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On the stability of the gravitational system of many bodies

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In this paper, a gravitational system is understood as a set of point bodies that interact according to Newton's law of attraction and have a negative value of the total energy. The question of the stability (nonstability) of a gravitational system of general position is discussed by direct computational experiment. A gravitational system of general position is a system in which the masses, initial positions, and velocities of bodies are chosen randomly from given ranges. A new method for the numerical solution of ordinary differential equations at large time intervals has been developed for the computational experiment. The proposed method allowed, on the one hand, to ensure the fulfillment of all conservation laws by a suitable correction of solutions, on the other hand, to use standard methods for the numerical solution of systems of differential equations of low approximation order. Within the framework of this method, the trajectory of a gravitational system in phase space is assembled from parts, the duration of each of which can be macroscopic. The constructed trajectory, generally speaking, is discontinuous, and the points of joining of individual pieces of the trajectory act as branch points. In connection with the latter circumstance, the proposed method, in part, can be attributed to the class of Monte Carlo methods. The general conclusion of a series of computational experiments has shown that gravitational systems of general position with a number of bodies of 3 or more, generally speaking, are unstable. In the framework of the proposed method, special cases of zero-equal angular momentum of a gravitational system with a number of bodies of 3 or more, as well as the problem of motion of two bodies, are specially considered. The case of numerical modeling of the dynamics of the solar system in time is considered separately. From the standpoint of computational experiments based on analytical methods, as well as direct numerical methods of high-order approximation (10 and higher), the stability of the solar system was previously demonstrated at an interval of five billion years or more. Due to the limitations on the available computational resources, the stability of the dynamics of the planets of the solar system within the framework of the proposed method was confirmed for a period of ten million years. With the help of a computational experiment, one of the possible scenarios for the disintegration of the solar systems is also considered.

Keywords: numerical methods, ordinary differential equations, Monte Carlo method

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МОДЕЛИ В ФИЗИКЕ И ТЕХНОЛОГИИ

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Об устойчивости гравитационной системы многих тел

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В работе под гравитационной системой понимается множество точечных тел, взаимодействующих согласно закону притяжения Ньютона и имеющих отрицательное значение полной энергии. Обсуждается вопрос об устойчивости (о неустойчивости) гравитационной системы общего положения путем прямого вычислительного эксперимента. Под гравитационной системой общего положения понимается система, у которой массы, начальные позиции и скорости тел выбираются случайными из заданных диапазонов. Для проведения вычислительного эксперимента разработан новый метод численного решения обыкновенных дифференциальных уравнений на больших интервалах времени. Предложенный метод позволил, с одной стороны, обеспечить выполнение всех законов сохранения путем подходящей коррекции решений, с другой — использовать стандартные методы численного решения систем дифференциальных уравнений невысокого порядка аппроксимации. В рамках указанного метода траектория движения гравитационной системы в фазовом пространстве собирается из частей, длительность каждой из которых может быть макроскопической. Построенная траектория, вообще говоря, является разрывной, а точки стыковки отдельных кусков траектории выступают как точки ветвления. В связи с последним обстоятельством предложенный метод отчасти можно отнести к классу методов Монте-Карло. Общий вывод проведенной серии вычислительных экспериментов показал, что гравитационные системы общего положения с числом тел 3 и более, вообще говоря, неустойчивы. В рамках предложенного метода специально рассмотрены частные случаи равенства нулю момента импульса гравитационной системы с числом тел 3 и более, а также задача движения двух тел. Отдельно рассмотрен случай численного моделирования динамики во времени Солнечной системы. С позиций вычислительного эксперимента на базе аналитических методов, а также прямых численных методов высокого порядка аппроксимации (10 и выше) устойчивость Солнечной системы ранее продемонстрирована на интервале в пять и более миллиардов лет. В силу ограничений на имеющиеся вычислительные ресурсы устойчивость динамики планет Солнечной системы в рамках использования предлагаемого метода удалось подтвердить на срок десять миллионов лет. С помощью вычислительного эксперимента рассмотрен также один из возможных сценариев распада Солнечной системы.

Ключевые слова: численные методы, обыкновенные дифференциальные уравнения, метод Монте-Карло

1. Introduction

The present article studies the stability of a gravitational system made up of multiple bodies. Here, the term "gravitational system" refers to a set of bodies whose total energy is negative. In order to study the stability of such systems, a new numerical method was developed for solving ordinary differential equations that describe the dynamics of multiple bodies interacting according to Newton's gravitational law.

The issue involving the (in)stability of gravitational systems has been studied by Newton, Laplace, Euler, Lagrange, and several other researchers for a long time using the Solar System dynamics as an example. While classical analytical solutions indicated stability and almost periodic motion, numerical and numerical-analytical solutions suggested the chaotic motion of planets [Kholshevnikov, Kuznetsov, 2007]. The concept of dynamic chaos in relation to the Solar System [Sussman, Wisdom, 1992] enabled a systematic study of the chaotic motion of planets in the long-term and secular perspective [Laskar 1996]. An inextricable link between chaos and the dynamics of solar planets, as well as other infinitesimal bodies, was discovered [Rezonansy …, 2006]. The focus in studying the dynamics of planets and other Solar System bodies was determined. On the one hand, it was essential for practice to develop a digital, highly deterministic short-term behavior pattern of the Solar System [Pit'eva et al., 2019]; on the other, to study dynamics in the long term calculated in the time of the Solar System's existence, i. e., five or more billion years [Zink et al., 2020].

The main reason for developing the new numerical method is that traditional numerical methods for solving a system of ordinary differential equations [Trenti, Hut, 2008] cannot be used in long-term calculations. This factor is attributable to the fact that sooner or later, the numerical solution "falls apart" due to the failure to conserve the energy of a gravitational system. Thus, researchers developed two ways in order to overcome this difficulty. The first way consists in the developing methods for the numerical solution of the systems of differential equations exhibiting high-order approximations (ten or above) [Aarseth, 2003; Rein, Spiege, 2015], allowing the dynamics to be studied for a period of five or more billion years while ensuring acceptable compliance with the energy conservation law of the gravitational system for the entire integration period. The second way is associated with introducing multiple corrections of numerical solutions to comply with all conservation laws during the numerical calculation period [Nacozy, 1971; Fukushima, 2003]. It will become clear later on that the present work can be referred to the second way.

Henceforth, bodies comprising a gravitational system will be assumed to be non-extended, i. e., point bodies. Here, let us use the dimensionless system of units. As characteristic values of mass, length and time, we adopted the mass of the Sun $M_{\odot} = 1.9855 \cdot 10^{30}$ kg, the distance from the Sun to Neptune, $L = 4.503 \cdot 10^{12}$ m, and time $T = \frac{L^{3/2}}{\gamma^{1/2} M_{\odot}^{1/2}} = 8.3008 \cdot 10^{8}$ s = 26.3217 years, where the gravitational constant is $\gamma = 6.674184 \cdot 10^{-11}$ m^{3}s^{-2} mg^{-1}. In this case, the characteristic value of speed amounts to V = 5.4248 km/s.

Let us write down the dimensionless form of Newton's equations describing the dynamics of the gravitational system comprising N point bodies of mass $m_1, \ldots, m_N$

$$\begin{cases} \dot{\mathbf{r}}_i = \mathbf{v}_i, \\ \dot{\mathbf{v}}_i = -\sum_{j=1, j \neq i}^{N} \frac{m_j}{r_{i,j}^3} \mathbf{r}_{i,j}, \end{cases} \tag{1}$$

where $\mathbf{r}_i = (x_i, y_i, z_i)$, $\mathbf{v}_i = (v_{x,i}, v_{y,i}, v_{z,i})$ — positions and velocities of the i-th body, $\mathbf{r}_{i,j} = \mathbf{r}_i - \mathbf{r}_j$, $r_{i,j} = |\mathbf{r}_{i,j}|$, $i, j = 1, \ldots, N$, the point above the quantities denotes a time derivative.

In order to solve the system of equations (1), one of the standard solvers of the MATLAB environment was applied in an extended time interval [0, 10^8] ($\approx 2.63 \cdot 10^9$ years in dimensional units) primarily to study the implementation of the energy conservation law. Specifically, the ode23 solver was used, implementing the second- and third-order Runge–Kutta method [Bogacki, Shampino, 1989], with relative and absolute accuracy values of 10^{-3} and 10^{-6}, respectively. For the sake of clarity, it was assumed that $N = 15$, $L = 1$, and $V = 1$. The masses, initial positions in space, and velocities were chosen uniformly as random from [0, 1], $[-L, L]^3$, and $[-V, V]^3$, respectively. The total energy was found not to be conserved over time; a noticeable "bounce" was observed with energy going down sharply, then turning positive and reaching a certain plateau. Note that all the other solvers in the MATLAB environment also do not conserve energy in such extensive time intervals.

Let us move to the coordinate system of the center of mass having the following position and velocity: $\mathbf{R} = \frac{1}{M_\Sigma}\sum_{i=1}^{N} m_i \mathbf{r}_i$, $\mathbf{V} = \frac{1}{M_\Sigma}\sum_{i=1}^{N} m_i \mathbf{v}_i$ where $M_\Sigma = \sum_{i=1}^{N} m_i$, $\mathbf{R} = \mathbf{V}t + \mathbf{R}_0$, t is time, $\mathbf{R}_0$ is some fixed vector. In order to describe the bodies of the gravitational system, new coordinates $\mathbf{q}_i, i = 1, \ldots, N$ and velocities $\mathbf{u}_i, i = 1, \ldots, N$ are introduced according to variable substitution: $\mathbf{r}_i = \mathbf{R} + \mathbf{q}_i$, $\mathbf{v}_i = \mathbf{V} + \mathbf{u}_i$, $i = 1, \ldots, N$. Following variable substitution, the system of equations (1) takes the following form

$$\begin{cases} \dot{\mathbf{q}}_i = \mathbf{u}_i, \\ \dot{\mathbf{u}}_i = -\sum_{j=1, j\neq i}^{N} \frac{m_j}{q_{i,j}^3} \mathbf{q}_{i,j}, \end{cases} \tag{2}$$

where $\mathbf{q}_{i,j} = \mathbf{q}_i - \mathbf{q}_j$, $q_{i,j} = |\mathbf{q}_{i,j}|$, $i, j = 1, \ldots, N$. Taking into account the substitution of variables when transitioning to the center of mass system, as well as the law of momentum conservation, the following vector equalities hold

$$\sum_{i=1}^{N} m_i \mathbf{q}_i = \mathbf{0}, \ \sum_{i=1}^{N} m_i \mathbf{u}_i = \mathbf{0}. \tag{3}$$

In addition to equations (3), the laws of angular momentum and energy conservation must be implemented:

$$\mathbf{k}_0 = \sum_{i=1}^{N} m_i \mathbf{q}_i \times \mathbf{u}_i, \tag{4}$$

$$e_0 = \sum_{i=1}^{N} \tfrac{1}{2} m_i \mathbf{u}_i^2 - \tfrac{1}{2}\sum_{i,j=1, i\neq j}^{N} \frac{m_i m_j}{q_{i,j}}. \tag{5}$$

A numerical solution of the system of equations (2) intended to overcome energy non-conservation when using standard solvers combines a special procedure for correcting solutions to comply with conservation laws with a Monte Carlo method.

The concept of "complete conservatism" as applied to the difference schemes of partial differential equations was actively discussed by A. A. Samarskii [Samarskii, 1977]. Note that it is quite simple to construct a finite-difference scheme for the system of Equations (2), implementing the law of momentum and angular momentum conservation. Conversely, it is somewhat problematic to fulfill the energy conservation law at the finite-difference level. Here we should note the so-called symplectic schemes [Wisdom, Holman, 1991; Feng and Qin, 2010], which, under certain restrictions, provide complete conservatism, as well as the Kustaanheim–Stiefel transformation known in celestial mechanics [Stiefel, Scheifele, 1975; Kozlov, 2007], offering a new way of presenting and describing the Keplerian orbits of two interacting bodies.

Next, we will show through the computational experiment that a general-position gravitational system comprising three or more bodies is unstable. Here, general position refers to a situation where the masses of bodies, their initial positions, and velocities are chosen randomly from specific ranges. The well-known Lagrange solutions to the three-body problem and several other solutions indicate that stable gravitational systems exist given a particular choice of masses, initial positions, and velocities.

Taking the work [Arnol'd et al., 1985] into account, let us adopt the following formal definition of the stability of a gravitational system. A gravitational system is stable when $0 < q_{i,j}(t) \le C$ for all points in time starting from some moment, i.e., at $t \ge t_0$ where $i \ne j$; $i, j = 1, \ldots, N$; C — some nonnegative constant, t_0 — the initial moment of time. According to Jacobi's theorem, if the gravitational system is stable in the sense indicated above, the total energy is negative. However, the converse is generally not true. Moreover, even if the total energy is negative, a general-position gravitational system is generally unstable at $N > 2$. In order to illustrate the last statement, let us consider a procedure for generating phase space points via the Monte Carlo method that satisfy all conservation laws.

2. Algorithm for generating phase space points

Let us prepare an algorithm for generating the points of $6N$-dimensional phase space in the form of a Monte Carlo procedure. The proposed algorithm must ensure the implementation of all conservation laws in the form of equations (3)–(5) without any restrictions that could prevent the generation of any possible points of the hypersurface of conservation laws (dimension $6N - 10$).

Let a set of $2N$ three-dimensional vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_N, \mathbf{b}_1, \ldots, \mathbf{b}_N\}$ taken uniformly at random from $[-1, 1]^3$ be prepared through random sampling. In order to implement the transition to the center of mass system, this set of vectors undergoes the following operation

$$\mathbf{A}_i = \mathbf{a}_i - \tfrac{1}{M_\Sigma} \sum_{j=1}^{N} m_j \mathbf{a}_j \,, \ \mathbf{B}_i = \mathbf{b}_i - \tfrac{1}{M_\Sigma} \sum_{j=1}^{N} m_j \mathbf{b}_j \,. \tag{6}$$

Next, let us assume that

$$\mathbf{q}_i = (\lambda_x A_{x,i}, \lambda_y A_{y,i}, \lambda_z A_{z,i}) \,, \ \mathbf{u}_i = (\mu_x B_{x,i}, \mu_y B_{y,i}, \mu_z B_{z,i}) \,, \tag{7}$$

where $i = 1, \ldots, N$; $\lambda_x, \lambda_y, \lambda_z, \mu_x, \mu_y, \mu_z$ — as yet undetermined coefficients. Note that the presentation of the required set of vectors in the form of (6)–(7) fulfills the condition (3), including the momentum conservation law.

By substituting (7) into the law of angular momentum conservation (4), we obtain

$$\mathbf{k}_0 = (c_{11}\lambda_y \mu_z - c_{12}\lambda_z \mu_y, c_{21}\lambda_z \mu_x - c_{22}\lambda_x \mu_z, c_{31}\lambda_x \mu_y - c_{32}\lambda_y \mu_x) \,, \tag{8}$$

where

$$\begin{aligned} c_{11} &= \sum_{i=1}^{N} m_i A_{y,i} B_{z,i} \,, \ c_{12} = \sum_{i=1}^{N} m_i A_{z,i} B_{y,i} \,; \\ c_{21} &= \sum_{i=1}^{N} m_i A_{z,i} B_{x,i} \,, \ c_{22} = \sum_{i=1}^{N} m_i A_{x,i} B_{z,i} \,; \\ c_{31} &= \sum_{i=1}^{N} m_i A_{x,i} B_{y,i} \,, \ c_{32} = \sum_{i=1}^{N} m_i A_{y,i} B_{x,i} \,. \end{aligned} \tag{9}$$

Let us assume that the angular momentum vector $\mathbf{k}_0$ is given. Then, the system of three equations takes the following form according to (8)

$$\begin{cases} c_{11}\lambda_y\mu_z - c_{12}\lambda_z\mu_y = k_{0,x}, \\ c_{21}\lambda_z\mu_x - c_{22}\lambda_x\mu_z = k_{0,y}, \\ c_{31}\lambda_x\mu_y - c_{32}\lambda_y\mu_x = k_{0,z}. \end{cases} \tag{10}$$

Let us solve the system of linear equations (10) for λ_x, λ_y, and λ_z. When the determinant of the system (10) $\Delta = \mu_x\mu_y\mu_z(c_{11}c_{21}c_{31} - c_{12}c_{22}c_{32})$ is nonzero, it has a unique solution. Given (9), direct verification will show that $c = c_{11}c_{21}c_{31} - c_{12}c_{22}c_{32} \equiv 0$ at $N = 2$. In other words, the two-body problem stands alone, requiring special consideration within this approach. In addition, the case where the angular momentum of the gravitational system equals zero, i. e. $\mathbf{k}_0 = \mathbf{0}$, needs special consideration.

Let us write down the solution to the system of linear equations (10) for λ_x, λ_y, and λ_z when the determinant Δ is nonzero

$$\begin{aligned} \lambda_x &= \tfrac{\mu_x}{\Delta}(k_{0,x}\mu_x c_{21}c_{32} + k_{0,y}\mu_y c_{12}c_{32} + k_{0,z}\mu_z c_{11}c_{21}), \\ \lambda_y &= \tfrac{\mu_y}{\Delta}(k_{0,x}\mu_x c_{21}c_{31} + k_{0,y}\mu_y c_{12}c_{31} + k_{0,z}\mu_z c_{12}c_{22}), \\ \lambda_z &= \tfrac{\mu_z}{\Delta}(k_{0,x}\mu_x c_{22}c_{32} + k_{0,y}\mu_y c_{11}c_{31} + k_{0,z}\mu_z c_{11}c_{22}). \end{aligned} \tag{11}$$

Let $\mu_x = V\xi_x$, $\mu_y = V\xi_y$, $\mu_z = V\xi_z$ where λ_x, λ_y, and λ_z are uniformly random numbers from [– 1,1], then

$$\begin{aligned} \lambda_x &= \tfrac{1}{V\xi_y\xi_z c}(k_{0,x}\xi_x c_{21}c_{32} + k_{0,y}\xi_y c_{12}c_{32} + k_{0,z}\xi_z c_{11}c_{21}) = \tfrac{p_x}{V}, \\ \lambda_y &= \tfrac{1}{V\xi_x\xi_z c}(k_{0,x}\xi_x c_{21}c_{31} + k_{0,y}\xi_y c_{12}c_{31} + k_{0,z}\xi_z c_{12}c_{22}) = \tfrac{p_y}{V}, \\ \lambda_z &= \tfrac{1}{V\xi_x\xi_y c}(k_{0,x}\xi_x c_{22}c_{32} + k_{0,y}\xi_y c_{11}c_{31} + k_{0,z}\xi_z c_{11}c_{22}) = \tfrac{p_z}{V}. \end{aligned} \tag{12}$$

Taking (12) into account, let us express the phase space point found using the Monte Carlo method through a single unknown parameter V

$$\mathbf{q}_i = \tfrac{1}{V}(p_x A_{x,i}, p_y A_{y,i}, p_z A_{z,i}),\ \mathbf{u}_i = V(\xi_x B_{x,i}, \xi_y B_{y,i}, \xi_z B_{z,i}),\ i = 1, \ldots, N. \tag{13}$$

By substituting the coordinates of the point (13) into Equation (5), determining the total energy e_0 of the gravitational system, we obtain

$$V^2 \sum_{i=1}^{N} \tfrac{1}{2} m_i (\xi_x^2 B_{x,i}^2 + \xi_y^2 B_{y,i}^2 + \xi_z^2 B_{z,i}^2) - \tfrac{V}{2} \sum_{i,j=1, i\neq j}^{N} \frac{m_i m_j}{\sqrt{p_x^2 A_{x,i,j}^2 + p_y^2 A_{y,i,j}^2 + p_z^2 A_{z,i,j}^2}} - e_0 = 0\,. \tag{14}$$

Since (14) is a quadratic equation for the unknown characteristic velocity V, two real solutions $V_{1,2}$ can be obtained at $D = e_{pot}^2 + 4e_{kin}e_0 \geq 0$

$$V_1 = \frac{-e_{pot} - \sqrt{D}}{2e_{kin}},\ V_2 = \frac{-e_{pot} + \sqrt{D}}{2e_{kin}}, \tag{15}$$

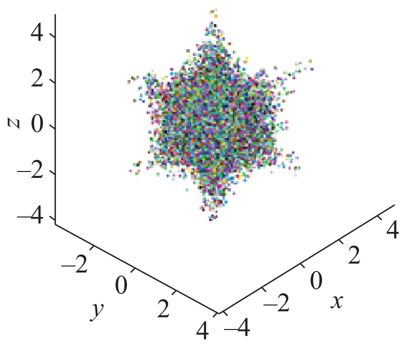
where $e_{kin} = \sum_{i=1}^{N} \tfrac{1}{2} m_i (\xi_x^2 B_{x,i}^2 + \xi_y^2 B_{y,i}^2 + \xi_z^2 B_{z,i}^2)$, $e_{pot} = -\tfrac{1}{2}\sum_{i,j=1,i\neq j}^{N} \frac{m_i m_j}{\sqrt{p_x^2 A_{x,i,j}^2 + p_y^2 A_{y,i,j}^2 + p_z^2 A_{z,i,j}^2}}$ — kinetic and potential energies.

Thus, a Monte Carlo algorithm for generating a set of phase space points lying on the hypersurface of all required conservation laws is developed, taking equation (15) into account. By varying the random variables ξ_x, ξ_y, and ξ_z, we obtain an unlimited number of required points (13). Henceforth, the letter M will denote the number of Monte Carlo experiments.

Note that the phase points lying on the hypersurface of the conservation laws are simultaneously the points of some trajectories-solutions of the system of equations (2). It remains to connect suitable pairs of phase points into a single trajectory, obtaining a finite-difference and exact solution at the same time [Dorodnitsyn, Kaptsov, 2013; Dorodnitsyn, 2001]. With some simplification and reservations, this very scenario is implemented in the next section.

Figure 1*a* shows a typical positioning pattern for the bodies comprising a gravitational system at $N = 100$, $M = 300$, $|\mathbf{k}_0| = 2.5$, and $e_0 = -10^3$. The masses in the gravitational system are chosen uniformly at random from [0, 1], while the characteristic velocity V is selected equally randomly from the two values of (15), i. e., $V = \{V_1, V_2\}$.

Figure 1*b* shows a positioning fragment of Solar System bodies comprising the Sun and eight planets in the center of mass system. The positions of the planets are found using the Monte Carlo generation procedure (6)–(15) at the given angular momentum and energy of the gravitational system. Among other things, the following constants are used in the calculation: $N = 9$, $M = 100$, $k_0 = 6.4549 \cdot 10^{-4}$, $e_0 = -0.0034$, and $V = \{V_1, V_2\}$. In Figure 1*b*, the Sun is denoted by a circle positioned in the center of the figure. On the periphery, the largest marker in the form of a dot represents Jupiter. Smaller dots represent the other planets. The accumulation of points (bodies) in space clearly indicates the presence of a plane similar to the ecliptic plane of the Solar System.



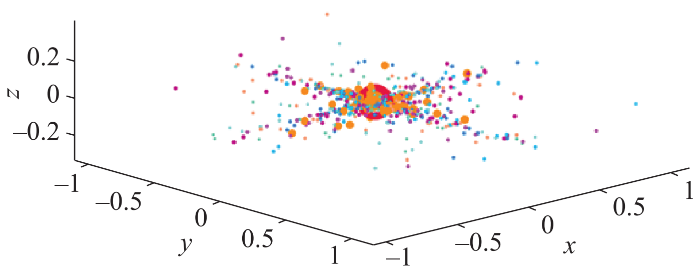


Figure 1. Typical positioning patterns in space: a) for one hundred bodies having a random mass; b) for the Sun and eight Solar System planets

Note that the accumulation of points in Figure 1*a* does not look compact; six cone-shaped extensions are clearly visible, potentially going to infinity. This factor indicates that a gravitational system comprising over two bodies is generally unstable. A similar situation is observed for Figure 1*b*, with a correction for the quasi-two-dimensionality of the positioning of Solar System bodies.

Two more pieces of indirect evidence showing the instability of the general-position gravitational system can be provided by plotting the dependences of the lower and upper limits of the stability criterion on the energy of the gravitational system e_0 and the number of Monte Carlo experiments M.

Let us rewrite the stability criterion of the gravitational system in the following form: $0 < q_{\min} \le q_{i,j} \le q_{\max}$ where $i \ne j$; $i, j = 1, \ldots, N$. The dependences of $q_{\min} = \min\limits_{\alpha=1,\ldots,M;\, i<j} q_{i,j}^{(\alpha)}$ and $q_{\max} =$

$= \max\limits_{\alpha=1,\ldots,M;i<j} q_{i,j}^{(\alpha)}$ on the energy of the gravitational system e_0 are plotted taking into account the procedure (6)–(15) for generating phase points $(\mathbf{q}_1^{(\alpha)}, \mathbf{u}_1^{(\alpha)}, \ldots, \mathbf{q}_N^{(\alpha)}, \mathbf{u}_N^{(\alpha)})$, $\alpha = 1,\ldots,M$.

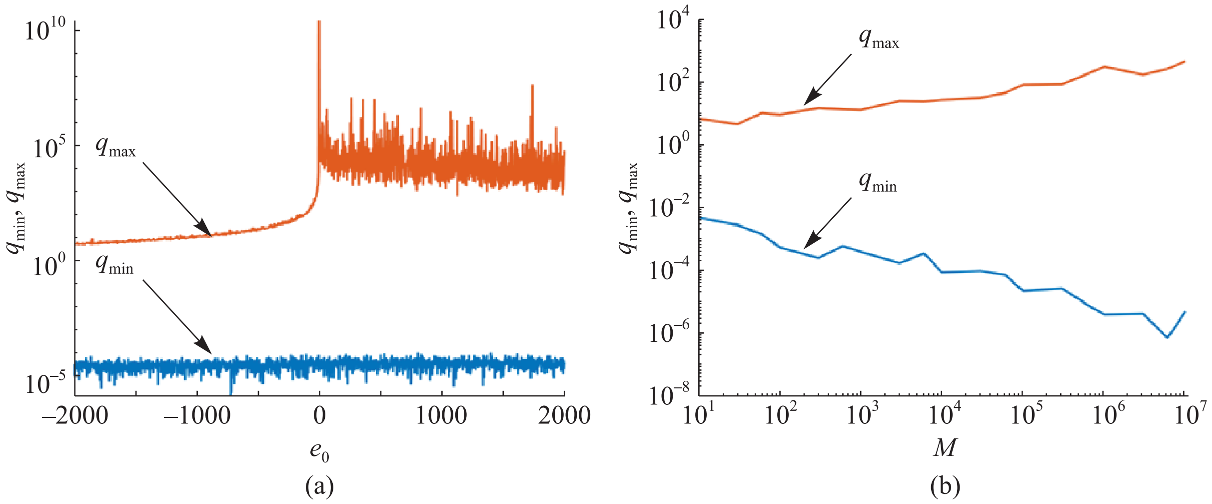


Figure 2. Dependences of the lower q_{min} and upper q_{max} limits of the stability criterion on: a) energy of the gravitational system e_0; b) number of Monte Carlo experiments M

Figure 2*a* shows a typical example of dependences $q_{min}(e_0)$ and $q_{max}(e_0)$. The following values of parameters are adopted in the calculation: $N = 10^2$, $M = 10^3$, and $k_0 = 2.5$. Thus, without factoring in small random fluctuations, the lower limit of the stability criterion $q_{min}(e_0)$ is virtually independent of the energy e_0 and its sign. With the energy e_0 transitioning from negative to positive values, the upper limit of the stability criterion $q_{max}(e_0)$, as well as its variability taking the logarithmic scale into account, significantly increases. Given Jacobi's theorem on the stability of the gravitational system, this factor is easily explained.

Figure 2*b* presents typical curves showing the dependence of the lower and upper limits of the stability criterion on the number of Monte Carlo experiments M. Other adopted values of the calculation parameters are as follows: $N = 10$, $e_0 = -10$, and $k_0 = 0.5$. As can be seen, the lower limit of the stability criterion quasi-monotonically decreases, while the upper limit quasi-monotonically increases with a rise in the number of Monte Carlo experiments.

3. Trajectory construction

Suppose that for a given number of bodies N in a gravitational system, a configuration $(\mathbf{q}_1^{(0)}, \ldots, \mathbf{u}_N^{(0)})$ is considered as initial taking (3) into account. Then the indicated initial configuration determines the angular momentum $\mathbf{k}_0$ and the total energy e_0 of the gravitational system.

Through one of the standard numerical algorithms, the system of equations (2) is solved in the time interval $[0, T_1]$ starting from the initial data $(\mathbf{q}_1^{(0)}, \ldots, \mathbf{u}_N^{(0)})$. Let a configuration $(\mathbf{q}_1^{(1)}, \ldots, \mathbf{u}_N^{(1)})$ be found at time $t = T_1$ that does not satisfy the conservation laws (3)–(5). It is assumed that the configuration $(\mathbf{q}_1^{(1)}, \ldots, \mathbf{u}_N^{(1)})$ is not too far from the hypersurface of the conservation laws. The meaning of "not too far" will become clear later on in the text.

Let us consider a procedure for projecting an arbitrary configuration $(\mathbf{q}_1,...,\mathbf{u}_N)$ onto the hypersurface of conservation laws (3)–(5). In order to satisfy (3), it is necessary to perform the following transformation

$$\mathbf{q}_i \to \mathbf{q}_i - \tfrac{1}{M_\Sigma}\sum\nolimits_{j=1}^{N} m_j \mathbf{q}_j \,,\ \mathbf{u}_i \to \mathbf{u}_i - \tfrac{1}{M_\Sigma}\sum\nolimits_{j=1}^{N} m_j \mathbf{u}_j \,,\ i = 1,\ \dots,\ N. \tag{16}$$

Transformation (16) brings us back to the center of mass coordinate system of the gravitational system where the equalities $\sum\nolimits_{i=1}^{N} m_i \mathbf{q}_i = \mathbf{0}$ and $\sum\nolimits_{i=1}^{N} m_i \mathbf{u}_i = \mathbf{0}$ are valid.

In order to satisfy the laws of conservation of angular momentum (4) and energy (5), let us consider the following transformation

$$\mathbf{q}_i \to (H_x q_{x,i}, H_y q_{y,i}, H_z q_{z,i}),\ \mathbf{u}_i \to (G_x u_{x,i}, G_y u_{y,i}, G_z u_{z,i})\,,\ i = 1,\ \dots,\ N, \tag{17}$$

where the parameters H_x, H_y, H_z, G_x, G_y, and G_z have not been determined yet. By applying (17) in (4) and (5), we obtain

$$\begin{gathered} c_{11} H_y G_z - c_{12} H_z G_y = k_{0,x}, \quad c_{21} H_z G_x - c_{22} H_x G_z = k_{0,y}, \quad c_{31} H_x G_y - c_{32} H_y G_x = k_{0,z}, \\ \tfrac{1}{2} c_{41} G_x^2 + \tfrac{1}{2} c_{42} G_y^2 + \tfrac{1}{2} c_{43} G_z^2 - \sum\nolimits_{i<j} \frac{m_i m_j}{\sqrt{H_x^2 q_{x,i,j}^2 + H_y^2 q_{y,i,j}^2 + H_z^2 q_{z,i,j}^2}} = e_0. \end{gathered} \tag{18}$$

where

$$c_{11} = \sum\nolimits_i m_i q_{y,i} u_{z,i}\,,\ c_{12} = \sum\nolimits_i m_i q_{z,i} u_{y,i}\,;\ c_{21} = \sum\nolimits_i m_i q_{z,i} u_{x,i}\,,\ c_{22} = \sum\nolimits_i m_i q_{x,i} u_{z,i}\,;$$

$$c_{31} = \sum\nolimits_i m_i q_{x,i} u_{y,i}\,,\ c_{32} = \sum\nolimits_i m_i q_{y,i} u_{x,i}\,;\ c_{41} = \sum\nolimits_i m_i u_{x,i}^2\,,\ c_{42} = \sum\nolimits_i m_i u_{y,i}^2\,,\ c_{43} = \sum\nolimits_i m_i u_{z,i}^2\,.$$

According to (18), the implementation of the laws of angular momentum and energy conservation requires finding a solution to the system of four nonlinear algebraic equations (18) for six unknowns: H_x, H_y, H_z, G_x, G_y, and G_z.

Here, we should note that the transformation procedure (16)–(17) intended to ensure the implementation of conservation laws is somewhat similar to the method originally proposed in [Nacozy, 1971]. In this work, numerical errors were compensated by correcting the positions and velocities of all bodies comprising the gravitational system so that the required conservation laws were satisfied. In our case, the choice of the procedure for correcting solutions to comply with the conservation laws follows from the procedure for randomly generating points in the phase space of the hypersurface of the conservation laws. It is also associated with the renormalization of positions and velocities of bodies comprising the gravitational system. At the first stage, the positions and velocities of each of the bodies are corrected to the center of mass according to (16). At the second stage, a unified correction of positions and velocities of all bodies comprising the gravitational system is performed according to (17), (18) to comply with angular momentum and energy conservation laws. The approach presented in [Nacozy, 1971] was further developed in [Fukushima, 2003], introducing a unified coefficient for the positions and velocities of bodies. This coefficient was selected by solving the corresponding cubic equation to ensure the implementation of the total energy conservation law. In our model, the above procedure in terms of equations (17), (18) involves introducing six coefficients to fulfill the angular momentum and energy conservation laws. In this case, instead of a cubic equation, it is necessary to solve a quadratic equation when selecting an appropriate coefficient to satisfy the energy conservation law in the gravitational system.

Let us solve the system of equations (18), assuming that the phase space point $(\mathbf{q}_1,...,\mathbf{u}_N)$ is slightly spaced from the surface, exhibiting the given momentum and energy values of the gravitational system. This factor suggests that the parameters H_x, H_y, H_z, G_x, G_y, and G_z do not differ much from unity, i. e., $H_x = 1 + h_x$, …, $G_z = 1 + g_z$ where $|h_x| \ll 1$, …, $|g_z| \ll 1$. Leaving first-order infinitesimals h_x, h_y, h_z, g_x, g_y, and g_z, we obtain

$$\begin{gathered}
c_{11}(h_y + g_z) - c_{12}(h_z + g_y) = \Delta k_x, \quad c_{21}(h_z + g_x) - c_{22}(h_x + g_z) = \Delta k_y, \\
c_{31}(h_x + g_y) - c_{32}(h_y + g_x) = \Delta k_z, \quad c_{41} g_x + c_{42} g_y + c_{43} g_z + c_{44} h_x + c_{45} h_y + c_{46} h_z = \Delta e, \\
c_{44} = \sum\nolimits_{i<j} \tfrac{m_i m_j}{q_{i,j}^3} q_{x,i,j}^2, \quad c_{45} = \sum\nolimits_{i<j} \tfrac{m_i m_j}{q_{i,j}^3} q_{y,i,j}^2, \quad c_{46} = \sum\nolimits_{i<j} \tfrac{m_i m_j}{q_{i,j}^3} q_{z,i,j}^2.
\end{gathered} \tag{19}$$

In (19), the values of $\Delta k_x, \Delta k_y, \Delta k_z$ and Δe are considered small, characterizing the deviation of the phase point $(\mathbf{q}_1,...,\mathbf{u}_N)$ from the hypersurface at the given values of the angular momentum $\mathbf{k}_0$ and energy e_0, with $\Delta k_x = k_{0,x} - c_{11} + c_{12}$, $\Delta k_y = k_{0,x} - c_{21} + c_{22}$, and $\Delta k_z = k_{0,z} - c_{31} + c_{32}$;

$$\Delta e = e_0 - \tfrac{1}{2} c_{41} - \tfrac{1}{2} c_{42} - \tfrac{1}{2} c_{43} + \sum\nolimits_{i<j} \tfrac{m_i m_j}{q_{i,j}}.$$

Let us solve the system of the first three equations in (19) for a set of unknowns h_x, h_y, and h_z. This can be done if $c = c_{11}c_{21}c_{31} - c_{12}c_{22}c_{32} \neq 0$. Note that the determinant $c \equiv 0$ at $N = 2$, i. e., in the case of the two-body problem. Similar to the case of generating phase points on a given hypersurface of the conservation laws, the special cases include the two-body problem, as well as the case of $\mathbf{k}_0 = \mathbf{0}$.

According to (19), the set of unknowns (h_x, h_y, h_z) is expressed in terms of the set (g_x, g_y, g_z). The three latter quantities are related by a single equation of the form $\alpha_1 g_x + \alpha_2 g_y + \alpha_3 g_z = \beta$ where $\alpha_1, \alpha_2, \alpha_3$, and β are expressed through the known quantities; the corresponding formulas are rather cumbersome to provide here. In order to solve the latter equation, let us sample three quantities ξ_1, ξ_2, and ξ_3 taken uniformly at random from [–1, 1] and produce the following expressions

$$g_x = \tfrac{\xi_1 \beta}{\alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3}, \; g_y = \tfrac{\xi_2 \beta}{\alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3}, \; g_z = \tfrac{\xi_3 \beta}{\alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3}. \tag{20}$$

Since it is evident that (20) provides the solution, we can find the set of h_x, h_y, and h_z and obtain a phase point $\{(1 + h_x) q_{x,1},...,(1 + g_z) u_{z,N}\}$ lying on the surface determined by the given angular momentum and energy.

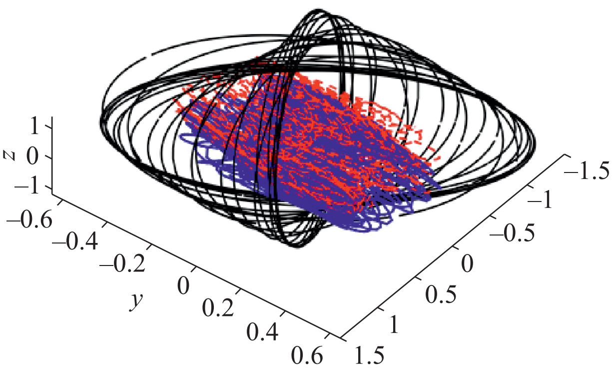
Note that random numbers in (20) make the procedure for correcting the phase point to the conservation laws partly random. In this case, the correction procedure can be regarded as a kind of Monte Carlo methods. Moreover, stochasticity is introduced into the dynamics directly rather than arising from the equations due to the so-called dynamic chaos [Sussman, Wisdom, 1992], which is clearly present in the initial equations.

Now, let us proceed to construct the motion trajectory. By applying the correction procedure (16)–(20) to the phase point $(\mathbf{q}_1^{(1)},...,\mathbf{u}_N^{(1)})$ found at $t = T_1$, we obtain a new phase point $(\tilde{\mathbf{q}}_1^{(1)},...,\tilde{\mathbf{u}}_N^{(1)})$ lying on the surface determined by the conservation laws. With the point $(\tilde{\mathbf{q}}_1^{(1)},...,\tilde{\mathbf{u}}_N^{(1)})$ further considered as initial, the initial system of differential equations (2) is solved numerically in $[T_1, T_1 + T_2]$. Then, the resulting solution $(\mathbf{q}_1^{(2)},...,\mathbf{u}_N^{(2)})$ at $t = T_1 + T_2$ is adjusted to $(\tilde{\mathbf{q}}_1^{(2)},...,\tilde{\mathbf{u}}_N^{(2)})$, etc. Finally, the last calculation for the system of equations (2) in $[T_1+...+T_{n-1}, T_1+...+T_n]$ completes the process of motion trajectory unfolding; the obtained solution $(\mathbf{q}_1^{(n)},...,\mathbf{u}_N^{(n)})$ is corrected to $(\tilde{\mathbf{q}}_1^{(n)},...,\tilde{\mathbf{u}}_N^{(n)})$. As a result, we obtain a numerical solution of the system of equations (2) in the following form:

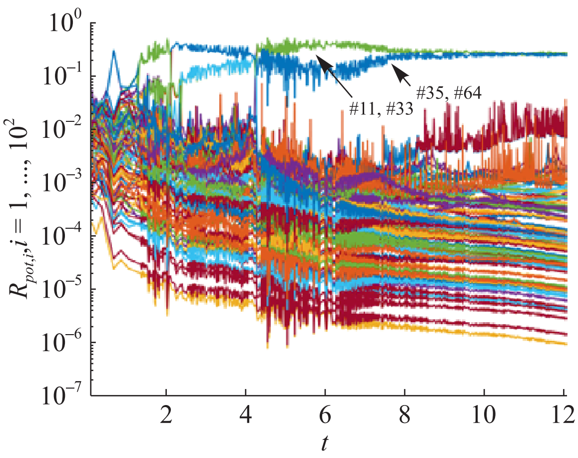
$$(\mathbf{q}_1^{(0)},...)\underset{[0,T_1]}{\rightarrow}(\mathbf{q}_1^{(1)},...)\Rightarrow(\tilde{\mathbf{q}}_1^{(1)},...)\underset{[T_1,T_1+T_2]}{\rightarrow}(\mathbf{q}_1^{(2)},...)\Rightarrow(\tilde{\mathbf{q}}_1^{(2)},...)\rightarrow...$$
$$\underset{[T_1+...+T_{n-1},T_1+...+T_n]}{\rightarrow}(\mathbf{q}_1^{(n)},...)\Rightarrow(\tilde{\mathbf{q}}_1^{(n)},...). \quad (21)$$

The integration segments $\{T_1,...,T_n\}$ of the system of equations (2) are found according to the following algorithm. First, the initial value $T_1 = T_{\max}$ of the integration interval is selected, growing slowly from step T_k to step T_{k+1} at $T_k < T_{\max}$. Then, if the modulo coefficients h_x, h_y, h_z, g_x, g_y, and g_z exceed a certain threshold value δ at the *k*-th stage of adjusting the solution to the conservation laws, the interval T_k is reduced, for example, by half, while the calculation is repeated in a reduced time interval.

Note that the numerical result of solving the system of equations (2) is not a separate deterministic curve in the phase space since the time points $t = T_1, T_1+T_2,...,T_1+...+T_n$ at which the solution is adjusted to the conservation laws act as solution branch points. Thus, the described procedure for assembling the motion trajectory generally generates an ensemble of trajectories. In addition, the obtained trajectories are not continuous; the size of the jumps serves as a control parameter *δ*.



(a)



(b)

Figure 3. Example of dynamics: a) a three-body gravitational system (trajectories are marked with solid, dotted, and dashed lines, respectively); b) contribution to the potential energy of each body comprising the gravitational system, $N = 10^2$

Figure 3*a* presents an application of the algorithm (16)–(21) to calculating the dynamics of a three-body gravitational system ($N = 3$). The choice of this example is intentional. The aim is to reveal the presence of non-trivial dynamics over a noticeable period of time when all three bodies are actively interacting, with none of the bodies going to infinity. Among other things, the following parameters are selected: $n = 100$, $T_1 = 1$, and $\delta = 0.025$. In addition, the body masses, initial positions, and velocities are chosen uniformly at random from [0, 1], as well as from $[-L, L]^3$ and $[-V, V]^3$ at $L = 1$ and $V = 1$. As a result, the calculation yields a noticeable value of $T_\Sigma = T_1 + ... + T_n = 228.26$. In

Figure 3*a*, one of the bodies (solid line) is rotating around another pair of bodies (dotted and dashed lines, respectively). However, we should note that the statistics of many computational experiments indicates that, for most solutions, one of the three bodies goes to infinity.

In calculations involving a noticeable number of bodies in a gravitational system, the dynamics typically develop in such a way that bodies go to infinity over time, with one or more pairs of bodies accumulating ≈ 95 % of the total potential energy. Thus, Figure 3*b* shows the typical temporal dynamics of the contribution of each body $R_{pot,i}, i=1,...,N$ to the potential energy of the gravitational system. These values are calculated using the formula $R_{pot,i}=-\frac{m_i}{2e_{pot}}\sum_j \frac{m_j}{q_{i,j}}$ where $e_{pot}=-\frac{1}{2}\sum_{i,j}\frac{m_i m_j}{q_{i,j}}$. Among other things, it is assumed that: $N=10^2$, $n=1300$, $T_1=1$, and $\delta=0.025$. The mass of bodies, their initial position, and velocity are chosen the same as in the previous calculation. Figure. 3*b* shows two pairs of bodies (Nos. 11 and 33; Nos. 35 and 64), dividing the ≈ 96 % of potential energy approximately equally between them.

4. Angular momentum of the gravitational system amounting to zero

In Section 2 discussing the Monte Carlo algorithm for the generation of phase space points lying on the surface of the conservation laws, several special cases are noted that require special consideration. For example, one special case consists in the absence of angular momentum in a gravitational system comprising three or more bodies, i.e., the case where $N>2$ and $\mathbf{k}_0=\mathbf{0}$.

At $\mathbf{k}_0=\mathbf{0}$, the system of equations (10) can be solved for λ_x,λ_y, and λ_z in six ways, i. e., there are generally six solutions. Taking into account the energy conservation law (5) and choosing the values of ξ_x,ξ_y,ξ_z, and ξ uniformly at random from [–1, 1], we obtain

1) $\mu_x=0,\ \mu_y=V\xi_y,\ \mu_z=V\xi_z;\ \lambda_x=0,\ \lambda_y=c_{12}\xi_y\frac{\xi}{|\xi|}\nu_1;\ \lambda_z=c_{11}\xi_z\frac{\xi}{|\xi|}\nu_1$;

$$\nu_1=\sum_{i<j}\frac{m_i m_j}{\sqrt{c_{12}^2\xi_y^2A_{y,i,j}^2+c_{11}^2\xi_z^2A_{z,i,j}^2}}\Big/\left[V^2\sum_i\frac{1}{2}m_i(\xi_y^2B_{y,i}^2+\xi_z^2B_{z,i}^2)-e_0\right];$$

2) $\mu_x=V\xi_x,\ \mu_y=0,\ \mu_z=V\xi_z;\ \lambda_x=c_{21}\xi_x\frac{\xi}{|\xi|}\nu_2,\ \lambda_y=0;\ \lambda_z=c_{22}\xi_z\frac{\xi}{|\xi|}\nu_2$;

$$\nu_2=\sum_{i<j}\frac{m_i m_j}{\sqrt{c_{21}^2\xi_x^2A_{x,i,j}^2+c_{22}^2\xi_z^2A_{z,i,j}^2}}\Big/\left[V^2\sum_i\frac{1}{2}m_i(\xi_x^2B_{x,i}^2+\xi_z^2B_{z,i}^2)-e_0\right];$$

3) $\mu_x=V\xi_x,\ \mu_y=V\xi_y,\ \mu_z=0;\ \lambda_x=c_{32}\xi_x\frac{\xi}{|\xi|}\nu_3,\ \lambda_y=c_{31}\xi_y\frac{\xi}{|\xi|}\nu_3,\ \lambda_z=0$;

$$\nu_3=\sum_{i<j}\frac{m_i m_j}{\sqrt{c_{32}^2\xi_x^2A_{x,i,j}^2+c_{31}^2\xi_y^2A_{y,i,j}^2}}\Big/\left[V^2\sum_i\frac{1}{2}m_i(\xi_x^2B_{x,i}^2+\xi_y^2B_{y,i}^2)-e_0\right]$$

where $c_{11},c_{12},c_{21},c_{22},c_{31},c_{32}$ are obtained according to formulas (9),

4) $\mu_x=0,\ \mu_y=0,\ \mu_z=V\xi_z;\ \lambda_x=0,\ \lambda_y=0,\ \lambda_z=\frac{\xi}{|\xi|}\nu_4$;

$$\nu_4=\sum_{i<j}\frac{m_i m_j}{|A_{z,i,j}|}\Big/\left(V^2\sum_i\frac{1}{2}m_i\xi_z^2B_{z,i}^2-e_0\right);$$

5) $\mu_x=0,\ \mu_y=V\xi_y,\ \mu_z=0;\ \lambda_x=0,\ \lambda_y=\frac{\xi}{|\xi|}\nu_5,\ \lambda_z=0$;

$$\nu_5=\sum_{i<j}\frac{m_i m_j}{|A_{y,i,j}|}\Big/\left(V^2\sum_i\frac{1}{2}m_i\xi_y^2B_{y,i}^2-e_0\right);$$

6) $\mu_x=V\xi_x,\ \mu_y=0,\ \mu_z=0;\ \lambda_x=\frac{\xi}{|\xi|}\nu_6,\ \lambda_y=0,\ \lambda_z=0$;

$$\nu_6=\sum_{i<j}\frac{m_i m_j}{|A_{x,i,j}|}\Big/\left(V^2\sum_i\frac{1}{2}m_i\xi_x^2B_{x,i}^2-e_0\right).$$

The solutions presented above contain one free parameter V, which, on the one hand, determines the characteristic velocity of the bodies of the gravitational system and, on the other, can be considered as arbitrary. Factoring in the above solutions, we can determine the required phase space point:

$$\mathbf{q}_i = (\lambda_x A_{x,i}, \lambda_y A_{y,i}, \lambda_z A_{z,i}), \quad \mathbf{u}_i = (\mu_x B_{x,i}, \mu_y B_{y,i}, \mu_z B_{z,i}), \tag{22}$$

where $A_{x,i}, A_{y,i}, A_{z,i}, B_{x,i}, B_{y,i}, B_{z,i}$, $i = 1,\dots,N$ are obtained according to (6).

When obtaining a set of phase space points (22) lying on a hypersurface with a given total energy e_0 and zero angular momentum $\mathbf{k}_0 = \mathbf{0}$, the set of quantities $\{\lambda_x, \lambda_y, \lambda_z, \mu_x, \mu_y, \mu_z\}$ should be chosen with equal probability from the six solutions presented above.

The phase space points (22) obtained on the basis of six solutions, in terms of spatial positioning, lie either on one of the three coordinate planes, or on one of the three axes of the Cartesian coordinate system. Thus, the existence of the ecliptic plane in the Solar System can be attributed to small angular momentum. Figure 4*a* shows a typical format for the positioning of bodies comprising a gravitational system in a set of statistical experiments at $\mathbf{k}_0 = \mathbf{0}$. Other adopted calculation parameters are as follows: $N = 25, M = 10^3, V = 1$, and $e_0 = -50$. Figure 4*a* shows the positioning of bodies in a cube $[-1.5, 1.5]^3$. With an increase in the side of the indicated cube, the positioning pattern of points becomes similar to that shown in Figure 1*a* (case where $\mathbf{k}_0 \neq \mathbf{0}$). Thus, the cluster of points is clearly non-compact in both cases (at $\mathbf{k}_0 \neq \mathbf{0}$ and $\mathbf{k}_0 = \mathbf{0}$), i. e., the gravitational system is generally unstable.

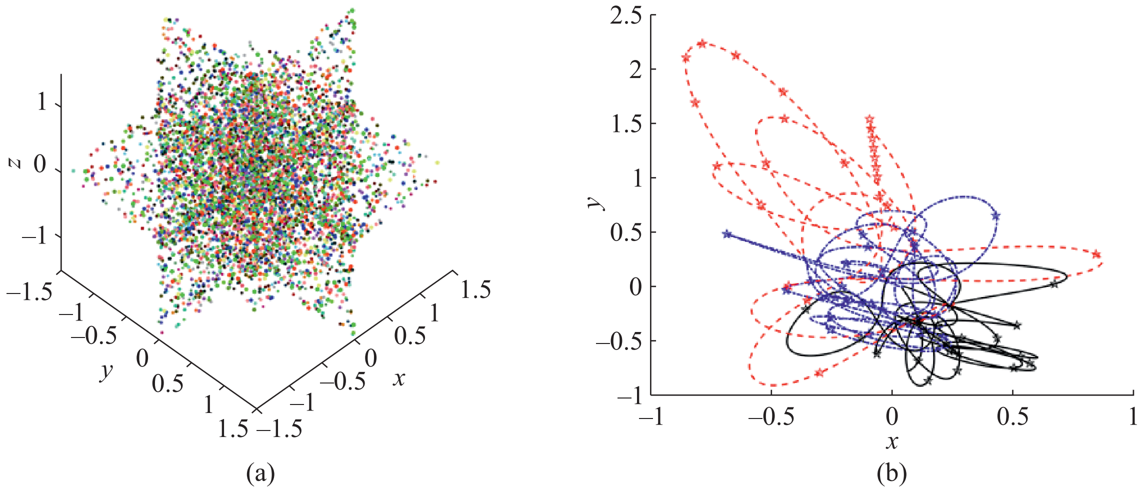


Figure 4. Typical format for positioning bodies comprising a gravitational system in a set of statistical experiments at $\mathbf{k}_0 = \mathbf{0}$ (a); an example showing the motion trajectories of three bodies in the *x*-*y* plane (trajectories are marked with a solid, dotted, and dash-dotted lines, respectively) (b)

Let us assemble the trajectory of the bodies comprising the gravitational system for one of the first three special cases. Here, we will consider solution number 3, in which the dynamics of the gravitational system occur in the *x*-*y* plane. In this case, it is assumed that $z = 0$. As a result, it is necessary to solve the following system of differential equations:

$$\begin{aligned} &\dot{q}_{x,i} = u_{x,i}, \ \dot{q}_{y,i} = u_{y,i}; \\ &\dot{u}_{x,i} = -\sum_j \frac{m_j (q_{x,i} - q_{x,j})}{(q_{x,i,j}^2 + q_{y,i,j}^2)^{3/2}}, \ \dot{u}_{y,i} = -\sum_j \frac{m_j (q_{y,i} - q_{y,j})}{(q_{x,i,j}^2 + q_{y,i,j}^2)^{3/2}}; \end{aligned} \tag{23}$$

where $i = 1,\dots,N$.

Prior to constructing the trajectory of the gravitational system described by the system of equations (23), the initial configuration $q_{x,i}^{(0)}, q_{y,i}^{(0)}, u_{x,i}^{(0)}, u_{y,i}^{(0)}, i=1,\dots,N$ is obtained. Taking the condition $\mathbf{k}_0 = \mathbf{0}$ into account in the system of equations (10), we obtain

$$q_{x,i}^{(0)} = \lambda_x A_{x,i},\ q_{y,i}^{(0)} = \lambda_y A_{y,i};\ u_{x,i}^{(0)} = V\xi_x B_{x,i},\ u_{y,i}^{(0)} = V\xi_y B_{y,i}; \tag{24}$$

where

$$\lambda_x = c_{32}\xi_x \tfrac{\xi}{|\xi|}v_3,\ \lambda_y = c_{31}\xi_y \tfrac{\xi}{|\xi|}v_3,\ \xi_x, \xi_y, \xi \in [-1,1].$$

The trajectory is assembled according to the procedure (16)–(20). Taking (19) into account, we obtain from equations (18)

$$\begin{cases} c_{31}h_x - c_{32}h_y - c_{32}g_x + c_{31}g_y = c_{32} - c_{31}, \\ c_{44}h_x + c_{45}h_y + c_{41}g_x + c_{42}g_y = \Delta e, \end{cases} \tag{25}$$

where

$$c_{44} = \sum\nolimits_{i<j} \tfrac{m_i m_j}{(q_{x,i,j}^2 + q_{y,i,j}^2)^{3/2}} q_{x,i,j}^2,\ c_{45} = \sum\nolimits_{i<j} \tfrac{m_i m_j}{(q_{x,i,j}^2 + q_{y,i,j}^2)^{3/2}} q_{y,i,j}^2;$$
$$\Delta e = e_0 - \tfrac{1}{2}c_{41} - \tfrac{1}{2}c_{42} + \sum\nolimits_{i<j} \tfrac{m_i m_j}{\sqrt{q_{x,i,j}^2 + q_{y,i,j}^2}}.$$

Let us assume that $h_x = 0$ and $h_y = 0$ in the procedure for correcting the obtained solution to the given conservation laws ($\mathbf{k}_0 = \mathbf{0}, e_0$). By substituting the latter equalities in (25) and solving the resulting system for the unknowns g_x and g_y we obtain

$$g_x = \tfrac{(c_{31} - c_{32})c_{42} + c_{31}\Delta e}{c_{31}c_{41} + c_{32}c_{42}},\quad g_y = \tfrac{(c_{32} - c_{31})c_{41} + c_{32}\Delta e}{c_{31}c_{41} + c_{32}c_{42}}. \tag{26}$$

Figure 4*b* provides an example showing the motion trajectories in the plane of three bodies ($N = 3$) interacting according to equations (23)–(26). The pentagrams on the trajectories indicate the moments when the procedure for adjusting the motion to the given conservation laws is carried out. The other adopted calculation parameters are as follows: $\mathbf{k}_0 = \mathbf{0}, e_0 = -0.5$, $n = 25$, $T_1 = 1$, $\delta = 0.025$, and $V = 1$. The motion trajectories are marked by different lines: solid, dotted, and dash-dotted. The total time interval for integrating the system of equations (23), including correction procedures to the conservation laws, is [0, 28.46].

Let us construct the motion trajectories of bodies comprising the gravitational system for one of the special cases: No. 4, No. 5, or No. 6. We will consider solution number 6, in which the dynamics of the gravitational system occur on the coordinate line x. In this case, it is assumed that $y, z = 0$. As a result, it is necessary to solve the following system of differential equations:

$$\begin{aligned} \dot{x}_i &= u_i, \\ \dot{u}_i &= -\sum\nolimits_j \tfrac{m_j x_{i,j}}{|x_{i,j}|^3}, \end{aligned} \tag{27}$$

where $x_{i,j} = x_i - x_j$, $i, j = 1,\dots N$.

The first step is to determine the initial configuration $x_i^{(0)}, u_i^{(0)}, i = 1,\dots,N$; then, taking into account the particular solution No. 6, we obtain

$$x_i^{(0)} = \tfrac{\xi}{|\xi|}v_6 A_{x,i},\ u_i^{(0)} = V\xi_x B_{x,i};\ \xi, \xi_x \in [-1,1]. \tag{28}$$

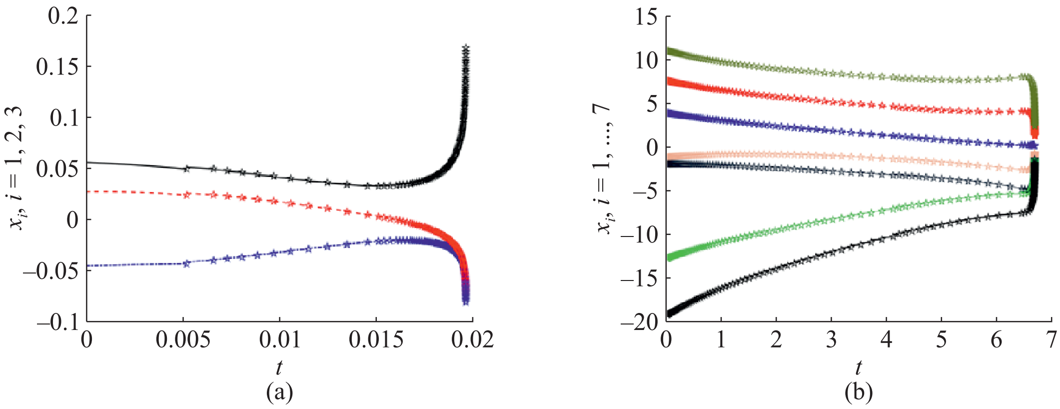


Figure 5. Examples showing the dynamics of positions on the x-axis: a) three bodies, $N = 3$, $e_0 = -1.5$; b) seven bodies, $N = 7$, $e_0 = -1$

When assembling trajectories by analogy with (17), we will follow the transformations $x_i \to H_x x_i$ and $u_i \to G_x u_i$ to comply with the energy conservation law. In this case, we obtain

$$G_x^2 e_{kin} + \tfrac{1}{H_x} e_{pot} = e_0 , \tag{29}$$

where $e_{kin} = \sum_i \frac{1}{2} m_i u_i^2$, $e_{pot} = -\sum_{i<j} \frac{m_i m_j}{|x_{i,j}|}$.

By solving Equation (29) for H_x, we get

$$H_x = \tfrac{e_{pot}}{e_0 - G_x^2 e_{kin}} . \tag{30}$$

Assuming $G_x = 1$, we can find $H_x = \frac{e_{pot}}{e_0 - e_{kin}}$ according to (30). The integration interval T_k is reduced when the following inequality holds: $| H_x - 1 | = | \frac{e_0 - e_{pot} - e_{kin}}{e_0 - e_{kin}} | > \delta$.

Figure 5 illustrates the calculations of the position dynamics of bodies comprising a gravitational system performed according to the algorithm for assembling trajectories (27) – (30) for two cases: $N = 3$ and $N = 7$. The other adopted parameters are as follows: $\delta = 0.025$ and $V = 1$. Pentagram markers on the graphs indicate the moments when the procedure for correcting the trajectory to the set value of the total energy e_0 is carried out.

The dynamics format in Figure 5*a* is specially selected to demonstrate that a pair of bodies are approaching each other while the third body is going to infinity. A typical format for a large number of bodies is shown in Figure 5*b*. The latter case is characterized by the bodies of the gravitational system accelerating and approaching each other, taking the energy conservation law into account. Note that such dynamics can be referred to a class characterized by escalation, with the bodies of the gravitational system falling into a common center at some finite point in time and their velocity reaching infinity. We will consider this case in more detail below, using the two-body problem as an example.

5. Two-body problem

Let us focus on a special case where $N = 2$ and the angular momentum of a pair of bodies is absent, i. e., $\mathbf{k}_0 = \mathbf{0}$. By solving the system of equations (10), we obtain

$$\lambda_x = \xi \mu_x c_{12} c_{32}, \quad \lambda_y = \xi \mu_y c_{12} c_{31}, \quad \lambda_z = \xi \mu_z c_{11} c_{31} , \tag{31}$$

where ξ — some as yet undetermined parameter.

Let us sample the quantities μ_x, μ_y, and μ_z according to

$$\mu_x = V\xi_x, \quad \mu_y = V\xi_y, \quad \mu_z = V\xi_z, \tag{32}$$

where ξ_x, ξ_y, ξ_z — uniformly random numbers from [–1, 1]; V — characteristic speed scale.

After substituting (32) into (31), we get

$$\lambda_x = V\xi\xi_x c_{12}c_{32}, \quad \lambda_y = V\xi\xi_y c_{12}c_{31}, \quad \lambda_z = V\xi\xi_z c_{11}c_{31}. \tag{33}$$

Taking (13)–(14) into account, let us specify the form of phase-space vectors:

$$\mathbf{q}_i = V\xi\mathbf{q}_i' = V\xi(\xi_x c_{12}c_{32}A_{x,i}, \xi_y c_{12}c_{31}A_{y,i}, \xi_z c_{11}c_{31}A_{z,i}), \tag{34}$$

$$\mathbf{u}_i = V\mathbf{u}_i' = V(\xi_x B_{x,i}, \xi_y B_{y,i}, \xi_z B_{z,i}), \; i = 1,2. \tag{35}$$

Let us find the unknown quantity ξ using the energy conservation law (5), i. e.

$$V^2\sum_{i=1}^{2}\tfrac{1}{2}m_i\mathbf{u}_i'^2 - \tfrac{1}{|\xi|}\tfrac{m_1m_2}{V|q_1'-q_2'|} - e_0 = 0. \tag{36}$$

By solving Equation (36) for $|\xi|$, we obtain

$$\nu = |\xi| = \tfrac{m_1m_2}{|q_1'-q_2'|}\Big/\left(V^2\sum_{i=1}^{2}\tfrac{1}{2}m_i\mathbf{u}_i'^2 - e_0\right). \tag{37}$$

Taking (34), (35), and (37) into account, we can write down the Monte Carlo procedure for generating an unlimited number of phase space points, with all of them having the given values of angular momentum ($\mathbf{k}_0 = \mathbf{0}$) and energy ($e_0$), i. e.

$$\mathbf{q}_i = V\tfrac{\zeta}{|\zeta|}\nu\mathbf{q}_i', \quad \mathbf{u}_i = V\mathbf{u}_i', \quad i = 1,2; \tag{38}$$

where ζ — uniformly random number from [–1,1].

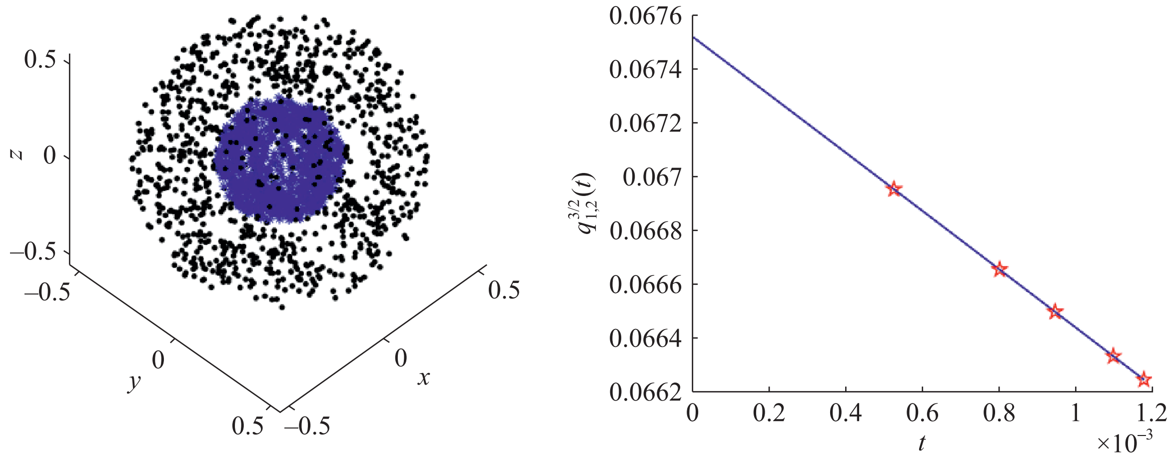
Figure 6*a* illustrates the positioning of a pair of bodies within a gravitational system obtained using the Monte Carlo method according to (31)–(38). The dots and asterisks indicate bodies Nos. 1 and 2, respectively. The following values of other parameters are adopted: $M = 10^3$, $e_0 = -0.5$, $V = 1$, $m_1 \cong 0.41$, and $m_2 \cong 0.97$.

According to formulas (37) and (38), the positions of phase points in the configuration space cannot go to infinity. In other words, the cluster of points in Figure 6*a* has a compact shape, indicating the stable nature of interaction in a two-body gravitational system.

The next step is to find an analytical solution of the two-body problem and compare it with a numerical solution obtained according to the solution assembly procedure, taking the implementation of the conservation laws into account $\mathbf{k}_0 = \mathbf{0}$ and e_0.

From equations (2), we obtain $\mathbf{q}_2 = -\frac{m_1}{m_2}\mathbf{q}_1$ and $\mathbf{u}_2 = -\frac{m_1}{m_2}\mathbf{u}_1$ at $N = 2$. In this case, the angular momentum (3) for the pair of bodies takes the form of $\mathbf{k}_0 = m_1\mathbf{q}_1 \times \mathbf{u}_1 + m_2\mathbf{q}_2 \times \mathbf{u}_2 = \left(m_1 + \frac{m_1^2}{m_2}\right)\mathbf{q}_1 \times \mathbf{u}_1 = \mathbf{0}$. The latter equation can be satisfied assuming that $\mathbf{u}_1 = \varphi\mathbf{q}_1$ where $\varphi = \varphi(t)$ is some as yet undetermined scalar time function. In order to determine $\varphi(t)$, we obtain equations similar to (2) for this case

$$\begin{cases} \dot{\mathbf{q}}_1 = \mathbf{u}_1, \\ \dot{\mathbf{u}}_1 = -\frac{m_2}{(1+m_1/m_2)^2 q_1^3}\mathbf{q}_1. \end{cases} \tag{39}$$



(a) (b)

Figure 6. Example showing the positioning of a pair of bodies in a set of statistical experiments at $\mathbf{k}_0 = \mathbf{0}$ (dots and asterisks indicate bodies Nos. 1 and 2, respectively) (a); a typical graph showing the temporal dynamics of the distance between the pair of bodies (b)

With $\mathbf{u}_1 = \varphi \mathbf{q}_1$ taken into account in (39), we obtain the following differential equation after some transformations

$$\ddot{\varphi} + 5\varphi\dot{\varphi} + 3\varphi^3 = 0 . \tag{40}$$

The solution of Equation (40) that we are interested in can be easily found, i. e., $\varphi = \varphi(t) = = \left(\varphi_0^{-1} + \frac{3}{2}t\right)^{-1}$ where $\varphi(0) = \varphi_0$. The latter expression allows a solution to the system of equations (39) to be obtained

$$\mathbf{q}_1 = \left(1 + \tfrac{3}{2}\varphi_0 t\right)^{2/3} \mathbf{q}_{1,0}, \ \mathbf{u}_1 = \left(1 + \tfrac{3}{2}\varphi_0 t\right)^{-1/3} \mathbf{q}_{1,0}, \tag{41}$$

where $\varphi_0 \neq 0$ and $\mathbf{q}_{1,0} = \mathbf{q}_1(0)$ is some constant vector. After substituting (41) into (39), we obtain $\varphi_0 = \pm\sqrt{\frac{2m_2}{(1+m_1/m_2)^2 q_{1,0}^3}}$. At $\varphi_0 = 0$, the solution (41) takes the following form

$$\mathbf{q}_1 = t^{2/3}\mathbf{q}_{1,0}, \quad \mathbf{u}_1 = \tfrac{2}{3}t^{-1/3}\mathbf{q}_{1,0}, \tag{42}$$

with $q_{1,0} = \frac{(9m_2)^{1/3}}{2^{1/3}(1+m_1/m_2)^{2/3}}$.

Let us determine the distance vector between the pair of bodies $\mathbf{q}_{1,2} = \mathbf{q}_1 - \mathbf{q}_2 = \left(1 + \frac{m_1}{m_2}\right)\mathbf{q}_1$. Then taking (41) into account, we obtain

$$q_{1,2} = \left|\mathbf{q}_{1,2}\right| = \left(1 + \tfrac{m_1}{m_2}\right)\left(1 + \tfrac{3}{2}\varphi_0 t\right)^{2/3} \left|\mathbf{q}_{1,0}\right|. \tag{43}$$

On physical grounds, it is clear that a pair of bodies are attracted to each other, i. e., the distance $q_{1,2}$ between them decreases. Taking (41) into account, $q_{1,2} \to 0$, $\varphi_0 < 0$ while the time increases, remaining below $t_f = -\frac{2}{3\varphi_0}$, i. e. $t < t_f$. The time t_f will be referred to as focusing time. Zero distance $q_{1,2} = 0$ between a pair of bodies can be interpreted as their agglomeration.

Here, the procedure for constructing a trajectory (16)–(18) is applied, taking into account the fact that we are considering the two-body problem and $\mathbf{k}_0 = \mathbf{0}$. Let there be a certain configuration $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{u}_1, \mathbf{u}_2\}$; the idea is to modify it to comply with the angular momentum conservation law, $\mathbf{k}_0 = \mathbf{0}$, and the given value of the total energy e_0. To this end, we can apply transformation (17); then, given (33), we obtain:

$$H_x = \tfrac{\xi}{|\xi|}\nu c_{12}c_{32}G_x, \quad H_y = \tfrac{\xi}{|\xi|}\nu c_{12}c_{31}G_y, \quad H_z = \tfrac{\xi}{|\xi|}\nu c_{11}c_{31}G_z, \tag{44}$$

where ξ is an arbitrary random number from [–1, 1];

$$\nu = \frac{m_1 m_2}{\sqrt{G_x^2 c_{12}^2 c_{32}^2 q_{x,1,2}^2 + G_y^2 c_{12}^2 c_{31}^2 q_{y,1,2}^2 + G_z^2 c_{11}^2 c_{31}^2 q_{z,1,2}^2}} \Big/ \left(G_x^2 \sum_i \tfrac{1}{2} m_i u_{x,i}^2 + G_y^2 \sum_i \tfrac{1}{2} m_i u_{y,i}^2 + G_z^2 \sum_i \tfrac{1}{2} m_i u_{z,i}^2 - e_0 \right).$$

According to (44), we can find the set of H_x, H_y, and H_z knowing G_x, G_y, and G_z. Since there are no restrictions on G_x, G_y, and G_z, we can assume that $G_x = G_y = G_z = 1$. In this case, the integration segment T_k is reduced when one of the three inequalities holds: $|H_x - 1| > \delta$, $|H_y - 1| > \delta$, and $|H_z - 1| > \delta$.

Figure 6*b* shows a typical trajectory sample describing the time dynamics of distances between a pair of bodies. In Figure 6*b*, the distance to the power of 3/2 is plotted on the ordinate axis since, according to the analytical solution (43), $q_{1,2}^{3/2}$ is a linear time function. The constructed trajectory is consistent with a straight line, i. e., fully corresponds to the analytical solution (43). The remaining parameters of the computational experiment are as follows: $e_0 = -0.5$, $V = 1$, and $\delta = 0.025$. In Figure 6*b*, pentagrams, as above, indicate the moments when the solution is corrected to satisfy the conservation laws.

Let us continue our consideration of the two-body problem at $\mathbf{k}_0 \neq \mathbf{0}$. In this case, the calculation procedure (6)–(15) does not work. It is necessary to introduce the relative vectors of position $\mathbf{q} = \mathbf{q}_1 - \mathbf{q}_2$ and velocity $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$; thus, the positions and velocities of each of the bodies, as well as the law of angular momentum and energy conservation, take the following form:

$$\mathbf{q}_1 = \frac{\mathbf{q}}{(1+m_1/m_2)}, \mathbf{q}_2 = -\frac{\mathbf{q}}{(1+m_2/m_1)}; \ \mathbf{u}_1 = \frac{\mathbf{u}}{(1+m_1/m_2)}, \mathbf{u}_2 = -\frac{\mathbf{u}}{(1+m_2/m_1)}; \tag{45}$$

$$\frac{m_1 m_2}{m_1 + m_2} \mathbf{q} \times \mathbf{u} = \mathbf{k}_0, \tag{46}$$

$$\frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} u^2 - \frac{m_1 m_2}{q} = e_0, \tag{47}$$

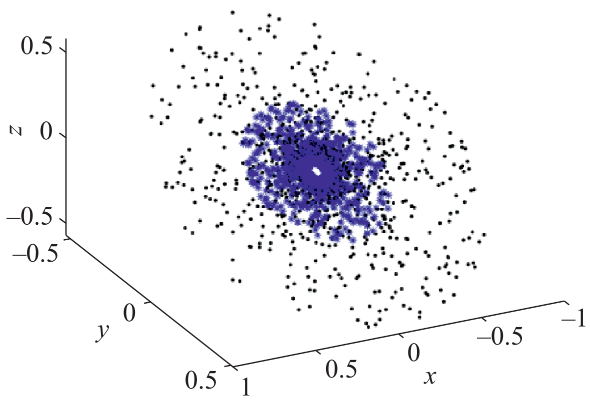
where $u = |\mathbf{u}|$ and $q = |\mathbf{q}|$.

The angular momentum $\mathbf{k}_0$ and the energy e_0 in (46)–(47) cannot be considered arbitrary independently of each other. Thus, equation (46) yields constraint 1: $k_0 \le \frac{m_1 m_2}{m_1 + m_2} qu$; whereas constraint 2 follows from equation (47): $e_0 \ge -\frac{m_1 m_2}{q}$. Constraints 1 and 2 imply constraint 3: $\frac{k_0 (m_1 + m_2)}{m_1 m_2 u} \le q \le \frac{m_1 m_2}{-e_0}$, which, given (47), leads to constraint 4 on the possible values of the relative velocity modulus: $\frac{m_1^2 m_2^2 - \sqrt{D}}{m_1 m_2 k_0} \le u \le \frac{m_1^2 m_2^2 + \sqrt{D}}{m_1 m_2 k_0}$. The latter inequality is valid only under constraint 5: $D = m_1^4 m_2^4 + 2 m_1 m_2 (m_1 + m_2) k_0^2 e_0 \ge 0$.

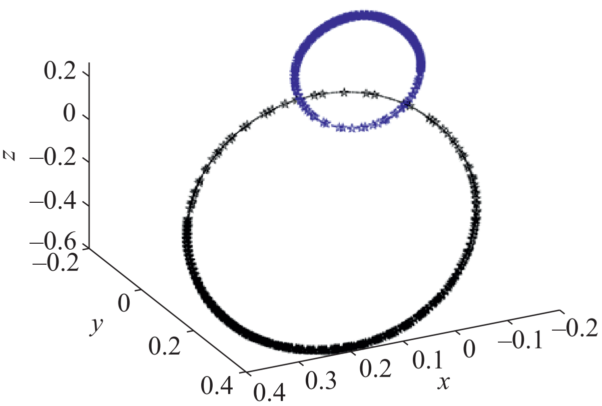
Let us introduce a pair of vectors of unit length: $\boldsymbol{\alpha} = \frac{\mathbf{q}}{q}$ and $\boldsymbol{\beta} = \frac{\mathbf{u}}{u}$; then the following scheme for obtaining phase space points via the Monte Carlo method is implemented. Step 1: Select the vector $\mathbf{k}_0$

and the constant e_0 to fulfill the following inequality: $D = m_1^4 m_2^4 + 2m_1 m_2 (m_1 + m_2) k_0^2 e_0 \ge 0$. Step 2: Select a relative velocity modulus u uniformly at random from $\left[\frac{m_1^2 m_2^2 - \sqrt{D}}{m_1 m_2 k_0}, \frac{m_1^2 m_2^2 + \sqrt{D}}{m_1 m_2 k_0}\right]$. Step 3: From equation (47), find the modulus of the relative distance $q = \frac{m_1 m_2}{\frac{1}{2}\frac{m_1 m_2}{m_1 + m_2} u^2 - e_0}$. Step 4: The ort $\boldsymbol{\alpha}$ is chosen randomly provided it is orthogonal to the angular momentum vector, i. e. $(\boldsymbol{\alpha}, \mathbf{k}_0) = 0$. Step 5: Taking (46) into account, find the angle θ between a pair of vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}$ that has two possible values: $\theta = \arcsin \frac{(m_1 + m_2) k_0}{m_1 m_2 q u}$ and $\theta = \pi - \arcsin \frac{(m_1 + m_2) k_0}{m_1 m_2 q u}$. Step 6: Obtain the vector $\boldsymbol{\beta}$ by rotating the vector $\boldsymbol{\alpha}$ by one of the two angles relative to the angular momentum vector; the choice of one of the two angles is considered equally probable. Step 7: Given that $\mathbf{q} = q\boldsymbol{\alpha}$ and $\mathbf{u} = u\boldsymbol{\beta}$, find the positions and velocities of each of the two bodies using formulas (45).

Figure 7*a* illustrates all possible positions of a pair of bodies in space (with asterisks and dots indicating the first and second bodies, respectively); these bodies interact according to Newton's law and have the given values of the angular momentum and energy ($|\mathbf{k}_0| = 0.125$, $e_0 = -0.25$). Figure 7*a* shows the results of $M = 10^3$ statistical experiments on positioning a pair of bodies comprising a gravitational system, obtained under the procedure described above.



(a)



(b)

Figure 7. Examples: a) random positions of a pair of bodies comprising the gravitational system at the given values of the angular momentum and energy (asterisks and dots indicate the first and second bodies, respectively); b) space motion trajectory of bodies comprising the gravitational system

Taking Step 3 into account, it is clear that a relative distance between a pair of bodies cannot become infinite, i.e., the phase points in the configuration space cannot go to infinity. This factor, in turn, means that the cluster of points in Figure 7*a* is compact, with the pair of bodies exhibiting stable motion.

Let us proceed to illustrate the presented algorithm by solving a system of equations describing the motion of a pair of bodies in terms of relative positions and velocities:

$$\dot{\mathbf{q}} = \mathbf{u}, \quad \dot{\mathbf{u}} = -\frac{m_1 + m_2}{q^3}\mathbf{q}. \tag{48}$$

The system of equations (48) permitting the conservation laws (46)–(47) can be solved using standard measures [Landau, Lifshitz, 1976]. They consist of the following steps: 1) a coordinate system is introduced, in which, for example, the applicate axis is directed along the angular momentum vector; 2) a polar coordinate system is considered for relative positions and velocities in the plane of the abscissa and ordinate axes; 3) for example, an equation describing the dynamics of the distance q between a pair of bodies as a time function is derived, then

$$\dot{q} = \pm \tfrac{(m_1+m_2)k_0}{m_1 m_2} \sqrt{(q_1^{-1} - q^{-1})(q^{-1} - q_2^{-1})}\,, \tag{49}$$

where $q_{1,2}^{-1} = \frac{m_1^2 m_2^2 \pm \sqrt{D}}{(m_1+m_2)k_0^2}$. Two signs before the radical in the right side of (49) indicate two solutions that intersect at the rotation points $q_{1,2}$, with the solution q oscillating within the segment $[q_1, q_2]$ for a period of $T = 2\pi\sqrt{\frac{m_1^3 m_2^3}{8(m_1+m_2)(-e_0)^3}}$.

One of our aims is to test a numerical method for solving the system of equations (49) along with a procedure for correcting the solution, which ensures that the laws of angular momentum (46) and energy (47) conservation are satisfied. Given (17), we obtain the following transformation:

$$\mathbf{q} \to (H_x q_x, H_y q_y, H_z q_z), \quad \mathbf{u} \to (G_x u_x, G_y u_y, G_z u_z). \tag{50}$$

Let us select the undetermined coefficients H_x, H_y, H_z, G_x, G_y, and G_z such that the laws of angular momentum and energy conservation are fulfilled, i. e., the following equations hold:

$$\begin{aligned}
&c_{11} H_y G_z - c_{12} H_z G_y = k_{0,x},\\
&c_{21} H_z G_x - c_{22} H_x G_z = k_{0,y},\\
&c_{31} H_x G_y - c_{32} H_y G_x = k_{0,z},\\
&\tfrac{1}{2} c_{41} G_x^2 + \tfrac{1}{2} c_{42} G_y^2 + \tfrac{1}{2} c_{43} G_z^2 - \tfrac{m_1 m_2}{\sqrt{H_x^2 q_{x,1,2}^2 + H_y^2 q_{y,1,2}^2 + H_z^2 q_{z,1,2}^2}} = e_0,
\end{aligned} \tag{51}$$

where

$$c_{11} = \mu q_y u_z, \quad c_{12} = \mu q_z u_y, \quad c_{21} = \mu q_z u_x, \quad c_{22} = \mu q_x u_z, \quad c_{31} = \mu q_x u_y, \quad c_{32} = \mu q_y u_x;$$

$$c_{41} = \mu u_x^2, \quad c_{42} = \mu u_y^2, \quad c_{43} = \mu u_z^2;$$

$\mu = \frac{m_1 m_2}{m_1 + m_2}$ — reduced mass.

Let us solve the nonlinear algebraic system of four equations (51) for the six unknowns in the linear approximation. Suppose that $H_x = 1 + h_x, |h_x| \ll 1$, …, $G_x = 1 + g_x, |g_x| \ll 1$; then equations (51) take the following form in the linear approximation:

$$\begin{aligned}
&f_1 = c_{11}(h_y + g_z) - c_{12}(h_z + g_y) - \Delta k_x = 0,\\
&f_2 = c_{21}(h_z + g_x) - c_{22}(h_x + g_z) - \Delta k_y = 0,\\
&f_3 = c_{31}(h_x + g_y) - c_{32}(h_y + g_x) - \Delta k_z = 0,\\
&f_4 = c_{41} g_x + c_{42} g_y + c_{43} g_z + c_{44} h_x + c_{45} h_y + c_{46} h_z - \Delta e = 0,
\end{aligned} \tag{52}$$

where

$$c_{44} = \tfrac{m_1 m_2}{q^3} q_x^2, \quad c_{45} = \tfrac{m_1 m_2}{q^3} q_y^2, \quad c_{46} = \tfrac{m_1 m_2}{q^3} q_z^2;$$

$$\begin{aligned}
&\Delta k_x = k_{0,x} - \mu(q_y u_z - q_z u_y), \quad \Delta k_y = k_{0,y} - \mu(q_z u_x - q_x u_z),\\
&\Delta k_z = k_{0,z} - \mu(q_x u_y - q_y u_x), \quad \Delta e = e_0 - \tfrac{1}{2}\mu \mathbf{u}^2 + \tfrac{m_1 m_2}{q}.
\end{aligned}$$

The undetermined linear system of equations (52) for the six unknowns h_x, h_y, h_z, g_x, g_y, and g_z is solved by minimizing a function of the following form:

$$\Phi(h_x, h_y, h_z, g_x, g_y, g_z) = \frac{f_1^2 + f_2^2 + f_3^2}{2(\Delta k_x^2 + \Delta k_y^2 + \Delta k_z^2)} + \frac{f_4^2}{2\Delta e^2}. \tag{53}$$

The minimization procedure performed, for example, via the gradient descent method typically implies that the initial values of the unknowns $h_x^{(0)}, h_y^{(0)}, h_z^{(0)}, g_x^{(0)}, g_y^{(0)}$, and $g_z^{(0)}$ are given. Let us choose the latter as uniformly random numbers from the interval [–δ, δ]. The system of equations (48) is solved numerically in [0, T_1] using conventional methods. If, after applying the procedure (51)–(53), it turns out that at least one of the six inequalities $|h_x| < \delta, ..., |g_z| < \delta$ is violated, the time interval is reduced, and the calculation is repeated from the same initial data. If all six inequalities remain valid, the solutions of equations (48) are corrected according to the formulas

$$\mathbf{q} \to ((1+h_x)q_x, (1+h_y)q_y, (1+h_z)q_z), \quad \mathbf{u} \to ((1+g_x)u_x, (1+g_y)u_y, (1+g_z)u_z).$$

Next, a new integration interval $[T_1, T_1 + T_2]$ is selected, and the entire procedure is repeated. At this point, the presentation of the algorithm for solving the system of equations (48) and the procedure for reducing solutions to the given values of the laws of angular momentum $\mathbf{k}_0$ and energy e_0 conservation can be considered complete.

Figure 7*b* illustrates the construction of motion trajectories in space for a pair of bodies comprising a gravitational system, taking into account the procedure (50)–(53) for correcting the solutions to the given values of the laws of angular momentum $\mathbf{k}_0$ and energy e_0 conservation. The following parameter values are adopted: $k_0 = 0.125$, $e_0 = -0.25$, $T_1 = 1$, and $\delta = 0.005$. The final integration interval is [0, 646.55], while the oscillation period corresponding to the analytical solution amounts to $T = 2.67$. In Figure 7*b*, the pentagrams indicate when the procedure for reducing the solution to the specified angular momentum and energy values is applied. In total, $n = 500$ procedures are performed.

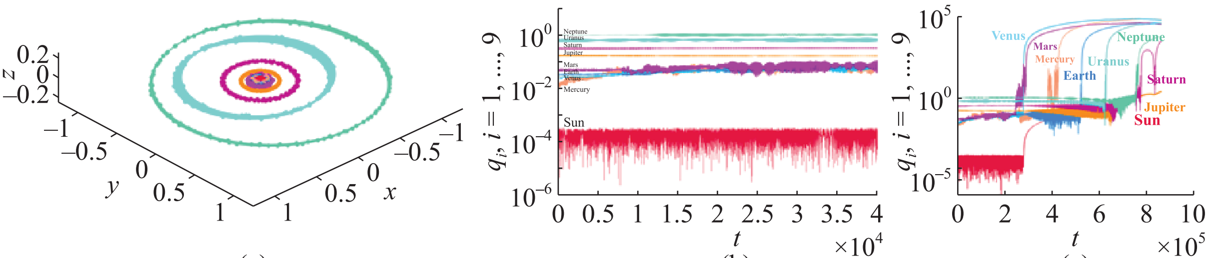
6. Stability of the Solar System

Let us move on to consider the results of modeling the Solar System dynamics. Actual data are adopted as the initial positions and velocities of bodies, i.e., when the planets (given the barycenter of the "Earth + Moon" system) move in the vicinity of the ecliptic plane, i.e., in a three-dimensional barycentric coordinate system. Several authors [Simon et al., 2013] present ephemerides obtained over a long interval in [Index of…, accessed November 2020]. For our calculations, a Julian ephemeris date of JD 2405730.5 is chosen, which corresponds to 06/25/1874 of the Gregorian calendar. The correction of solutions to comply with conservation laws is carried out according to the procedure (16)–(20).

Figure 8*a* presents a typical position of the orbits of the Solar System planets after performing calculations for a period of $\approx 1.05\times10^7$ years, i.e., more than ten million years. The orbits of the planets are plotted according to the latter $\approx 9.01\times10^4$ years. The points on the graphs of planetary trajectories indicate the moments of correcting solutions to comply with the conservation laws. The other adopted calculation parameters are as follows: $T_1 = T_{\max} = 30$, $n = 12'984$, $\delta = 0.025$, as well as the relative and absolute accuracy values of the solver of the system of differential equations amounting to $2\cdot10^{-7}$ and $2\cdot10^{-8}$, respectively. Figure 8*b* shows the time dependences of the distances from the center of mass to the Sun and the planets during the entire calculation period of $[0, 1.05\cdot10^7]$ years.

The graphs in Figure 8*b* show that the orbits of the lightest planets, i. e., Mercury and Mars, vary considerably over time, while the entire planetary system remains stable.

The dynamics of the Solar System planets calculated from actual initial positions, but with rougher relative and absolute accuracies of 10^{-6} and 10^{-7}, respectively, revealed that a noticeable transformation of the Solar System began after $\approx 6.5 \cdot 10^6$ years. Figure 8*c* shows the result of the transformation lasting for $[0, 2.27 \cdot 10^7]$ years, i.e., more than twenty million years. The transformation scenario for the Solar System is reduced to the order in which the planets leave the Solar System: Venus → → Mars → Mercury → Earth → Uranus → Neptune → Saturn, followed by the formation of the double system "Sun + Jupiter." The other adopted calculation parameters are as follows: $T_1 = T_{\max} = 30$, $n = 29'500$, and $\delta = 0.025$. All possible transformation scenarios for the Solar System can be divided into three groups: 1) most of the planets leave the Solar System; a double system consisting of the Sun and one of the heavy planets remains; 2) two planets crash into each other, or one of the planets crashes into the Sun; 3) a mixed version.



(a) (b) (c)

Figure 8. Fragment showing the dynamics of the Solar System planets (a); time dependence of the distances from the center of mass to the Sun and the planets (b); dynamics of the Solar System transformation (c)

Note that the onset of a noticeable transformation of the Solar System can be significantly pushed forward in time if the values of the absolute and relative accuracy of the used calculation scheme are reduced. However, the computational resources available to the present author do not allow the above calculation scheme to be applied to the relative and absolute accuracies of much smaller values of $2 \cdot 10^{-7}$ and $2 \cdot 10^{-8}$, respectively.

7. Conclusion

The present article studies the stability of a gravitational system comprising multiple bodies by means of a computational experiment. In order to perform a long-term calculation of the systems of differential equations, a new method was developed. This method combines the use of conventional numerical methods for solving differential equations, as well as a specially designed procedure for correcting solutions to the given integrals of motion. This correction procedure makes the method conservative while introducing a random component into the calculations. As a result, this method can be referred to the class of Monte Carlo methods, with the entire computation scheme becoming stochastic-deterministic.

The paper presents a generator of phase space points from the hypersurface of the conservation laws of a gravitational system. The performed computational experiment indicates that the accumulation of phase space points in the configuration space is not compact when the number of bodies comprising the gravitational system exceeds two. This factor means that a general-position gravitational system is unstable (at $N > 2$), including when the total energy is negative. In the present work, the general position refers to a situation where the masses, as well as the initial positions and velocities of bodies, are random variables selected from certain fixed ranges.

The method described in this paper is applied to calculating the Solar System dynamics, drawing on the actual values of ephemerides. Due to the limited computational resources, the stability of the Solar System is confirmed by the performed calculation only for a period of about ten million years. At the end of the specified period, the structure of the Solar System is generally preserved, except for a noticeable orbital realignment of Mercury and Mars. At rougher values of the relative and absolute accuracies of the calculation algorithm, the full transformation cycle of the Solar System can be traced, which includes the release of the planets and the final formation of the “Sun + Jupiter” pair.

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