product $\pi_x(x_i, y_i)$ times x_i. I. e. when $\pi_x(x_i, y_i) = 0$ the probability of victory $p_x(x_i, y_i) = 0.5$, because $\beta_i x_i = y_i$. It is natural to assume that $p_x(x_i, y_i) = 1$ for $y_i = 0$. Let's consider the simplest — linear in $p_x(x_i, y_i)$ case of a victory function form: $2p_x(x_i, y_i) - 1 = \pi_x(x_i, y_i)$. Then we get:

$$2p_x(x_i, y_i) - 1 = \frac{\beta_i x_i - y_i}{\beta_i x_i}, \quad p_x(x_i, y_i) = \frac{2\beta_i x_i - y_i}{2\beta_i x_i}, \quad 2\beta_i x_i \ge y_i. \tag{2.3}$$

It is difficult to meaningfully explain the presence of the factor 2 in the numerator and denominator in the victory function of Yu. Germeyer and the use of the forces' difference function.

Let's specify the drawbacks of the goal function of the form (2.2). It has proven in military science [Osipov, 1915] that every battle is a psychological act, ending with the refusal of one of the parties. Therefore, during the battle, the fighters are divided into three groups: actively participating in the battle; killed or wounded; those who refused to participate in the battle (deserters, illness simulators, etc.). In model (2.2), the third group is not taken into account; by default, it is assumed that the number of units that have broken through is equal to the difference between their total number and the number of units that have been hit.

In A. A. Vasin and N. I. Tsyganov works [Vasin, Tsyganov, 2021; Vasin, Tsyganov, 2021a] the analytical dependence of the form of the victory function on the battle scale (root square of geometric mean of the initial numbers of sides) was studied for the first time.

S. Skaperdas et al. note that only a small number of publications address the issues of verifying conflict functions on real data in spite of a significant number of publications on modeling conflicts, competitions and auctions in various fields of activity [Jia, Skaperdas, Vaidya , 2013].

Let's consider the G. Tullock conflict function's extension in order to modeling fight, battle, operation. Let us define the probability of victory of the first side by the formula [Shumov, 2020]:

$$p_x(x,y)=\frac{(\beta x)^m}{(\beta x)^m+(y)^m}=\frac{q^m}{q^m+1},\ \ q=\frac{\beta x}{y}>0, \tag{2.4}$$

where: $\beta > 0$ — the parameter of the combat (moral and technological) superiority of the first side over the second; q is the ratio of the forces of sides; m is the form parameter (scale of combat operations). In the general case, technological (tactical) superiority (and the value of the parameter β) is determined, firstly, by the tactical and technical characteristics of forces and facilities of the sides, and, secondly, by the characteristics of the ground and the degree of its preparedness for combat operations.

If we make substitution of a variable q, we get the Pareto distribution:

$$p_x(z)=\frac{q^m}{q^m+1}=\frac{z^m-1}{z^m}=1-z^{-m},\ \ z^m=q^m+1,\ z\geq 1, \tag{2.5}$$

which has the property of self-similarity (the distribution of values exceeding $z_0 \geq 1$ is also the Pareto distribution). In essence, this means that the combat operations of a battalion, regiment can be described by the same distribution as the combat operations of the division in which they operate. The mathematical property of self-similarity corresponds to the most important principle of military art, which requires considering the same factors that determine the success of any fight, battle, and operation [Rech, 1985].

Let's list features of the victory function (2.4). First, the probability of the victory of the second side is the probability of the loss of the first, i.e. $p_y(x, y) = 1 - p_x(x, y)$. Second, the function $p_x(x, y)$ is strictly increasing in x and strictly decreasing in y. Thirdly, the function is symmetrical or anonymous (if the efforts of the parties are reversed, then their probabilities of victory will also change) and belongs to the class of models based on the ratio of forces, which fits to the established practice of operational-tactical calculations [Morse, Kimball, 1951; Tsygichko, Stoili, 1997]. Fourthly, within one form of combat (military, special) operations, the victory function is a homogeneous function of zero degree, i. e. $p_y(tx, ty) = p_y(x, y)$, $t > 0$. Fifth, the shape parameter m of the victory function allows us to consider the features of special operations (fight against irregular enemy formations), combat operations on the tactical and operational levels, military operations (strategic operations).

An analytical estimation of the parameter β of combat superiority follows from the definition of a battle (a set of strikes, fires and maneuvers of troops coordinated in terms of purpose, place and time) and was studied earlier [Shumov, 2020]. Statistical estimation of the parameter β was performed by the maximum likelihood method.

For a statistical estimation of the form parameter, it is necessary to collect and process data about the results of fights, battles, operations (outcome of the conflict, list of units, etc.). Statistical data of operational and strategic levels are presented in literature (see [Osipov, 1915], [Velikaya, 2010], etc.), but at the tactical level (battles involving joint arms and other units) we have little reliable data due to the following reasons: firstly, it is necessary to compare archival data about number and combat composition of the parties, including Inter-Entity Boundary Lines; secondly, one formation (unit, small/large unit) can operate on the edge of the enemy, and it is not always possible to reveal from archival data the strength of the enemy in a battle even if the initial data were available; thirdly, combat operations are dynamic and during a battle strengths of two sides can change significantly (introduction of new units into battle, withdrawal of units to the reserve or moving to a new area). In this regard, to assess the shape parameter at the tactical level, an international database of incidents in maritime space was used, according to which it is possible to accurately determine strengths of the parties and the outcome of a conflict.

On fig. 2 the proportion of successful piracy and robbery acts at sea, according to international statistics of incidents in maritime space for 2010–2020 are shown (sample size n = 714, estimate of the pirate superiority parameter β = 1,42) [Shumov, Sidorenko, Caesar, 2021].

The shape parameter is equal to m = 1 with a significance level of 0,05 according to the Fisher's chi-square statistical test (at 6 degrees of freedom).

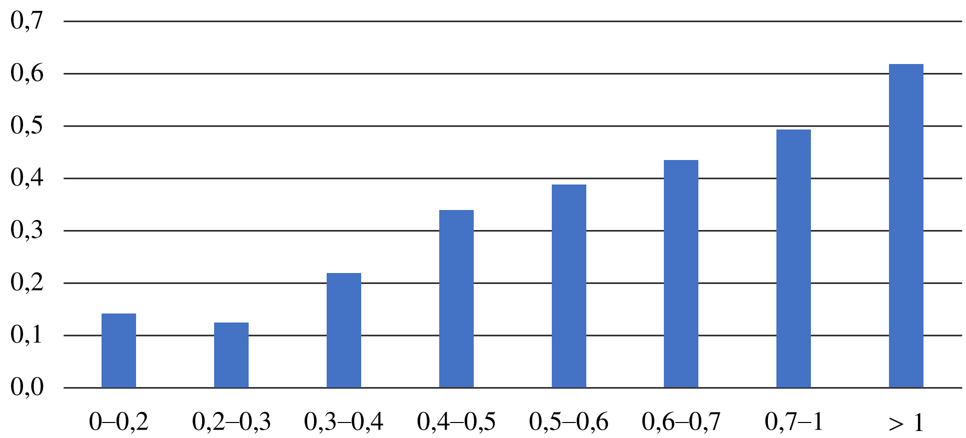


Fig. 2. Shares (vertical axis) of successful piracy and robbery at sea on q — the ratio of forces of sides (horizontal axis)

On fig. 3 shares of victories of the strongest side in terms of numbers of the XIX – early. XX centuries are shown (data taken from [Osipov, 1915], parameter β = 1, sample size n = 38). The shape parameter is m = 3 with a significance level of 0,01 according to Pearson's chi-square statistical test (at 4 degrees of freedom).

On fig. 4 shares of the victories of the strongest side in terms of numbers in strategic defensive and offensive operations during the Great Patriotic War of 1941–1945 are shown (data taken from [Velikaya, 2010], parameter β = 1, sample size n = 46).

The shape parameter is equal to m = 3 with a significance level of 0,01 according to the Pearson chi-square statistical test (at 4 degrees of freedom).

From formula (2.4) we find the required ratio of forces q to defeat the enemy with a given probability p_x:

$$q = \sqrt[m]{\frac{p_x}{1 - p_x}} . \tag{2.6}$$

Results of calculations are presented in table 1.

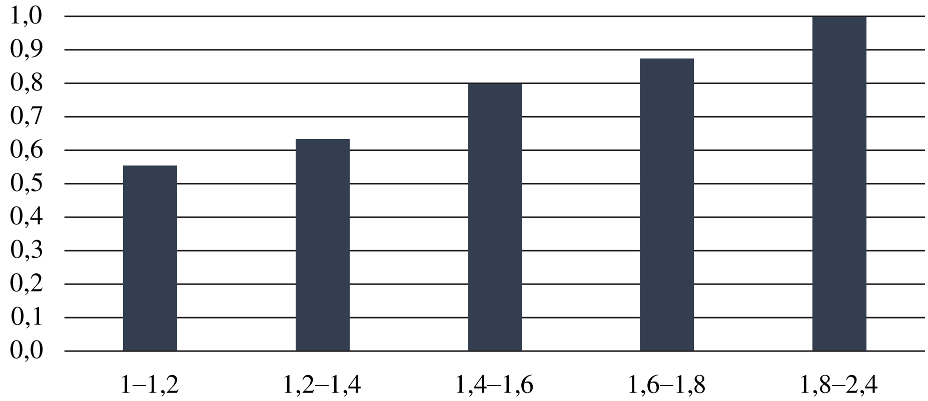


Fig. 3. Shares (vertical axis) of victories of the strongest side in the battles of the XIX – early. XX centuries on q (horizontal axis)

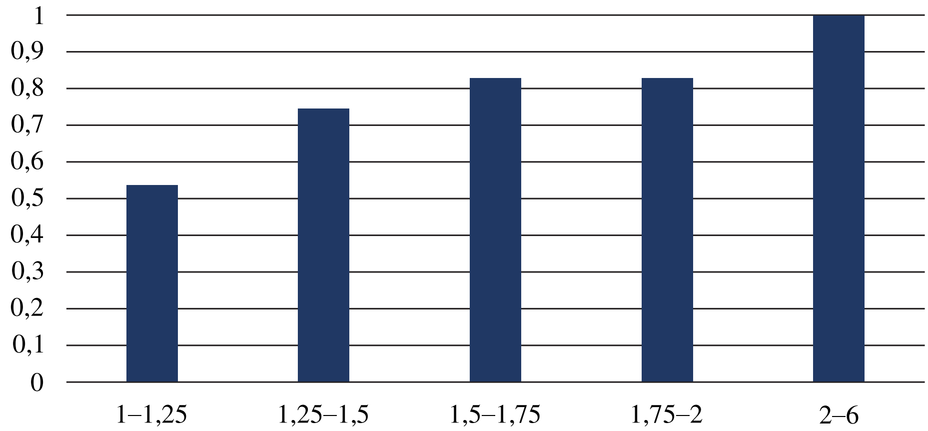


Fig. 4. Shares (vertical axis) of victories of the strongest side in strategic operations during the Great Patriotic War on q (horizontal axis)

Table 1. Necessary superiority over the enemy

Probability of defeating an enemy, p_x	Parameter m of model shape (2.6)			
	$m = 0{,}5$	$m = 1$	$m = 2$	$m = 3$
0,7	5,4 : 1	2,3 : 1	1,5 : 1	1,3 : 1
0,75	9,0 : 1	3,0 : 1	1,7 : 1	1,4 : 1
0,8	16,0 : 1	4,0 : 1	2,0 : 1	1,6 : 1
0,9	81,0 : 1	9,0 : 1	3,0 : 1	2,1 : 1

With the prevalence of non-traditional forms of combat (attacks from ambushes, partisan actions, etc.) and when modeling counter terrorist and special military operations, it is advisable to use the value of the form parameter $m = 0{,}5$. To achieve a high probability of victory, it is necessary to ensure multiple superiority in forces and facilities over the enemy. For example, a probability of victory of 0,75 is achieved with a combat superiority over the enemy $q = 9$. This result is confirmed by the practice of counter terrorist and special military operations: the experience of internal conflicts indicates that the ratio of the number of government troops to insurgents should be between 8 : 1 to 10 : 1 (i. e. eight to ten units to one). Many Western states proceed precisely with such indicators when determining the size of law enforcement forces [Kontrterroristicheskaya, 2000].

The actions of subunits and units in the offensive and defense can be described by the model of the relation of forces with the value of the form parameter $m = 1$. In this case, the probability of victory of 0,75 is achieved with a triple superiority in forces and facilities over the enemy, which corresponds to the prevailing ideas about conducting joint arms combat.

When modeling the actions of divisions (corps, armies) in a battle (operation), it seems reasonable to use the value of the form parameter m = 2–3. Here the success of the battle (operation) is almost guaranteed with a 2–3x overall superiority over the enemy forces and facilities. President of the Academy of Military Science, army General M. A. Gareev noted that during the Great Patriotic War there was not a single successful defensive operation carried out by much smaller forces than that of the advancing enemy. It is possible to repulse attacks of superior enemy forces at the tactical level, but not at the operational-strategic level [Ionin, 2005].

Consequently, meaningful and statistical estimates of the form parameter of the victory function give reason to believe that the form parameter reflects the nature and scale of combat actions. The current scientific task is a statistical estimation of the form parameter of the victory function in counter-terrorist and special operations, characteristic of which are the following: firstly, the enemy's desire to get lost in a crowd of civilians, if possible, and secondly, the purpose of the operation is not only to defeat the enemy, but also its neutralization, preventing exit from a certain area, for which special elements of the battle order are created (blocking and cover groups, filtration points, etc.).

2.2. Statement of the modeling problem

Let there be n defensive points (districts, sections, lanes) numbered i = 1, ..., n, where a breakthrough by the offense is possible. Let's denote R_x and R_y — the number of combat facilities of the offensive (player O) and defenders (player D). Resources R_x and R_y are assumed to be infinitely divisible, which will make it possible to consider the actions of their own, attached and supporting facilities/units, when their efforts are alternately directed to various points and tasks.

The offense side consists of combat units designed to solve the nearest (breaking through the enemy defense) and subsequent (repulse a counterattack of the enemy's reserves, occupying a line or an object in the depths of the defense) tasks. The player O facilities vector:

$$x = (x_1, \ldots, x_n, u) \in X = \left\{ x \mid \sum_{i=1}^{n} x_i + u = R_x \right\}, \;\; r_x = R_x - u, \;\; x_i, u \in \Re^+ , \tag{2.7}$$

where: $x_i \geq 0$ — the number of facilities for solving the nearest task (the first echelon) operating at the point i; r_x — the total number of facilities for solving the nearest problem; $u > 0$ — the number of facilities for solving the subsequent problem (second echelon).

The defending side consists of troops of first echelon and reserve (or second echelon). The task of the first echelon is to prevent a breakthrough of defense points, the task of the reserve (second echelon) is to counterattack in the event of a breakthrough of the defense or to hold the second line of defense. Facilities vector of player D:

$$y = (y_1, \ldots, y_n, w) \in Y = \left\{ y \mid \sum_{i=1}^{n} y_i + w = R_y \right\}, \;\; r_y = R_y - w, \;\; y_i, w \in \Re^+ , \tag{2.8}$$

where $y_i \geq 0$ is the number of first-echelon facilities defending point i; r_y is the total number of facilities for solving the first objective (holding defense points); $w > 0$ — the number of reserve facilities intended for counterattack in the event of a breakthrough (second objective).

Let's assume that the sides have common knowledge, make decisions simultaneously and independently. Then we have an antagonistic game (the gain of the first side is the loss of the second), and in order to find optimal solutions, one should find the Nash equilibrium.

Using known methods of game theory [Germeyer, 1971; Vasin and Morozov, 2005; Pisaruk, 2019] the authors have developed a game-theoretic model of combat operations, which is presented below.

2.3. Optimal distribution of forces and facilities when breaking through defense points (criterion — breaking through the weakest point)

When modeling a battle, the goal function of Yu. B. Germeyer [Germeier, 1971] is often used as a criterion:

$$f(x,y)=\sum_{i=1}^{n}\max[x_i-\mu_i y_i,0], \tag{2.9}$$

where μ_i is the number of facilities of offensive that one unit of facilities of defense at the point i can destroy. It has been shown (see [Shumov, Korepanov, 2021]) that this goal function corresponds to the probability of victory in a battle of the form:

$$p_x(x_i,y_i)=\frac{2\beta_i x_i-y_i}{2\beta_i x_i},\ \ 2\beta_i x_i\geq y_i,\ \beta_i=1/\mu_i\,. \tag{2.10}$$

It is meaningfully difficult to explain the presence of the factor 2 in the numerator and denominator in the victory function of Yu. Germeyer (2.10) and the use of the forces difference. Another drawback of the goal function (2.9) is the assumption that the number of advancing combat units that have broken through is equal to the number of undestroyed ones. This assumption is valid when modeling unmanned combat units. At the same time, this assumption contradicts the principles of combined arms management and modeling. According to M. Osipov, «victory does not depend on the duration of the battle, but mainly on the losses suffered by the sides; therefore, it would be more correct to assume that the battle lasts until the losses of one of the sides reach a certain percentage. On average, 20 % can be considered as such percentage...» [Osipov, 1915]. In other words, combat units are divided into three groups: struck (wounded), fighting and evading combat [ibid].

The probability of solving the first-phase objective by offensive is defined as (m = 1, tactical level)

$$f(x,y)=\max_{i=1,\dots,n}\frac{\beta_i x_i}{\beta_i x_i+y_i},\ \ \sum_{i=1}^{n}x_i=r_x,\ \ \sum_{i=1}^{n}y_i=r_y \tag{2.11}$$

(strike at the weakest point of the enemy's defense). The goal function (2.11) uses the probabilistic function of victory in the conflict, which corresponds to the tradition of military modeling and doesn't have disadvantages noted above.

Meaningfully, the goal function (2.11) reflects the desire of the offensive to break through defense at the enemy's weakest point. The goal of the defenders is to prevent a breakthrough at this point, the goal function is $1-f(x,y)$. We have an antagonistic game. Let's assume that the durations of the cycles of combat operations of the sides are approximately equal, then there is reason to believe that the sides make decisions independently and simultaneously and to find a solution using the Nash equilibrium.

It was proven [Shumov, Korepanov, 2021] that the optimal strategy of the defender (distribution of the resource among defense points) is equal to

$$y^0:y_i^0=\frac{\beta_i}{\sum_{j=1}^{n}\beta_j}r_y=\frac{\beta_i}{B}r_y,\ \ B=\sum_{j=1}^{n}\beta_j,\ i=1,\dots,n, \tag{2.12}$$

and the offensive uses a mixed strategy, allocating the entire resource to the point i with probability

$$\pi_i^0=\frac{\beta_i}{B},\ i=1,\dots,n. \tag{2.13}$$

In this case, the value of the game is equal to

$$v = \frac{r_x B}{r_x B + r_y}. \quad (2.14)$$

If the points of defense are homogeneous ($\beta = \beta_1 = \beta_2 = \ldots = \beta_n$), then the value of the game is equal to

$$v = \frac{n\beta r_x}{n\beta r_x + r_y}. \quad (2.15)$$

The weakness of self-defense directly follows from the last expression — with the growth of the number of defense points, the effectiveness of the offensive increases significantly.

It is important to note that the optimal solutions of the sides (2.12) and (2.13) do not depend on the expected number of enemy combat units. The solutions are wholly and completely determined by the characteristics of the terrain and the structure of the formations of the sides involved in combat operations taken into account in the parameter β.

2.4. Optimal distribution of forces and facilities when breaking through defense points (criterion — breaking through at least one point)

In some cases, the task of breaking through defense points at the tactical level (m = 1) can be described by the goal function of the player O in the form:

$$G(x,y) = 1 - \prod_{i=1}^{n}(1 - p_x(x_i, y_i)) = 1 - \prod_{i=1}^{n}\left(\frac{y_i}{\beta_i x_i + y_i}\right), \ \sum_{i=1}^{n} x_i = r_x, \ \sum_{i=1}^{n} y_i = r_y \quad (2.16)$$

(the probability of breaking through at least at one point of the enemy's defense).

The solution of the problem will not change if the goal function is written in an equivalent form:

$$1 - \prod_{i=1}^{n}\left(\frac{y_i}{\beta_i x_i + y_i}\right) \Rightarrow -\prod_{i=1}^{n}\left(\frac{y_i}{\beta_i x_i + y_i}\right) \Rightarrow -\sum_{i=1}^{n}\ln\left(\frac{y_i}{\beta_i x_i + y_i}\right) \Rightarrow$$

$$g(x,y) = \sum_{i=1}^{n}\ln\left(\frac{\beta_i x_i + y_i}{y_i}\right). \quad (2.17)$$

Let's consider function:

$$h(x,y) = \ln\left(a\frac{x}{y} + 1\right), \ x,y,c > 0.$$

Its partial derivatives with respect to the variable y are

$$h'_y = -ax[axy + y^2]^{-1}, \ h''_y = ax(ax + 2y)[axy + y^2]^{-2} \geq 0.$$

Consequently, $h(x, y)$ is convex and the function (2.17) is convex in y (sum of convex functions is convex). Similarly, it is easy to show that the function (2.17) is concave in x. Therefore, there exists a solution to the game in pure strategies (see [Vasin, Morozov, 2005]), and the value of the game is equal to the upper and lower costs of the game (which are the same):

$$v = \underline{v} \max_{x\in X} \min_{y\in Y} g(x,y) = \min_{y\in Y} \max_{x\in X} g(x,y) = \overline{v}.$$

To find the game solution, we compose two Lagrange functions, find their derivatives and set them equal to zero:

$$L(x,\lambda) = -\sum_i \ln\left(\beta_i \frac{x_i}{y_i}+1\right)+\lambda\left(\sum_i x_i - r_x\right),\ L(y,\mu) = \sum_i \ln\left(\beta_i \frac{x_i}{y_i}+1\right)+\mu\left(\sum_i y_i - r_y\right),$$

$$\frac{\partial L(x,\lambda)}{\partial x_i} = -\frac{\beta_i}{\beta_i x_i + y_i}+\lambda = 0,\quad i = 1, \ldots, n, \tag{2.18}$$

$$\frac{\partial L(y,\mu)}{\partial y_i} = -\frac{\beta_i x_i}{y_i(\beta_i x_i + y_i)}+\mu = 0,\quad i = 1, \ldots, n. \tag{2.19}$$

Let's transfer λ and μ to the right parts of the equations and divide (2.19) by (2.18):

$$\frac{x_i}{y_i} = \frac{\mu}{\lambda}. \tag{2.20}$$

With the restrictions $\sum_{i=1}^{n} x_i = r_x$ and $\sum_{i=1}^{n} y_i = r_y$ from (2.20) we have:

$$\frac{r_x}{r_y} = \frac{\mu}{\lambda}. \tag{2.21}$$

Then from (2.18) and (2.21) we find

$$x_i^0 = \frac{\beta_i r_x}{S(\beta_i r_x + r_y)},\ i = 1, \ldots, n,\ S = \sum_{k=1}^{n} \frac{\beta_k}{\beta_k r_x + r_y}, \tag{2.22}$$

$$y_i^0 = \frac{\beta_i r_y}{S(\beta_i r_x + r_y)},\ i = 1, \ldots, n. \tag{2.23}$$

The game value equal to:

$$v = 1 - \prod_{i=1}^{n} \frac{r_y}{\beta_i r_x + r_y}. \tag{2.24}$$

2.5. Optimal distribution of forces and facilities when breaking through defense points (criterion — weighted average probability of a breakthrough)

Let defense points be characterized by values $V_i > 0$, $i = 1, \ldots, n$, then the goal functions of the sides at the tactical level ($m = 1$) will have the form (weighted average values of the probabilities of capturing/breakthrough points):

$$f_x(x,y) = \sum_{i=1}^{n} V_i \frac{\beta_i x_i}{\beta_i x_i + y_i},\quad \sum_{i=1}^{n} x_i = r_x, \tag{2.25}$$

$$f_y(x,y) = \sum_{i=1}^{n} V_i \frac{y_i}{\beta_i x_i + y_i},\quad \sum_{i=1}^{n} y_i = r_y \tag{2.26}$$

(a game with constant sum).

In the antagonistic game (2.25), (2.26), the optimal strategies of the sides are equal to (see the probabilistic model [Novikov, 2012, pp. 33–36])

$$x_i^0 = \frac{V_i \beta_i r_x}{S(\beta_i r_x + r_y)^2} = \frac{V_i \beta_i r_x}{(\beta_i r_x + r_y)^2 \sum_{i=1}^{n} \frac{V_i \beta_i}{(\beta_i r_x + r_y)^2}},\ i = 1, \ldots, n, \tag{2.27}$$

$$y_i^0 = \frac{V_i\beta_i r_y}{S(\beta_i r_x + r_y)^2} = \frac{V_i\beta_i r_y}{(\beta_i r_x + r_y)^2 \sum_{i=1}^{n} \frac{V_i\beta_i}{(\beta_i r_x + r_y)^2}}, \quad i = 1, \ldots, n. \tag{2.28}$$

In an equilibrium situation, the values of the goal functions are equal to:

$$f_x(x^0, y^0) = \sum_{i=1}^{n} V_i \frac{\beta_i r_x}{\beta_i r_x + r_y}, \quad f_y(x^0, y^0) = \sum_{i=1}^{n} V_i \frac{r_y}{\beta_i r_x + r_y}. \tag{2.29}$$

Example 2.1. Let $r_x = 200$, $r_y = 100$, $\beta_1 = 1$, $\beta_2 = 0{,}5$, $\beta_3 = 0{,}5$, $V_1 = V_2 = V_3 = 1/3$, $n = 3$. Let’s find the optimal strategies of the sides and the game value according to three criteria for breaking through points of defense.

The calculation results are presented in the form of a table:

Criteria	Breaking through the weakest point (2.11)	Breaking through at least one point (2.16)	Weighted average probability of a breakthrough (2.25), (2.26)
Offensive optimal strategy (player O)	Probabilities of choosing a point to strike with all forces: 0,5; 0,25; 0,25	Units distribution on defense points: 80; 60; 60	Units distribution on defense points: 61,5; 69,2; 69,2
Defense optimal strategy (player D)	Units distribution on defense points: 50; 25; 25	Units distribution on defense points: 40; 30; 30	Units distribution on defense points: 30,8; 34,6; 34,6
The value of a game	0,8	0,92	0,56; 0,44

In the example considered, it is expedient for the offensive at the stage of preparation for combat to be guided by the criterion of breaking through at least one point of defense.

Assuming that the defense points are homogeneous ($\beta = \beta_1 = \beta_2 = \ldots = \beta_n$), we compare the values of the game by the criterion of breaking through the weakest point v_1 and the criterion of breaking through at least one point v_2.

$$v_1 = 1 - \frac{r_y}{n\beta r_x + r_y} = 1 - s_1, \quad v_2 = 1 - \left(\frac{r_y}{\beta r_x + r_y}\right)^n = 1 - s_2,$$

$$\frac{s_1}{s_2} = \frac{(\beta r_x / r_y + 1)^n}{n\beta r_x / r_y + 1}.$$

It follows from the last expression that the criterion of breaking through at least one point becomes obvious choice for the offensive with an increase in the number of defense points n.

2.6. Optimal distribution of forces and facilities between the first-phase and subsequent objectives

The criterion of offensive in the «offensive – defense» model can be formulated as follows: maximizing the probability of breaking through defense points (the first-phase objective) and capturing a goal/object in the depths of the defense (destroying enemy reserves — the subsequent objective).

If both sides, when solving the first-phase objective, are guided by the criterion of breaking through the weakest point of defense, then we have at the tactical level ($m = 1$) the following goal function of player O:

$$F(u,w) = \frac{B(R_x - u)}{B(R_x - u) + (R_y - w)} \times \frac{\delta u}{\delta u + w}, \quad B = \sum_{j=1}^{n} \beta_j, \tag{2.30}$$

where δ is the parameter of the combat superiority of the offensive in solving their subsequent objective. The first multiplier represents the solution of the first-phase objective, the second — the next one.

It has been proved [Shumov, Korepanov, 2021] that it is expedient for the sides to use pure strategies:

$$u^0 = R_x D, \quad w^0 = R_y D, \quad D = \frac{R_y + BR_x}{2R_y + (B+\delta)R_x} = \frac{1 + BR_x / R_y}{2 + (B+\delta)R_x / R_y}. \tag{2.31}$$

Informatively, the parameter D value is the proportion of troops assigned to the second echelon (reserve). This share essentially depends on the value of the parameter δ and, to a lesser extent, on the value of the parameter B and the ratio of the resources of the parties.

If all defense points are homogeneous ($\beta = \beta_1 = \beta_2 = \ldots = \beta_n$), then the parameter D is equal to:

$$D = \frac{R_y + n\beta R_x}{2R_y + (n\beta + \delta)R_x}.$$

These dependencies for the distribution of combat units of the defending side by objectives (echelons) correspond to the views of US military experts in the preparation and conduct of defensive operations. In particular, when the defenders are not conceding to the offensive in mobility then a hastily taken up defense is organized with a significant part of the forces and facilities (up to two-thirds) in the second echelon (reserve) in order to defeat the wedged enemy during counterattacks. Positional defense is based on the firm holding for a certain time of defensive positions prepared in advance in engineering terms, the maximum use of fire weapons support, and the location of the main forces and facilities in the base area of defense formation.

The sides may use not the criterion of the maximization (minimization) of the probability by solving the first and second tasks (2.30), but the criterion of achieving a guaranteed result:

$$F(u,w) = \min\left(\frac{B(R_x - u)}{B(R_x - u) + R_y - w}; \frac{\delta u}{\delta u + w} \right), \quad 0 \le u \le R_x, \quad 0 \le w \le R_y, \quad B > \delta \tag{2.32}$$

(the offensive estimates the probability of breaking through the weakest point of defense and the probability of completing the subsequent task and take as a criterion the minimum value to be maximized). Accordingly, the goal of the defenders can be evaluated by criterion $1 - F(u, w)$.

It has been proved [Shumov, Korepanov, 2021] that in an antagonistic game at the tactical level with the goal function (2.32), the optimal amount of forces and facilities allocated by the offensive to solve the subsequent objective is equal to:

$$u^0 = \frac{B}{B+\delta} R_x. \tag{2.33}$$

The share of forces and facilities allocated by the offensive to solve the subsequent objective is determined by the value of the parameter B of the combat superiority of the offensive when breaking through defense points and the parameter δ of the combat superiority of the offensive in the depth of the enemy's defense (when repelling his counterattack). Accordingly, the optimal amount of forces and facilities allocated for solving the first-phase objective is $R_x - u^0$.

The optimal mixed strategy for the defenders is as follows. Defenders with probability $\frac{\delta}{B+\delta}$ distribute all forces and facilities on the second line of defense, and with probability $\frac{B}{B+\delta}$ — on the first line. In this case, the optimal value of the game is equal to:

$$v = \frac{\delta B R_x}{\delta B R_x + (B + \delta) R_y} . \tag{2.34}$$

With homogeneity of defense points ($\beta = \beta_1 = \beta_2 = \ldots = \beta_n$), the value of the game is equal to

$$v = \frac{\delta n \beta R_x}{\delta n \beta R_x + (n\beta + \delta) R_y} .$$

Let's consider a meaningful interpretation of the problem. Speaking about the results of the Warsaw – Poznan operation of the troops of the 1st Belorussian Front, G. K. Zhukov noted that the enemy was able to determine the time and direction of the main attack, and there was no full guarantee of achieving operational-tactical surprise, so he, as a commander, assume for the worst case and also count for the worst. «What could the enemy do when he figured out our plan? He could leave in the first echelon of defense, that is, on his front line, reinforced cover, heavy machine guns, manual automatic weapons, individual cannons, and even deploy tanks. Any reconnaissance that we would conduct, he would reject and thereby create the impression that he is sitting firmly here. In the depths of the defense, the enemy could place decoys, have standby equipment, maneuvering through the trenches he could create the impression that positions adjacent to the front edge to a depth of 2–3 km were living and not only living, but also shooting. But he could keep the main forces 5–6 km from the front line. Having finally lost 5–6 km of territory from our first strike and forcing us to shoot artillery supplies, he would have succeeded in disrupting our operation» [Rech, 1985].

To achieve the success of operation, G. K. Zhukov proposed and implemented a plan for a false attack: «So, the strength of the artillery strike, the strength of the attack should not cause any suspicion in the enemy, and if it turns out that the enemy will be taken by surprise, falter and cannot withstand this strike, we will use this success, we will immediately go on the attack with all our forces and will carry out our general/main plan, that is, we will conduct a general attack/offense. Let's assume that the enemy still went for deception and cleared the territory for 3–5 km, gave our first echelon of attack an opportunity to approach the true front line, and then it would be stopped and the attack would have bogged down. In this case, a maximum of 1–1,5 hours after the transfer of the relevant commands and orders, we could proceed to the plan for the implementation of the artillery preparation of the general attack. Artillery from the main positions, without making any movements, because the artillery was placed so close to the front line (divisional artillery was located 700–1000 meters from the front line), could perform the tasks of artillery preparation» [Rech, 1985].

3. Reflexion and information control in the «offensive – defense» model

One of the fundamental properties of the decision-making process is the existence of the natural («objective») reality, and its reflexion in the mind. At the same time, in many cases, there is an inevitable gap, a mismatch between the natural reality and its image in the mind. Purposeful study of this phenomenon is traditionally associated with the term «reflexion».

As known, a fully informed game in normal form is defined by enumeration of the set of players, the sets of their admissible actions, and the set of their goal functions. However, the essential question is: does any of players know this description itself? For a long time in game theory, the «default» assumption was that the game is known to all its participants and, moreover, it is *common knowledge* among the participants. This technical term, the common knowledge, was introduced by the philosopher David Lewis [Lewis, 1969], and into game theory by Robert Aumann [Aumann, 1976] to denote a fact that all players know about, and all players know that it is known to all players, etc.

It is clear that the game is not always a common knowledge. For modeling such situations, the concept of a *reflexive game* has been introduced (see [Novikov, Chkhartishvili, 2012]). In contrast to the game with common knowledge, the goal functions of the players in the reflexive game depend (aside from the set of actions of the players) on an *uncertain parameter*, also called the state of nature. Each of the players may have their own belief of what state of nature has been happening. Further, each player can have his own belief of opponents' beliefs, beliefs about beliefs, and so on. The complex of all these representations forms the structure of awareness of the game. Thus, the description of a reflexive game differs from the description of a «regular» normal form game by the presence of the awareness structure (see [Novikov, Chkhartishvili, 2012; Fedyanin, 2020]).

If one of sides has the opportunity to form one or another awareness structure, we are dealing with the task of information control (see [Novikov, Chkhartishvili, 2012]) — the purposeful formation of beliefs that are beneficial to the control subject. In this section, examples of reflexive games and information control problems in the offensive – defense model at the tactical level ($m = 1$) with various uncertain parameters will be considered.

3.1. Reflexion and information control in the problem of breaking through defense points (criterion — breaking through at least one point)

Let's consider an example of a reflexive game of breaking through defense points at the tactical level with the criterion of breaking through at least one point (section 2.4), in which the total amount of the offensive side's facilities is an uncertain parameter. Let the defense side — player D — believe that this parameter takes the value ρ_x, while the true value is r_x, and all this is known to the offensive side — player O. Such an awareness structure can be represented as a graph with three nodes (see Fig. 5), where DO denotes player O from the point of view of player D, and the arrows indicate that the players are adequately informed about each other. The DO player is phantom, i. e. it exists only in the mind of player D and does not coincide with the real player O (player DO believe that the value of the uncertain parameter is ρ_x).

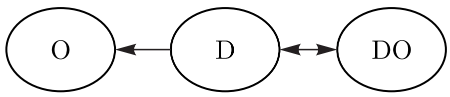


Fig. 5. The Graph of a reflexive game with an informed offensive side

Thus, subjectively, player D plays an antagonistic game with goal function

$$G(x,y)=1-\prod_{i=1}^{n}\left(\frac{y_i}{\beta_i x_i+y_i}\right),\quad \sum_{i=1}^{n}x_i=\rho_x,\quad \sum_{i=1}^{n}y_i=r_y$$

and own strategy $y = (y_1, \ldots, y_n)$. As shown in Section 2.4, the optimal strategy of player D in this game has the following form:

$$y_i^0(\rho_x)=\frac{\beta_i r_y}{\tilde{S}(\beta_i\rho_x+r_y)},\ i = 1, \ldots, n,\ \tilde{S}=\sum_{k=1}^{n}\frac{\beta_k}{\beta_k\rho_x+r_y}. \tag{3.1}$$

The player O uses the criterion specified by the goal function

$$G(x,y)=1-\prod_{i=1}^{n}\left(\frac{y_i}{\beta_i x_i+y_i}\right),\quad \sum_{i=1}^{n}x_i=r_x,\quad \sum_{i=1}^{n}y_i=r_y. \tag{3.2}$$

Let's find the optimal strategy of player O $x = (x_1, \ldots, x_n)$ for a fixed strategy (3.1) of player D. To do this, we use equations (2.18), which we will write in the following form:

$$x_i = \frac{1}{\lambda} - \frac{y_i}{\beta_i}, \quad i = 1, \ldots, n. \tag{3.3}$$

Summing up equations (3.3) and using the relation $\sum_{i=1}^{n} x_i = r_x$, we obtain

$$\frac{1}{\lambda} = \frac{1}{n}\left(r_x + \sum_{i=1}^{n} \frac{y_i}{\beta_i} \right),$$

from what, taking into account the same relations (3.3), it follows that

$$x_i = \frac{1}{n}\left(r_x + \sum_{i=1}^{n} \frac{y_i}{\beta_i} \right) - \frac{y_i}{\beta_i}, \quad i = 1, \ldots, n.$$

Substituting the value $y_i^0(\rho_x)$ from (3.1) instead of y_i in the last expression, we get the optimal strategy for O:

$$x_i^0 = \frac{1}{n}\left(r_x + \sum_{i=1}^{n} \frac{r_y}{\tilde{S}(\beta_i \rho_x + r_y)} \right) - \frac{r_y}{\tilde{S}(\beta_i \rho_x + r_y)}, \quad i = 1, \ldots, n. \tag{3.4}$$

Now let’s assume that player O has possibility to form any belief of player D about the value of an uncertain parameter within the set R. Then the answer to the question of what kind of belief it is beneficial for him to form will be the solution of the following optimization problem:

$$G(x^0, y^0(\rho_x)) \xrightarrow[\rho_x \in R]{} \max .$$

Considering (3.2), (3.1), and (3.4), this problem can be written as follows:

$$v(\rho_x) = 1 - \frac{(nr_y)^n}{\left(\sum_{i=1}^{n} \frac{r_x \beta_i + r_y}{\rho_x \beta_i + r_y} \right)^n \prod_{i=1}^{n} \left(\beta_i \rho_x + r_y \right)} \xrightarrow[\rho_x \in R]{} \max . \tag{3.5}$$

The solution of the problem (3.5) describes the following assertion.

Statement 3.1. If the superiority parameters for each defend point are the same, i. e. condition

$$\beta_1 = \ldots = \beta_n , \tag{3.6}$$

hold, then the function $v(\rho_x)$ is a constant. If condition (3.6) is not satisfied, the function $v(\rho_x)$ strictly decreases on the interval $0 < \rho_x < r_x$ and strictly increases for $\rho_x > r_x$.

Proof. With the help of direct substitution, it is easy to verify that when condition (3.6) is satisfied, the function $v(\rho_x)$ is a constant, so below we will assume that this condition is not satisfied.

Further, it is easy to see that the intervals of decrease and increase of the function $v(\rho_x)$ coincide with the intervals of decrease and increase, respectively, of the function

$$\left(\sum_{i=1}^{n} \frac{r_x \beta_i + r_y}{\rho_x \beta_i + r_y} \right)^n \prod_{i=1}^{n} \left(\beta_i \rho_x + r_y \right). \tag{3.7}$$

Let’s introduce notations

$$\gamma_i = r_x + \frac{r_y}{\beta_i}, \quad \Delta = \rho_x - r_x ,$$

with the help of which we write the function (3.7) in the following form:

$$\varphi(\Delta)=\left(\prod_{i=1}^{n}\beta_i\right)\left(\sum_{i=1}^{n}\frac{\gamma_i}{\gamma_i+\Delta}\right)^{n}\prod_{i=1}^{n}(\gamma_i+\Delta),\quad \Delta>-r_x .$$

The function $\varphi(\Delta)$ is continuously differentiable on the set under consideration, therefore the intervals of its monotonicity are determined by the sign of the derivative. Let's find the derivative:

$$\varphi'(\Delta)=\left(\prod_{i=1}^{n}\beta_i\right)\left(\sum_{i=1}^{n}\frac{\gamma_i}{\gamma_i+\Delta}\right)^{n-1}\prod_{i=1}^{n}(\gamma_i+\Delta)\left[-n\sum_{i=1}^{n}\frac{\gamma_i}{(\gamma_i+\Delta)^2}+\left(\sum_{i=1}^{n}\frac{\gamma_i}{\gamma_i+\Delta}\right)\left(\sum_{i=1}^{n}\frac{1}{\gamma_i+\Delta}\right)\right].$$

It is easy to verify that

$$\varphi'(\Delta)=0 \text{ при } \Delta = 0.$$

To complete the proof, it suffices to show that the expression

$$Z=-n\sum_{i=1}^{n}\frac{\gamma_i}{(\gamma_i+\Delta)^2}+\left(\sum_{i=1}^{n}\frac{\gamma_i}{\gamma_i+\Delta}\right)\left(\sum_{i=1}^{n}\frac{1}{\gamma_i+\Delta}\right) \tag{3.8}$$

takes negative values for $-r_x<\Delta<0$ and positive values for $\Delta>0$.

Let's use a well-known algebraic inequality (see, for example, [Radzivilovskii, 2006]): if the sets of numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are ordered in the same way, then the inequality holds:

$$n\,(a_1b_1+a_2b_2+\ldots+a_nb_n)\geq(a_1+a_2+\ldots+a_n)\,(b_1+b_2+\ldots+b_n), \tag{3.9}$$

if they are in reverse order, the inequality with the opposite sign is satisfied; moreover, if not all numbers $a_1, a_2, \ldots, a_n$ are equal to each other and simultaneously not all numbers $b_1, b_2, \ldots, b_n$ are equal to each other, then the inequality is strict.

Let's introduce notations

$$a_i=\frac{\gamma_i}{\gamma_i+\Delta},\quad b_i=\frac{1}{\gamma_i+\Delta},\quad i=1,\ldots,n .$$

Then the expression (3.8) can be written as follows:

$$Z=-n\,(a_1b_1+a_2b_2+\ldots+a_nb_n)+(a_1+a_2+\ldots+a_n)\,(b_1+b_2+\ldots+b_n).$$

Let us order the quantities γ_i, $i = 1, \ldots, n$. For $\Delta<0$ a_i and b_i are monotonically decreasing functions of γ_i on the interval $\gamma_i>r_x$, so not all a_i are equal, not all b_i are equal, a_i and b_i are equally ordered. Therefore, for $\Delta<0$, the inequality $Z<0$ is true.

Similarly, for $\Delta>0$ a_i is a monotonically increasing function of γ_i, and b_i is a monotonically decreasing function of γ_i, so not all a_i are equal to each other, not all b_i are equal to each other, a_i and b_i are inversely ordered. Therefore, for $\Delta>0$, the inequality $Z>0$ is true.

Corollary. Let the set R be a closed interval. Then the maximum of the function (3.5) is reached on one of the closed interval endpoints.

Example 3.1. Let $n = 3$, $r_x = 200$, $r_y = 100$, $\beta_1 = 1$, $\beta_2 = \beta_3 = 0{,}05$, $R = [20, 2000]$. Substituting r_x and endpoints values of the closed interval R into the formula (3.5), we obtain:

$$\nu(200)\approx 0{,}72,\quad \nu(20)\approx 0{,}78,\quad \nu(2000)\approx 0{,}83.$$

When $\beta_1 = 1$, $\beta_2 = 0{,}5$, $\beta_3 = 0{,}2$ we get:

$$\nu(200)\approx 0{,}88,\quad \nu(20)\approx 0{,}89,\quad \nu(2000)\approx 0{,}89.$$

Thus, the optimal information control by the side of player O is , and it allows increasing the probability of breaking through defense points by 0,11.

On fig. 6 a graph of the probability of breaking through points for values of ρ_x from 20 to 2000 and for two defense conditions (vector values $\beta = (\beta_1, \beta_2, \beta_3)$) is shown.

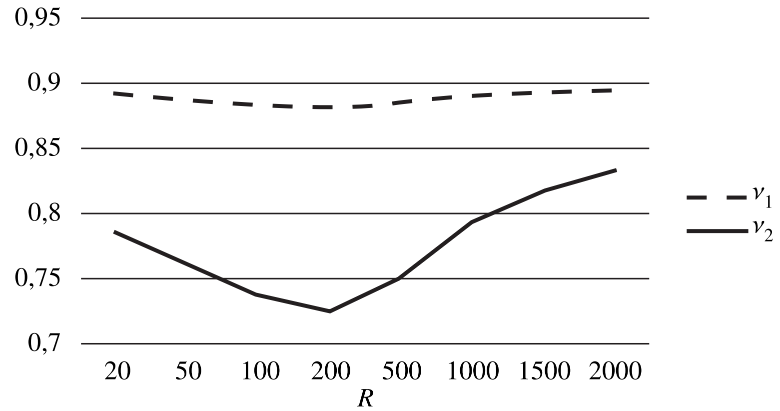


Fig. 6. The graph of the probability of breaking through defense points (vertical axis) depending on beliefs of the defender about the number of attackers (horizontal axis): v_1 at $\beta_1 = 1$, $\beta_2 = \beta_3 = 0{,}05$; v_2 at $\beta_1 = 1$, $\beta_2 = 0{,}5$, $\beta_3 = 0{,}2$

It can be seen from the figure that it is advisable to form false enemy (defender) beliefs about the number of our combat units when the possibilities for breaking through defense points (values of the vector β) are significantly different.

3.2. Reflexion and information control in the problem of distributing forces and facilities between the first-phase and subsequent objectives

Consider an example of a reflexive game of breaking through defense points at the tactical level in a problem in which the choice of players is the distribution of resources between two directions of their use — breaking through the defense and capturing an object in the depths of the defense (section 2.5). The uncertain parameter is the total amount of facilities of the defending side. Let the offensive side — player O — believes that this parameter takes the value ρ_y, while the true value is R_y, and the defense side — player D, knows about this. Such an awareness structure can be represented as a graph with three vertices (see Fig. 7), on which player D from the point of view of player O is denoted by OD, and the arrows indicate adequate awareness of the players about each other.

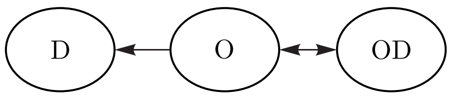


Fig. 7. Graph of a reflexive game with an informed defense side

Thus, subjectively, player O plays an antagonistic game with goal function

$$F(u,w) = \frac{B(R_x - u)}{B(R_x - u) + (\rho_y - w)} \times \frac{\delta u}{\delta u + w}, \quad B = \sum_{j=1}^{n} \beta_j ,$$

and own strategy u. It is known (see Section 2.5) that the optimal strategy of player O in this game has the following form:

$$u^0(\rho_y) = R_x \tilde{D}, \quad \tilde{D} = \frac{\rho_y + BR_x}{2\rho_y + (B + \delta)R_x}. \tag{3.10}$$

Player D, being informed about it, solves the problem of minimizing the goal function

$$F(u^0, w) = \frac{B(R_x - u^0)}{B(R_x - u^0) + (\rho_y - w)} \times \frac{\delta u^0}{\delta u^0 + w}, \quad B = \sum_{j=1}^{n} \beta_j, \tag{3.11}$$

by choosing its strategy w. It is easy to see that the optimal strategy of player O which minimizes the function (3.11) has the following form:

$$w^0 = \frac{BR_x + R_y - (B+\delta)R_x \tilde{D}}{2}. \tag{3.12}$$

Now suppose that player D has possibility to form any belief of player O about the value of the uncertain parameter within the set R. Then the answer to the question of what value of belief it is beneficial for D to form is the solution of the following optimization problem:

$$F(u^0(\rho_y), w^0) \xrightarrow[\rho_y \in R]{} \min.$$

Based on relations (3.10)–(3.12), this problem can be written, after simple transformations, as follows:

$$\psi(\rho_y) = \frac{4B\delta(1 - \tilde{D}(\rho_y))\tilde{D}(\rho_y)}{[B + R_y / R_x + (\delta - B)\tilde{D}(\rho_y)]^2} \xrightarrow[\rho_y \in R]{} \min. \tag{3.13}$$

The function minimized in (3.13) is rational function and allows one to study intervals of increase and decrease in the usual way (by comparing the sign of the derivative with zero). It is easy to verify the validity of the following statement.

Statement 3.2. If the sum of the superiority parameters coincides with the combat superiority parameter of the offensive when he solves the subsequent objective, i.e. condition is met

$$B = \delta, \tag{3.14}$$

then the function (3.13) is a constant. If condition (3.14) is not satisfied, the function (3.13) strictly increases on the interval $0 < \rho_y < R_y$ and strictly decreases for $\rho_y > R_y$.

Corollary. Let the set R be a closed interval. Then one of the closed interval endpoints gives the minimum of the function (3.13).

Example 3.2. Let $n = 3$, $R_x = 400$, $R_y = 300$, $B = 10$, $\delta = 1$, $R = [30, 3000]$. Substituting R_y and endpoints of closed interval R in expression (3.13), we get:

$$\psi(300) \approx 0{,}53, \quad \psi(30) \approx 0{,}51, \quad \psi(3000) \approx 0{,}40.$$

Thus, the optimal informational control of player D is $\rho_y = 3000$, and it makes possible to reduce the probability of capturing an object by player O in the defense depth (the probability of solving the first-phase and subsequent objective) by 0,13.

4. Conclusion and perspectives

Thus, the game-theoretic models of «offensive – defense» were studied, in which the sides solve the first-phase and subsequent objectives, having the formation of troops in one or two echelons.

The function of victory at an object (point, area, strip) used in the model has the property of self-similarity and makes it possible to model the actions of troops at the tactical, operational, and strategic levels.

At the first stage of modeling, the solution of the first-phase objective is found — a breakthrough (holding) of defense points, at the second — the solution of the subsequent objective — the defeat of the enemy in the depths of the defense (counterattack and restoration of defense).

It is assumed that the duration of the action cycles of the sides is approximately the same, which gives reason to use the Nash equilibrium (the sides make decisions simultaneously and independently). For each method of breaking through (holding) defense points we assign a goal function, and for the tactical level, solutions were found for the corresponding antagonistic games (Table 2).

Table 2. Solutions to antagonistic games (breaking through defense points, tactical level)

Offensive (O)	Defense (D)		
	Breakthrough the weakest point	Breakthrough at least one point	Average weighted probability
Breakthrough the weakest point	**(11)** **O — mixed, D — pure**	(12)	(13)
Breakthrough at least one point	(21)	**(22)** **O & D — pure**	(23)
Average weighted probability	(31)	(32)	**(33)** **O & D — pure**

Abbreviations: mixed — mixed strategies, pure — pure strategies.

It is shown that it is rational for the offensive side to use the criterion of «breakthrough at least one point», which, other things being equal, ensures the maximum probability of breaking through the defense points.

A promising direction of the research is the solution of non-antagonistic games (problems 12, 13, 21, 23, 31 and 32 in table 2), in which sides use different criteria of breaking through (holding) defense points.

The accepted assumption about the same duration of the cycles of actions of the sides (and using the Nash equilibrium), firstly, allows us to use found solutions to simulate counter fight that occurs during a march, offensive and defense, and secondly, excludes from the analysis, for example, well-prepared defense, when it is rational to use a hierarchical game for its formalization (defender makes the first move).

At the second stage of modeling for a particular case (the sides, when breaking through and holding defense points, are guided by the criterion of breaking through the weakest point, tactical level), the problem of distributing forces and facilities between tactical tasks (echelons) is solved:

– the first criterion is a product of the probability of breaking through the defense line and the probability of defeating the enemy in the depth of the defense; antagonistic game, the sides use pure strategies. The share of troops allocated to solve the subsequent task, firstly, does not depend much on the initial strengths of the sides and the values of superiority parameters at defense points, and secondly, decreases with an increase in the value of the superiority parameter of the offensive in the depth of defense;

– the second criterion (a guaranteed result) is the minimum value of the probability of breaking through the defense point and the probability of defeating the enemy in the depth of the defense; an antagonistic game, the offensive uses pure strategy, the defense — a mixed one, distributing all their facilities in the first line or the second (in the depth of defense).

A promising direction of research is the solution of game-theoretic problems of resource distribution between the first-phase and subsequent objectives with various methods of breaking through points and defeating in the depths of defense, and at all levels (tactical, operational and strategic).

An important aspect of combat operations is awareness, including mutual awareness. Therefore, the side that can influence the opponent's beliefs can gain an advantage. In this paper, we consider two examples of reflexive games (games characterized by complex mutual awareness) and the implementation of informational control. It is shown under what conditions the informational control increases the player's payoff, and the optimal informational control is found. A promising direction for further research is the consideration of more complex structures of players' awareness and the corresponding tasks of informational control.

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